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On the Schurer-Stancu approximation formula

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ABSTRACT. Let $p \geq 0$ be a given integer and let $\alpha, \beta \in \mathbb{R}$ be parameters satisfying the conditions $0 \leq \alpha \leq \beta$. In [1] was introduced the Schurer-Stancu operator $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$ defined for any $m \in \mathbb{N}$ and any $f \in C([0,1+p])$ by (1.1). Considering the Schurer-Stancu approximation formula (1.3), one studies its remainder term. As particular cases follow the remainder terms of Schurer, Stancu and respectively Bernstein approximation formulas.

1. PRELIMINARIES

Let $p \geq 0$ be a given integer and let α, β be real parameters satisfying $0 \leq \alpha \leq \beta$. The Schurer-Stancu operator (see [1]) $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$ is defined for any $m \in \mathbb{N}$ and any $f \in C([0,1+p])$ by

$$\text{(1.1)} \qquad \left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right),$$

where

$$\text{(1.2)} \qquad \widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k},$$

are the fundamental Schurer polynomials (see [7]).

Note that $\widetilde{S}_{m,p}^{(0,0)}$ is the operator introduced by F. Schurer in 1962 (see [7]), $\widetilde{S}_{m,0}^{(\alpha,\beta)}$ is the operator introduced and studied by D.D. Stancu in 1968 (see [11]) and $\widetilde{S}_{m,0}^{(0,0)}$ is the classical Bernstein operator (see [4]).

Some approximation properties of operator (1.1) were studied in our earlier papers [1], [2], [3], [4].

In what follows, we consider the Schurer-Stancu approximation formula

$$\text{(1.3)} \qquad f = \widetilde{S}_{m,p}^{(\alpha,\beta)} f + \widetilde{R}_{m,p}^{(\alpha,\beta)} f,$$

and we are dealing with the expression of remainder $\widetilde{R}_{m,p}^{(\alpha,\beta)} f$ using the first and second order divided differences of approximated function.

The brackets denote divided differences. We recall that if $I \subseteq \mathbb{R}$ is an interval, $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f : I \to \mathbb{R}$ is bounded on I, the first order divided differences of f is defined by:

$$\text{(1.4)} \qquad [x_1, x_2; f] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

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The divided difference of n-th order is defined by the recurrence relation:

$$[x_0, x_1, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}. \tag{1.5}$$

The function $f : I \to \mathbb{R}$, bounded on I, is convex (concave) of n-th order on I if and only if for any distinct points $x_0, x_1, \dots, x_{n+1} \in I$, the following

$$[x_0, x_1, \dots, x_{n+1}; f] \underset{(<)}{>} 0 \tag{1.6}$$

holds.

Clearly, a convex function of first order is monotonous increasing on I, a convex function of second order is convex in usually sense on I, etc.

The notion of convexity of n-th order was introduced by the great Romanian mathematician T. Popoviciu.

Let us to recall another important result due to T. Popoviciu.

Theorem 1.1. [6] *If the linear functional $A \in C[a, b]$ satisfies the conditions:*
(i) *$A(e_1) = A(e_2) = \cdots = A(e_n) = 0$, $A(e^{n+1}) \neq 0$;*
(ii) *for any $f \in C([a, b])$ convex of n-th order on $[a, b]$, $A(f) \neq 0$,*
then, for any $f \in C([a, b])$ there exist the points $a \le \xi_1 < \xi_2 < \cdots < \xi_{n+2} \le b$ such that the following

$$A(f) = A(e_{n+1})[\xi_1, \dots, \xi_{n+2}; f] \tag{1.7}$$

holds.

2. Main results

We shall prove

Theorem 2.2. *For any $f \in C([0, 1 + p])$ the remainder term of Schurer-Stancu approximation formula (1.3) can be expressed under the form*

$$\left(\widetilde{R}^{(\alpha,\beta)}_{m,p} f\right)(x) = \frac{1}{m+\beta}\{(\beta - p)x - \alpha\} \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) \left[x, \frac{k+\alpha}{m+\beta}; f\right] - \tag{2.8}$$
$$- \frac{m+p}{(m+\beta)^2} x(1-x) \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) \left[x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta}; f\right].$$

Proof. From (1.1) and (1.3) follows:

$$\left(\widetilde{R}^{(\alpha,\beta)}_{m,p} f\right)(x) = \frac{1}{m+\beta} \sum_{k=0}^{m+p} \{(m+\beta)x - (k+\alpha)\} \left[x, \frac{k+\alpha}{m+\beta}; f\right]. \tag{2.9}$$

But

$$(m+\beta)x - (k+\alpha) = (m+p-k)x + (\beta - p)x - \alpha - k(1-x) \tag{2.10}$$

