CARPATHIAN J. MATH.
21 (2005), No. 1 - 2, 21 - 26

Coincidence theorems for subadditive and superadditive functions

PÁL BURAI and ÁRPÁD SZÁZ

ABSTRACT. We give some necessary and sufficient conditions in order that a quasi-subadditive function should coincide with a quasi-superadditive one. Thus, we can easily prove a straightforward extension of a theorem of Marek Kuczma on the linearity of a subadditive function majorized by a differentiable one.

1. INTRODUCTION

The following theorem has been proved by Kuczma in [3, p. 409].

Theorem 1.1. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be subadditive, and let $g : \mathbb{R}^N \to \mathbb{R}$ be a function such that $g(0) = 0$ and g has the Stolz differential at 0.*

If $f(x) \le g(x)$ for all $x \in \mathbb{R}^N$, then there exists a $c \in \mathbb{R}^N$ such that $f(x) = cx$ for all $x \in \mathbb{R}^N$. In particular, f is of class C^1 in $\mathbb{R}^N$.

Remark 1.1. The $f = g$ and $N = 1$ particular case of this theorem was already observed by Hille [2, p. 144].

Moreover, an extension of the $N = 1$ particular case of the above theorem to functions defined only on a open interval about 0 was already established by Wetzel [6, p. 1068].

In the present paper, by using the reasonings of Kuczma, we shall prove the following coincidence theorem.

Theorem 1.2. *If p is a quasi-subadditive and q is a quasi-superadditive function of a vector space X over $\mathbb{Q}$ such that*

$$\underline{\lim}_{n\to\infty} n \left(p\left(\frac{1}{n}\,x\right) - q\left(\frac{1}{n}\,x\right)\right) \le 0,$$

for all $x \in X$, then $p = q$. Thus, in particular p is quasi-additive.

Remark 1.2. Because of the basic homogeneity properties of subadditive functions [3, p. 401], a real-valued function p of a group X will be called here quasi-subadditive if $-p(x) \le p(-x)$ and $p(nx) \le np(x)$ for all $x \in X$ and $n \in \mathbb{N}$. While, if the inequalities are reversed, then p will be called quasi-superadditive.

By using Theorem 1.2, we shall easily prove the following straightforward generalization of Theorem 1.1.

Received: 15.11.2004; In revised form: 05.05.2005
2000 *Mathematics Subject Classification:* 39B72, 26A24.
Key words and phrases: *Subadditive and superadditive functions.*

Theorem 1.3. *If p is a quasi-subadditive and g is a real-valued function of a normed space X over $\mathbb{R}$ such that $p \leq g$, and moreover $g(0) = 0$ and g is differentiable at 0, then $p = g'(0)$. Thus, in particular p is linear and continuous.*

Remark 1.3. Now, as an important particular case of this theorem, we can also state that if p is a quasi-subbadditive function of a normed space X such that $p\,(0) = 0$ and p is differentiable at 0, then $p = p'(0)$. Thus, in particular p is linear and continuous.

2. Subadditive and superadditive functions

According Hille [2, p. 131] and Rosenbaum [5, p. 227], we may naturally have the following

Definition 2.1. A real-valued function p of a group X is called subadditive if

$$p\,(x+y) \leq p\,(x) + p\,(y)\,,$$

for all $x, y \in X$. While, if the inequality is reversed, then p is called superadditive.

Remark 2.4. Note that thus p may be called additive if it is both subadditive and superadditive.

Moreover, p is superadditive if and only if $-p$ is subadditive. Therefore, superadditive functions need not be studied separately.

By using the arguments of Kuczma [3, p. 401], one can easily prove the following

Theorem 2.4. *If p is a subadditive function of a group X and $x \in X$, then*

(2.1) $\quad -p\,(x) \leq p\,(-x)\,;$

(2.2) $\quad p\,(nx) \leq np\,(x) \quad$ *for all* $\quad n \in \mathbb{N}$.

