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R-Preirresolute Functions

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ABSTRACT. A strong form of a preirresolute function, called R-preirresolute, is introduced. It is established that R-preirresolute implies strongly preirresolute and that R-preirresolute functions satisfy a strong form of the closed graph property. Also it shown that R-preirresolute functions are closely related to pre-regularity.

1. INTRODUCTION

Several generalizations of preirresoluteness have been developed recently. In 1996 Pal and Bhattacharyya [8] introduced strong preirresoluteness and quasi preirresoluteness and in 2002 Noiri [7] developed θ-preirresoluteness. The purpose of this note is to introduce a form of preirresoluteness stronger than all of these, which we call R-preirresoluteness. This concept is based on the notion of R-continuity developed by Konstadilaki-Savvopoulou and Janković [3] in 1992. Besides developing the basic properties of these functions, we establish that R-preirresolute functions satisfy a strong form of the closed graph property. Also we show that R-preirresoluteness is closely related to pre-regularity. For example, it is shown that the range of an R-preirresolute, M-preopen function is pre-regular and that a space Y is pre-regular if and only if every preirresolute function with codomain Y is R-preirresolute. In general we establish that R-preirresolute functions are related to preirresolute functions and pre-regular spaces in much the same way as R-continuous functions are related to continuous functions and regular spaces.

2. PRELIMINARIES

Throughout this paper X, Y, and Z represent topological spaces with no separation axioms assumed unless explicitly stated. All sets are assumed to be subsets of topological spaces. For a set A the closure and interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is pre-open [5] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. The collection of preopen subsets of a space X is denoted by $\mathrm{PO}(X)$ and the collection of all preopen subsets of X containing a fixed point x of X is denoted by $\mathrm{PO}(X, x)$. A set is preclosed provided its complement is preopen. The intersection of all preclosed sets containing a set A is called the preclosure [2] of A and is signified by $\mathrm{pCl}(A)$. The preinterior of A, denoted by $\mathrm{pInt}(A)$, is the union of all preopen sets contained in A. A set A is semi-open [4] provided there is an open set U such that $U \subseteq A \subseteq \mathrm{Cl}(U)$ or equivalently $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The collection

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of semi-open subsets of a space X is denoted by SO(X) and the collection of all semi-open subsets of X containing a fixed point x of X is denoted by SO(X, x).

Definition 2.1. A function $f : X \to Y$ is said to be R-continuous (Konstadilaki-Savvopoulou and Janković [3]) provided that for every $x \in X$ and every open subset V of Y containing $f(x)$, there is an open subset U of X containing x for which $\mathrm{Cl}(f(U)) \subseteq V$.

Definition 2.2. A function $f : X \to Y$ is said to be preirresolute (Reilly and Vamanamurthy [9]) provided that $f^{-1}(V) \in \mathrm{PO}(X)$ for every $V \in \mathrm{PO}(Y)$.

Definition 2.3. A function $f : X \to Y$ is said to be strongly preirresolute (resp. quasi preirresolute) (Pal and Bhattacharyya [8]) if for every $x \in X$ and every $V \in \mathrm{PO}(Y, f(x))$, there exists $U \in \mathrm{PO}(X, x)$ such that $f(\mathrm{pCl}(U)) \subseteq V$ (resp. $f(U) \subseteq \mathrm{pCl}(V)$).

Definition 2.4. A function $f : X \to Y$ is said to be θ-preirresolute (Noiri [7]) if for every $x \in X$ and every $V \in \mathrm{PO}(Y, f(x))$, there exists $U \in \mathrm{PO}(X, x)$ such that $f(\mathrm{pCl}(U)) \subseteq \mathrm{pCl}(V)$.

Definition 2.5. A function $f : X \to Y$ is said to be M-preopen (Mashhour et al. [6]) (M-preclosed) provided that for every preopen (preclosed) subset A of X, $f(A)$ is preopen (preclosed) in Y.

3. R-preirresolute Functions

We define a function $f : X \to Y$ to be R-preirresolute provided that for every $x \in X$ and every $V \in \mathrm{PO}(Y, f(x))$ there exists $U \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(f(U)) \subseteq V$. Obviously an R-preirresolute function is preirresolute.