Taking the simple identity (2.10) into account, from (2.9) yields:

$$(2.11)\quad \left(\widetilde{R}_{m,p}^{(\alpha,\beta)}f\right)(x)=\frac{(\beta-p)x-\alpha}{m+\beta}\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\left[x,\frac{k+\alpha}{m+\beta};f\right]+$$
$$+\frac{x}{m+\beta}\sum_{k=0}^{m+p-1}(m+p-k)\widetilde{p}_{m,k}(x)\left[x,\frac{k+\alpha}{m+\beta}f\right]-$$
$$-\frac{1-x}{m+\beta}\sum_{k=1}^{m+p}k\,\widetilde{p}_{m,k}(x)\left[x,\frac{k+\alpha}{m+\beta};f\right]$$

After some elementary transformations, (2.11) can be expressed under the form:

$$(2.12)\quad \left(\widetilde{R}_{m,p}^{(\alpha,\beta)}f\right)(x)=\frac{1}{m+\beta}\{(\beta-p)x-\alpha\}\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\left[x,\frac{k+\alpha}{m+\beta};f\right]+$$
$$+\frac{m+p}{m+\beta}x(1-x)\sum_{k=0}^{m+p-1}\widetilde{p}_{m-1,k}(x)\left\{\left[x,\frac{k+\alpha}{m+\beta}f\right]-\left[x,\frac{k+\alpha+1}{m+\beta};f\right]\right\}.$$

Applying (1.5) for $n=2$, $x_0=x$, $x_1=\dfrac{k+\alpha}{m+\beta}$, $x_2=\dfrac{k+\alpha+1}{m+\beta}$ follows:

$$(2.13)\quad \left[x,\frac{k+\alpha}{m+\beta};f\right]-\left[x,\frac{k+\alpha+1}{m+\beta};f\right]=$$
$$=-\frac{1}{m+\beta}\left[x,\frac{k+\alpha}{m+\beta},\frac{k+\alpha+1}{m+\beta};f\right].$$

Using (2.13), from (2.12) we get:

$$\left(\widetilde{R}_{m,p}^{(\alpha,\beta)}f\right)(x)=\frac{1}{m+\beta}\{(\beta-p)x-\alpha\}\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\left[x,\frac{k+\alpha}{m+\beta};f\right]-$$
$$-\frac{m+p}{(m+\beta)^2}x(1-x)\sum_{k=0}^{m+p-1}\widetilde{p}_{m-1,k}(x)\left[x,\frac{k+\alpha}{m+\beta},\frac{k+\alpha+1}{m+\beta};f\right],$$

which is the desired result (2.8). □

Lemma 2.1. *For any $f\in C([0,1+p])$ and any $x\in[0,1]$ there exist the points ξ_1, $\xi_2\in[0,1]$ so that*

$$(2.14)\quad \sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\left[x,\frac{k+\alpha}{m+\beta};f\right]=[\xi_1,\xi_2;f].$$

Proof. Let $x_0\in[0,1]$ be arbitrary given. We define the linear functional $A\in C^{\#}[0,1]$ by

$$(2.15)\quad A(f)=\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x_0)\left[x_0,\frac{k+\alpha}{m+\beta};f\right].$$

Because $\left[x_0, \dfrac{k+\alpha}{m+\beta}; 1\right] = 0$, $\left[x_0, \dfrac{k+\alpha}{m+\beta}; x\right] = 1$, follows that $A(e_0) = 0$, $A(e_1) = 1 \neq 0$.

For any monotonous increasing function $f \in C([0,1])$ $\left[x_0, \dfrac{k+\alpha}{m+\beta}; f\right] > 0$. Also, for any $x \in [0,1]$, $\widetilde{p}_{m,k}(x) \geq 0$.

Follows $A(f) > 0$ for any $f \in C([0,1])$, monotonous increasing on $[0,1]$.

We can then apply the Popoviciu's theorem and we get that there exist the distinct points $\xi_1, \xi_2 \in [0,1]$ such that

$$\sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x_0) \left[x_0, \frac{k+\alpha}{m+\beta}; f\right] = [\xi_1, \xi_2; f],$$

which is in fact the desired identity (2.14), because $x_0 \in [0,1]$ is arbitrary chosen. □

Lemma 2.2. *For any $f \in C([0,1+p])$ and any $x \in [0,1]$ there exist the points $\eta_1, \eta_2, \eta_3 \in [0,1]$, so that:*

$$\sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) \left[x, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta+1}\ f\right] = [\eta_1, \eta_2, \eta_3; f]. \tag{2.16}$$

Proof. Let $x_0 \in [0,1]$ be fixed and let $A \in C^{\#}([0,1])$ be defined by

$$A(f) = \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(\eta) \left[x_0, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta+1}; f\right].$$