Remark 2.5. Note that if p is superadditive, then just the opposite inequalities hold. Therefore, if in particular $p$is additive, then the corresponding equalities are also true.

Because of Definition 2.1 and Theorem 2.4, we may naturally introduce the following

Definition 2.2. A real-valued function p of a group X will be called subodd if

$$p\,(-x) \leq -p\,(x)$$

for all $x \in X$. While, if the inequality is reversed, then p will be called superodd.

Remark 2.6. Note that thus p may be called odd if it is both subodd and superodd.

Moreover, p is superodd if and only if $-p$ is subodd. Therefore, superodd functions need not be studied separately.

However, because of Theorem 2.4, it is now more convenient to state the following obvious properties of superodd functions.

Proposition 2.1. *If p is a superodd function of a group X, then*

(2.3) $\quad 0 \leq p(0);$

(2.4) $\quad -p(-x) \leq p(x) \quad$ *for all* $\quad x \in X.$

Remark 2.7. Note that if p is subodd, then just the opposite inequalities hold. Therefore, if in particular p is odd, then the corresponding equalities are also true.

Analogously to Definition 2.2, we may also naturally introduce the following

Definition 2.3. A real-valued function p of a group X will be called $\mathbb{N}$–subhomogeneous if

$$p(nx) \leq np(x),$$

for all $n \in \mathbb{N}$ and $x \in X$. While, if the inequality is reversed, then p will be called $\mathbb{N}$–superhomogeneous.

Remark 2.8. Note that thus p may be called $\mathbb{N}$–homogeneous if it is both $\mathbb{N}$–subhomogeneous and $\mathbb{N}$–superhomogeneous.

Moreover, p is $\mathbb{N}$–superhomogeneous if and only if $-p$ is $\mathbb{N}$–subhomogeneous. Therefore, $\mathbb{N}$–superhomogeneous functions need not be studied separately.

Concerning $\mathbb{N}$–subhomogeneous functions, we shall only quote here the following theorem of [1].

Theorem 2.5. *If p is an $\mathbb{N}$–subhomogeneous function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $l \in \mathbb{Z}$, then*

(2.5) $\quad \dfrac{1}{k}\, p(lx) \leq p\left(\dfrac{l}{k}\, x\right) \quad$ *for all* $\quad 0 < k \in \mathbb{Z};$

(2.6) $\quad -\dfrac{1}{k}\, p(-lx) \leq p\left(\dfrac{l}{k}\, x\right) \quad$ *for all* $\quad 0 > k \in \mathbb{Z}.$

Now, by using Definitions 2.2 and 2.3, we may also naturally introduce the following

Definition 2.4. A real-valued function p of a group X will be called quasi-subadditive if it is superodd and $\mathbb{N}$–subhomogeneous.

Moreover, the function p will be called quasi-superadditive if it is subodd and $\mathbb{N}$–superhomogeneous.

Remark 2.9. Note that thus p may be called quasi-additive if it is both quasi-subadditive and quasi-superadditive.

Moreover, p is quasi-superadditive if and only if $-p$ is quasi-subadditive. Therefore, quasi-superadditive functions need not be studied separately.

Concerning quasi-subadditive functions, we shall only quote the following theorem of [1].

Theorem 2.6. *If p is a quasi-subadditive function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $0 \neq k \in \mathbb{Z}$, then*

(2.7) $\quad \dfrac{1}{k}\, p(lx) \leq p\left(\dfrac{l}{k}\, x\right) \quad$ *for all* $\quad l \in \mathbb{Z};$

(2.8) $$p\left(\frac{l}{k}\,x\right) \leq lp\left(\frac{1}{k}\,x\right) \geq \frac{l}{k}\,p\,(x) \quad \textit{for all} \quad 0 < l \in \mathbb{Z};$$

(2.9) $$p\left(\frac{l}{k}\,x\right) \geq lp\left(\frac{1}{k}\,x\right) \leq \frac{l}{k}\,p\,(x) \quad \textit{for all} \quad 0 > l \in \mathbb{Z}.$$

Remark 2.10. Note that if p is quasi-superadditive, then just the opposite inequalities hold. Therefore, if in particular p is quasi-additive, then the corresponding equalities are also true.