Theorem 3.1. *For a function $f : X \to Y$ the following statements are equivalent:*

(a) *f is R-preirresolute.*

(b) *For every $x \in X$ and every $V \in PO(Y, f(x))$ there exists $U \in PO(X, x)$ such that $pCl(f(pCl(U))) \subseteq V$.*

(c) *For every $x \in X$ and every preclosed subset F of Y with $f(x) \notin F$, there exist sets $U \in PO(X, x)$ and $V \in PO(Y)$ with $F \subseteq V$ for which $f(pCl(U)) \cap V = \emptyset$.*

(d) *For every $x \in X$ and every preclosed subset F of Y with $f(x) \notin F$, there exist sets $U \in PO(X, x)$ and $V \in PO(Y)$ with $F \subseteq V$ for which $f(U) \cap V = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). Let $x \in X$ and let $V \in \mathrm{PO}(Y, f(x))$. Then there exists $U \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(f(U)) \subseteq V$. Since f is also preirresolute, $f(\mathrm{pCl}(U)) \subseteq \mathrm{pCl}(f(U))$ and therefore $\mathrm{pCl}(f(\mathrm{pCl}(U))) \subseteq \mathrm{pCl}(f(U)) \subseteq V$.

(b) $\Rightarrow$ (c). Let $x \in X$ and let F be a preclosed subset of Y with $f(x) \notin F$. Then $f(x) \in Y - F$ which is preopen. Hence there exists $U \in \mathrm{PO}(X, x)$ for which $\mathrm{pCl}(f(\mathrm{pCl}(U))) \subseteq Y - F$. Then for $V = Y - \mathrm{pCl}(f(\mathrm{pCl}(U)))$ we have $F \subseteq V$ and, since $\mathrm{pCl}(f(\mathrm{pCl}(U)))$ is preclosed, V is preopen. Also $f(\mathrm{pCl}(U)) \cap V = \emptyset$.

(c) $\Rightarrow$ (d). Clear.

(d) $\Rightarrow$ (a). Let $x \in X$ and let $V \in \mathrm{PO}(Y, f(x))$. Then $Y - V$ is preclosed and $f(x) \notin Y - V$. Therefore there exist sets $U \in \mathrm{PO}(X, x)$ and $W \in \mathrm{PO}(Y)$ with $Y - V \subseteq W$ for which $f(U) \cap W = \emptyset$. Since $f(U) \subseteq Y - W$ which is preclosed, it follows that $\mathrm{pCl}(f(U)) \subseteq Y - W \subseteq V$ which proves that f is R-preirresolute. $\square$

The next result is an immediate consequence of Theorem 3.1(b).

Theorem 3.2. *If $f : X \to Y$ is R-preirresolute, then f is strongly preirresolute.*

The following example shows that strongly preirresolute is not equivalent to R-preirresolute.

Example 3.1. Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$, and σ be the discrete topology on X. The identity mapping $f : (X, \sigma) \to (X, \tau)$ is strongly preirresolute, but not R-preirresolute.

Combining Theorem 3.2, Example 3.1 and Remark 2.2 in Noiri [7], yields the following implications, none of which are reversible: R-preirresolute $\Longrightarrow$ strongly preirresolute $\Longrightarrow$ preirresolute $\Longrightarrow$ θ-preirresolute $\Longrightarrow$ quasi preirresolute

Definition 3.6. A function $f : X \to Y$ is said to be weakly M-preclosed provided that for every $U \in \mathrm{PO}(X)$, $\mathrm{pCl}(f(U)) \subseteq f(\mathrm{pCl}(U))$.

Obviously M-preclosed implies weakly M-preclosed. The following example shows that the converse implication does not hold.

Example 3.2. Let $X = \{a, b\}$ be the Sierpinski space with the topology $\tau = \{X, \emptyset, \{a\}\}$. Then the function $f : X \to X$ given by $f(a) = b$ and $f(b) = a$ is weakly M-preclosed but not M-preclosed.