Because $A(e_0) = 0$, $A(e_1) = 0$ ($\Leftarrow$ follows from (2.12)) and

$$\begin{aligned} A(e_2) &= \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(\eta) \left[x_0, \frac{k+\alpha}{m+\beta}, \frac{k+\alpha+1}{m+\beta+1}; x^2\right] = \\ &= \sum_{k=0}^{m+p-1} p_{m-1,k}(x_0) = 1 \neq 0, \end{aligned}$$

and $A(f) > 0$ for any convex function of first order $f \in C([0,1])$, we can apply the Popoviciu's theorem and we get that there exist the distinct points η_1, η_2, $\eta_3 \in [0,1]$ such that

$$A(f) = [\eta_1, \eta_2, \eta_3; f]\, A\left(x_0^2\right) = [\eta_1, \eta_2, \eta_3; f]$$

and the proof ends. □

Theorem 2.3. *For any $f \in C([0,1+p])$ and any $x \in [0,1]$, the remainder term of Schurer-Stancu approximation formula (1.3) can be represented under the form:*

$$\begin{aligned} \left(\widetilde{R}_{m,p}^{(\alpha,\beta)} f\right)(x) &= \frac{1}{m+\beta} \{(\beta-p)x-\alpha\} [\xi_1, \xi_2; f] - \\ &- \frac{m+p}{(m+\beta)^2} x(1-x) [\eta_1, \eta_2, \eta_3; f] \end{aligned} \tag{2.17}$$

for any $m \geq 2$.

Proof. One applies Theorem 1.2, Lemma 2.1 and Lemma 2.2. □

Corollary 2.1. *The remainder term of Schurer approximation formula can be represented under the form*

$$\left(\widetilde{R}_{m,p}^{(0,0)} f\right)(x) = \frac{-px}{m}\,[\xi_1, \xi_2; f] - \frac{m+p}{m^2}\,x(1-x)\,[\eta_1, \eta_2, \eta_3; f] \tag{2.18}$$

for any $f \in C([0, 1+p])$ *and any* $x \in [0,1]$*, where* $\xi_1, \xi_2, \eta_1, \eta_2, \eta_3$ *are distinct points from* $[0,1]$.

Proof. In Theorem 2.2 one makes $\alpha = \beta = 0$. □

Corollary 2.2. *The remainder term of Stancu approximation formula can be represented under the form:*

$$\left(\widetilde{R}_{m,0}^{(\alpha,\beta)} f\right)(x) = \frac{1}{m+\beta}\,(\beta x - \alpha)\,[\xi_1, \xi_2; f] - \frac{m}{(m+\beta)^2}\,x(1-x)\,[\eta_1, \eta_2, \eta_3; f]\,. \tag{2.19}$$

Proof. One applies Theorem 2.2 for $p = 0$. □

Corollary 2.3. *The remainder term of Bernstein approximation formula can be represented under the form:*

$$\left(\widetilde{R}_{m,0}^{(0,0)} f\right)(x) = -\frac{1}{m}\,x(1-x)\,[\eta_1, \eta_2, \eta_3; f]\,.$$

Proof. In Theorem 2.2 one makes $\alpha = \beta = p = 0$. □

Corollary 2.4. *For any* $f \in C^2([0, 1+p])$ *and any* $x \in [0,1]$ *there exist the points* $\xi, \eta \in [0,1]$ *such that*

$$\left(\widetilde{R}_{m,p}^{(\alpha,\beta)} f\right)(x) = \frac{1}{m+\beta}\,\{(\beta - p)x - \alpha\}\,f'(\xi) - \frac{m+p}{2(m+\beta)^2}\,x(1-x)f''(\eta). \tag{2.20}$$

Proof. One applies Theorem 2.2 and the mean theorem for divided differences (see [15]). □

Corollary 2.5. *For any* $f \in C^2([0, 1+p])$ *and any* $x \in [0,1]$ *there exist the points* $\xi, \eta \in [0,1]$ *such that*

$$\left(\widetilde{R}_{m,p}^{(0,0)} f\right)(x) = -\frac{p}{m}\,xf'(\xi) - \frac{m+p}{2m^2}\,x(1-x)f''(\eta). \tag{2.21}$$

Proof. One applies Corollary 2.4 for $\alpha = \beta = 0$. □

Corollary 2.6. *For any* $f \in C^2([0,1])$ *and any* $x \in [0,1]$ *there exist the points* $\xi, \eta \in [0,1]$ *such that*

$$\left(\widetilde{R}_{m,0}^{(\alpha,\beta)} f\right)(x) = \frac{1}{m+\beta}\,(\beta x - \alpha)f'(\xi) - \frac{m}{2(m+\beta)^2}\,x(1-x)f''(\eta).$$

Proof. In Corollary 2.4 we make $p = 0$. □

Corollary 2.7. *For any* $f \in C^2([0,1])$ *and any* $x \in [0,1]$ *there exists the point* $\eta \in [0,1]$ *such that*

$$\left(\widetilde{R}_{m,0}^{(0,0)} f\right)(x) = -\frac{1}{m}\,x(1-x)f''(\eta). \tag{2.22}$$

Proof. One applies Corollary 2.4 for $\alpha = \beta = p = 0$. □

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