3. Coincidence of subadditive and superadditive fuctions

Theorem 3.7. *If p is a superodd and q is a subodd function of a group X such that $p \leq q$, then $p = q$. Thus, in particular p is odd.*

Proof. Now, by Remark 2.6,$-q$ is superodd. Moreover, we have $-q \leq -p$. Hence, by Definition 2.2 and Proposition 2.1, it is clear that

$$q(x) = -(-q)(x) \leq (-q)(-x) \leq (-p)(-x) = -p(-x) \leq p(x),$$

for all $x \in X$. Therefore, $q \leq p$, and thus $p = q$ is also true. □

Hence, it is clear that in particular we also have

Corollary 3.1. *If p is a superodd function of a group X such that $p \leq c$ for some $c \leq 0$, then $p = 0$.*

Proof. Define $q(x) = c$ for all $x \in X$. Then, q is a subodd function of X such that $p \leq q$. Therefore, by Theorem 3.7, we have $p = q$. Hence, by Proposition 2.1, it is clear that $0 \leq p\,(0) = q\,(0) = c \leq 0$, and thus $c = 0$. Therefore, $q = 0$, and thus $p = 0$ also holds. □

Remark 3.11. Concerning the above statements, note that if for instance $p\,(x) = |x|$ and $q\,(x) = 0$ (resp. $q\,(x) = x$) for all $x \in \mathbb{R}$, then p is a subadditive and q is an additive function of $\mathbb{R}$ such that $q \leq p$, but $q \neq p$.

Moreover, by using Theorem 3.7, we can also prove the following

Theorem 3.8. *If p is a quasi-subadditive and q is a quasi-superadditive function of a vector space X over $\mathbb{Q}$ such that*

$$\varliminf_{n\to\infty} n\left(p\left(\frac{1}{n}\,x\right) - q\left(\frac{1}{n}\,x\right)\right) \leq 0$$

for all $x \in X$, then $p = q$. Thus, in particular p is quasi-additive.

Proof. If $x \in X$, then by the $l = 1$ particular case of Theorem 2.6 (2.7) and Remark 2.9 we have

$$\begin{aligned} p\,(x) - q\,(x) &= p\,(x) + (-q)(x) \\ &\leq np\left(\frac{1}{n}\,x\right) + n(-q)\left(\frac{1}{n}\,x\right) = n\left(p\left(\frac{1}{n}\,x\right) - q\left(\frac{1}{n}\,x\right)\right). \end{aligned}$$

Hence, by the assumption of the theorem, it is already clear that

$$p\,(x) - q\,(x) \leq \varliminf_{n\to\infty} n\left(p\left(\frac{1}{n}\,x\right) - q\left(\frac{1}{n}\,x\right)\right) \leq 0,$$

and thus $p\,(x) \leq q\,(x)$.Therefore, $p \leq q$, and thus, by Theorem 3.7, $p = q$. □

Now, analogously to Corollary 3.1, we can also prove the following

Corollary 3.2. *If p is a quasi-subadditive function of a vector space X over $\mathbb{Q}$ such that*

$$\underline{\lim}_{n\to\infty} n\left(p\left(\frac{1}{n}x\right)-c\right)\leq 0$$

for some $c\leq 0$ and all $x\in X$, then $p=0$.

However, it is now more important to note that by using Theorem 3.8 we can also prove the following

Theorem 3.9. *If p is a quasi-subbadditive and q is a quasi-superadditive function of a normed space X over $\mathbb{R}$ such that $p(0)\leq q(0)$ and*

$$\overline{\lim}_{x\to 0}\frac{1}{\|x\|}\,(p(x)-q(x))\leq 0,$$

then $p=q$. Thus, in particular p is quasi-additive.