Theorem 3.3. *If $f : X \to Y$ is strongly preirresolute and weakly M-preclosed, then f is R-preirresolute.*

Proof. Let $x \in X$ and $V \in \mathrm{PO}(Y, f(x))$. Since f is strongly preirresolute, there exists a set $U \in \mathrm{PO}(X, x)$ for which $f(\mathrm{pCl}(U)) \subseteq V$. Because f is weakly M-preclosed, $\mathrm{pCl}(f(U)) \subseteq f(\mathrm{pCl}(U)) \subseteq V$, which proves that f is R-preirresolute. $\square$

The function in Example 3.2 is weakly M-preclosed but not R-preirresolute. Hence weakly M-preclosed alone does not imply R-preirresolute.

Definition 3.7. A function $f : X \to Y$ is said to be contra M-preopen provided that for every $U \in \mathrm{PO}(X)$, $f(U)$ is preclosed.

Theorem 3.4. *If $f : X \to Y$ is preirresolute and contra M-preopen, then f is R-preirresolute.*

Proof. Let $x \in X$ and $V \in \mathrm{PO}(Y, f(x))$. Then $f^{-1}(V) \in \mathrm{PO}(X, x)$ and, since $f(f^{-1}(V))$ is preclosed, $\mathrm{pCl}(f(f^{-1}(V))) = f(f^{-1}(V)) \subseteq V$. $\square$

4. Properties related to pre-regularity

Definition 4.8. A space X is said to be pre-regular (Pal and Bhattacharyya [8]) if for every preclosed set A and every $x \in X - A$ there exist disjoint sets $U, V \in \mathrm{PO}(X)$ with $x \in U$ and $A \subseteq V$.

Remark 4.1. It is also observed in [8] that a space X is pre-regular if and only if for every $x \in X$ and every $U \in \mathrm{PO}(X, x)$ there exists $V \in \mathrm{PO}(X, x)$ such that $x \in V \subseteq \mathrm{pCl}(V) \subseteq U$.

Theorem 4.5. *If $f : X \to Y$ is preirresolute and Y is pre-regular, then f is R-preirresolute.*

Proof. Let $x \in X$ and $V \in \mathrm{PO}(Y, f(x))$. Since Y is pre-regular, there exists $W \in \mathrm{PO}(Y, f(x))$ such that $f(x) \in W \subseteq \mathrm{pCl}(W) \subseteq V$. Since f is preirresolute, there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subseteq W$. Then $\mathrm{pCl}(f(U)) \subseteq \mathrm{pCl}(W) \subseteq V$, which proves that f is R-preirresolute. □

The following result is a slightly generalized version of Theorem 4.5.

Theorem 4.6. *For a space Y the following statements are equivalent:*

(a) *Y is pre-regular.*
(b) *The identity mapping $i : Y \to Y$ is R-preirresolute.*
(c) *For every space X every preirresolute function $f : X \to Y$ is R-preirresolute.*

Proof. (b) ⇒ (a) Clear.

(a) ⇒ (c) Follows from Theorem 4.5.

(c) ⇒ (b) Since the identity mapping is obviously preirresolute, the result follows immediately. □

Theorem 4.7. *If $f : X \to Y$ is an R-preirresolute, M-preopen, surjection, then Y is pre-regular.*

Proof. Let $y \in Y$ and $V \in \mathrm{PO}(Y, y)$. Then let $x \in X$ for which $y = f(x)$. Since f is R-preirresolute, there exists $U \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(f(U)) \subseteq V$. Then, since $f(U) \in \mathrm{PO}(Y)$ and $y \in f(U) \subseteq \mathrm{pCl}(f(U)) \subseteq V$, we see that Y is pre-regular. □

Remark 4.2. Theorem 4.3 can also be proved by replacing the requirement that f be M-preopen with the apparent weaker requirement that $f(U) \subseteq \mathrm{pInt}(\mathrm{pCl}(f(U)))$ for every $U \in \mathrm{PO}(X)$. However, it can be shown this condition is equivalent to M-preopenness.

Combining Theorems 4.5 and 4.7 yields the following result.

Theorem 4.8. *Let $f : X \to Y$ be an preirresolute, M-preopen surjection. Then f is R-preirresolute if and only if Y is pre-regular.*

Next we establish conditions under which pre-regularity is preserved. The following result from Pal and Bhattacharyya [8] will be required.