Proof. By using the corresponding properties of upper limits and the norm, we can see that

$$\begin{aligned}\underline{\lim}_{n\to\infty} n\left(p\left(\frac{1}{n}x\right)\right. & \left.-q\left(\frac{1}{n}x\right)\right)\leq \overline{\lim}_{n\to\infty} n\left(p\left(\frac{1}{n}x\right)-q\left(\frac{1}{n}x\right)\right)\\ &=\overline{\lim}_{n\to\infty}\|x\|\frac{1}{\|\frac{1}{n}x\|}\left(p\left(\frac{1}{n}x\right)-q\left(\frac{1}{n}\right)\right)\\ &=\|x\|\,\overline{\lim}_{n\to\infty}\frac{1}{\|\frac{1}{n}\|}\left(p\left(\frac{1}{n}x\right)-q\left(\frac{1}{n}x\right)\right)\\ &\leq\|x\|\,\overline{\lim}_{y\to 0}\frac{1}{\|y\|}\,(p(y)-q(y))\leq 0,\end{aligned}$$

for all $0\neq x\in X$. Moreover, since $(0)\leq q(0)$, we can also see that

$$\underline{\lim}_{n\to\infty} n\left(p\left(\frac{1}{n}0\right)-q\left(\frac{1}{n}0\right)\right)\underline{\lim}_{n\to\infty} n\,(p(0)-q(0))\leq 0.$$

Therefore, by Theorem 3.8, we necessarily have $p=q$. $\square$

Now, by using the above theorem, we can easily prove the following straightforward generalization of Kuczma's Theorem 1.1.

Theorem 3.10. *If p is a quasi-subbadditive and g is a real-valued function of a normed space X over $\mathbb{R}$ such that $p\leq g$, and moreover $g(0)=0$ and g is differentiable at 0, then $p=g'(0)$. Thus, in particular p is linear and continuous.*

Proof. Since g is differentiable at 0, there exists a continuous linear function q of X into $\mathbb{R}$ such that

$$\lim_{x\to 0}\frac{1}{\|x\|}\,(g(x)-g(0)-q(x))=0.$$

Hence, since $g(0)=0$, we can also state that

$$\overline{\lim}_{x\to 0}\frac{1}{\|x\|}\,(p(x)-q(x))=\lim_{x\to 0}\frac{1}{\|x\|}\,(p(x)-q(x))=0.$$

Therefore, by Theorem 3.9, we necessarily have $p = q$. Hence, since $q = g'(0)$, it is clear that $p = g'(0)$ is also true. $\square$

Now, as an important particular case of the above theorem we can also state

Corollary 3.3. *If p is a quasi-subbadditive function of a normed space X over $\mathbb{R}$ such that $p(0) = 0$ and p is differentiable at 0, then $p = p'(0)$. Thus, in particular p is linear and continuous.*

Remark 3.12. In [3, Example 1, p. 401], it is also shown that if

$$p(x) = 1 \text{ for } x \leq 0 \quad \text{and} \quad p(x) = e^{-x^2} \text{ for } x > 0,$$

then p is a nonlinear, differentiable subadditive function of $\mathbb{R}$. Therefore, the extra condition in Corollary 3.3 is indispensable.

REFERENCES

[1] Burai, P. and Száz, Á., *Homogeneity properties of subadditive and superadditive functions*, Tech. Rep., Inst. Math., Univ. Debrecen, **10** (2004), 1–15

[2] Hille, E., *Functional analysis and Semi-Groups*, Amer. Math. Soc. Coll. Publ. 31, New York, 1948

[3] Kuczma, M., *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Warszawa, 1985

[4] Matkowski, J., *On subadditive functions and ψ–additive mappings*, Central European J. Math., **4** (2003), 435–440

[5] Rosenbaum, R. A., *Sub-additive functions*, Duke Math. J., **17** (1950), 227–247

[6] Wetzel, J. E., *On the functional inequality $f(x+y) \geq f(x)f(y)$*, Amer. Math. Monthly, **74** (1967), 1065–1068

INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, PF. 12, HUNGARY
E-mail address: buraip@math.klte.hu
E-mail address: szaz@math.klte.hu