Lemma 4.1. (Pal and Bhattacharyya [8]) *If $f : X \to Y$ is a function and X is pre-regular, then f is preirresolute if and only if f is strongly preirresolute.*

Theorem 4.9. *If $f : X \to Y$ is a preirresolute, M-preopen, weakly M-preclosed surjection and X is pre-regular, then Y is pre-regular*

Proof. Since f is preirresolute and X is pre-regular it follows from Lemma 4.1 that f is strongly preirresolute. Then, because f is strongly preirresolute and weakly M-preclosed, Theorem 3.3 implies that f is R-preirresolute. Finally, since f is R-preirresolute, M-preopen, and surjective, it follows from Theorem 4.7 that Y is pre-regular. □

5. Additional properties

For a function $f : X \to Y$ the graph of f, $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Definition 5.9. The graph of a function $f : X \to Y$, $G(f)$, is said to be strongly preclosed with respect to X provided that for every $(x, y) \notin G(f)$ there exist $U \in \mathrm{PO}(X, x)$ and $V \in \mathrm{PO}(Y, y)$ such that $(\mathrm{pCl}(U) \times V) \cap G(f) = \emptyset$.

Definition 5.10. A space X is said to be pre-T_1 provided that for every pair of distinct points x and y in X there exist sets $U \in \mathrm{PO}(X, x)$ and $V \in \mathrm{PO}(X, y)$ such that $y \notin U$ and $x \notin V$.

Theorem 5.10. *If $f : X \to Y$ is R-preirresolute and Y is pre-T_1, then $G(f)$ is strongly preclosed with respect to X.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is pre-T_1, there exists $V \in \mathrm{PO}(Y, f(x))$ such that $y \notin V$. Since f is R-preirresolute, there exists $U \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(f(U)) \subseteq V$. Since $y \notin V$, obviously $y \notin \mathrm{pCl}(f(U))$. Thus $(x, y) \in \mathrm{pCl}(U) \times (Y - \mathrm{pCl}(f(U)))$. Finally, because f is preirresolute, $f(\mathrm{pCl}(U)) \subseteq \mathrm{pCl}(f(U))$. It then follows that $\mathrm{pCl}(U) \times (Y - \mathrm{pCl}(f(U))) \subseteq X \times Y - G(f)$, which proves that $G(f)$ is strongly preclosed with respect to X. □

Definition 5.11. A space X is said to be pre-R_0 provided that for every $x \in X$ and every $U \in \mathrm{PO}(X, x)$, $\mathrm{pCl}(\{x\}) \subseteq U$.

Theorem 5.11. *If $f : X \to Y$ is R-preirresolute and surjective, then Y is pre-R_0.*

Proof. Let $y \in Y$ and let $V \in \mathrm{PO}(Y, y)$. Then let $x \in X$ for which $y = f(x)$. Since f is R-preirresolute, there exists $U \in \mathrm{PO}(X, x)$ for which $\mathrm{pCl}(f(U)) \subseteq V$. Then we see that $\mathrm{pCl}(\{y\}) = \mathrm{pCl}(\{f(x)\}) \subseteq \mathrm{pCl}(f(U)) \subseteq V$ and hence Y is pre-R_0. □

We conclude this note with several results concerning the composition and restriction of R-preirresolute functions. The proof of the next result is straightforward and is omitted.

Theorem 5.12. *If $f : X \to Y$ is preirresolute and $g : Y \to Z$ is R-preirresolute, then $g \circ f : X \to Z$ is R-preirresolute.*

Corollary 5.1. *If $f : X \to Y$ is R-preirresolute and $g : Y \to Z$ is R-preirresolute, then $g \circ f : X \to Z$ is R-preirresolute.*

Theorem 5.13. *If $f : X \to Y$ is R-preirresolute and $g : Y \to Z$ is preirresolute and M-preclosed, then $g \circ f : X \to Z$ is R-preirresolute.*

Proof. Let $x \in X$ and $V \in \mathrm{PO}(Z, g(f(x)))$. Since g is preirresolute, $g^{-1}(V) \in \mathrm{PO}(Y, f(x))$ and, since f is R-preirresolute, there exists $W \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(f(W)) \subseteq g^{-1}(V)$ or $g(\mathrm{pCl}(f(W))) \subseteq V$. Because g is M-preclosed, $g(\mathrm{pCl}(f(W)))$ is preclosed. Therefore we have $\mathrm{pCl}(g(f(W))) \subseteq \mathrm{pCl}(g(\mathrm{pCl}(f(W))))$ $= g(\mathrm{pCl}(f(W))) \subseteq V$ which proves that $g \circ f$ is R-preirresolute. □

In the next result we modify the hypothesis of Theorem 5.13 by weakening the requirement on g and strengthening the requirement on f.

Theorem 5.14. *If $f : X \to Y$ is R-preirresolute and M-preopen and $g : Y \to Z$ is preirresolute and weakly M-preclosed, then $g \circ f : X \to Z$ is R-preirresolute.*

Proof. Let $x \in X$ and $V \in \mathrm{PO}(Z, g(f(x)))$. Since g is preirresolute, $g^{-1}(V) \in \mathrm{PO}(Y, f(x))$. Because f is R-preirresolute, there exists $W \in \mathrm{PO}(X, x)$ for which $\mathrm{pCl}(f(W)) \subseteq g^{-1}(V)$ or $g(\mathrm{pCl}(f(W))) \subseteq V$. Then, using the M-preopenness of f and the weakly M-preclosedness of g, we have $\mathrm{pCl}(g(f(W))) \subseteq g(\mathrm{pCl}(f(W))) \subseteq V$, which proves that $g \circ f$ is R-preirresolute. □

Theorem 5.15. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f : X \to Z$ is R-preirresolute and f is a M-preopen surjection, then g is R-preirresolute.*

Proof. Let $y \in Y$ and $V \in \mathrm{PO}(Z, g(y))$. Since f is surjective, there exists $x \in X$ for which $f(x) = y$. Because $g \circ f$ is R-preirresolute, we have $W \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(g(f(W))) \subseteq V$. Since f is M-preopen, $f(W) \in \mathrm{PO}(Y, y)$ and hence g is R-preirresolute. □

The following lemma will be useful.

Lemma 5.2. (Császár [1]) *If $A \in SO(X)$ and $B \in PO(X)$, then $A \cap B \in PO(A)$.*

Theorem 5.16. *If $f : X \to Y$ is R-preirresolute and $A \in SO(X)$, then $f|_A : A \to Y$ is R-preirresolute.*

Proof. Let $x \in A$ and $V \in \mathrm{PO}(Y, f(x))$. Then there exists $U \in \mathrm{PO}(X, x)$ such that $\mathrm{pCl}(f(U)) \subseteq V$. Since $A \in \mathrm{SO}(X)$, $U \cap A \in \mathrm{PO}(A, x)$. Then we see that $\mathrm{pCl}(f|_A(U \cap A)) = \mathrm{pCl}(f(U \cap A)) \subseteq \mathrm{pCl}(f(U)) \subseteq V$ which proves that $f|_A : A \to Y$ is R-preirresolute. □

References

[1] Császár, Á., *Generalized open sets*, Acta Math. Hungar., **75** (1997), 65–87
[2] El Deeb, N., Hasanein, I. A., Mashhour, A. S. and Noiri, T., *On p-regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **27** (1983), 311–315
[3] Konstadilaki-Savvopoulou, Ch. and Janković, D., *R-continuous functions*, Internat. J. Math. Math. Sci., **15** (1992), 57–64
[4] Levine, N., *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41
[5] Mashhour, A. S., Abd El-Monsef, and S. N. El-Deeb, S. N., *On Precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, **53** (1982), 47–53
[6] Mashhour, Abd El Monsef, M. E. and Hasanein, I. A., *On pretopological spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **28** (1984), 39–45
[7] Noiri, T., *On θ-preirresolute functions*, Acta Math. Hungar., **95** (2002), 287–298
[8] Pal, M. C. and Bhattacharyya, P., *Feeble and strong forms of preirresolute functions*, Bull. Malaysian Math. Soc., **19** (1996), 63–75
[9] Reilly, I. L. and Vamanamurthy, M. K., *On α-continuity in topological spaces*, Acta Math. Hungar., **45** (1985), 27–32

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