CARPATHIAN J. MATH.
21 (2005), No. 1 - 2, 27 - 38

Multivalued version of Radon-Nikodym theorem

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ABSTRACT. We defined in [7] a set-valued integral for multifunctions with respect to a multimeasure, where both the multifunctions and the multimeasure take values in $\mathcal{P}_{kc}(X)$, the family of nonempty compact convex subsets of a locally convex algebra X. But the construction of the integral and all the results remain valid if the multifunctions and the multimeasure take values in $\mathcal{P}_k(X)$, the family of nonempty compact subsets of X.

In this paper we establish a Radon-Nikodym theorem (for the integral described in [7], but using the family $\mathcal{P}_k(X)$ instead of $\mathcal{P}_{kc}(X)$) which bases on a construction of Maynard type [14], using the notion of exhaustion.

1. TERMINOLOGY AND NOTATIONS

Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S. Let X be a Hausdorff locally convex vector space and let Q be a filtering family of seminorms which defines the topology of X. We consider $(x, y) \mapsto xy$ having the following properties for every $x, y, z \in X$, $\alpha, \beta \in \mathbb{R}$, $p \in Q$:

(i) $x(yz) = (xy)z$,
(ii) $xy = yx$,
(iii) $x(y + z) = xy + xz$,
(iv) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$,
(v) $p(x, y) \leq p(x)p(y)$.

Examples 1.1.

(a) $X = \{f : T \to \mathbb{R} | f \text{ is bounded}\}$ with $(fg)(t) = f(t)g(t), \forall\, t \in T$, where T is a topological space. Let $\mathcal{K} = \{K \subset T | K \text{ is compact}\}$ and $Q = \{p_K | K \in \mathcal{K}\}$ where $p_K(f) = \sup_{t \in K} |f(t)|, \forall\, f \in X$.

(b) $X = \{f : T \to \mathbb{R}\}$ with $(fg)(t) = f(t)g(t)$, $\forall\, t \in T$, where T is a nonempty set and $Q = \{p_t \,|\, t \in T\}, p_t(f) = |f(t)|, \forall\, f \in X$.

We denote by $\mathcal{P}_k(X) = \mathcal{P}_k$ the family of all nonempty compact subsets of X. If $A, B \in \mathcal{P}_k, \alpha \in \mathbb{R}$,

$$A + B = \{x + y \,|\, x \in A, y \in B\},$$

$$\alpha A = \{\alpha x \,|\, x \in A\},$$

$$A \cdot B = \{xy \,|\, x \in A, y \in B\}.$$

For every $p \in Q, A, B \in \mathcal{P}_k$, let $e_p(A, B) = \sup_{x \in A} \inf_{y \in B} p(x - y)$ and $h_p(A, B) = \max\{e_p(A, B), e_p(B, A)\}$ - the Hausdorff - Pompeiu semimetric defined by p on

Received: 15.11.2004; In revised form: 10.10.2005
2000 *Mathematics Subject Classification:* 28B20.
Key words and phrases: *Multimeasure, integrable multifunctions, Radon-Nikodym type theorem for multimeasures.*

$\mathcal{P}_k$. We define $\|A\|_p = h_p(A, O) = \sup_{x \in A} p(x)$, $\forall A \in \mathcal{P}_k$, where $O = \{0\}$. Then $\{h_p\}_{p \in Q}$ is a filtering family of semimetrics on $\mathcal{P}_k$ which defines a Hausdorff topology on $\mathcal{P}_k$.

Let $Y \subset \mathcal{P}_k$ satisfying the conditions:

(y_1) Y is complete with respect to $\{h_p\}_{p \in Q}$,
(y_2) $O \in Y$,
(y_3) $\forall A, B \in Y \Rightarrow A + B, A \cdot B \in Y$,
(y_4) $A \cdot (B + C) = A \cdot B + A \cdot C$ for every $A, B, C \in Y$.

Examples 1.2.

(a) $Y = \{\{x\} | x \in X\}$ for X like in (a) and (b) of examples 1.1.
(b) $Y = \{[a, b] \mid a, b \in \mathbb{R}, 0 \leq a \leq b\}$ for $X = \mathbb{R}$.
(c) For X like in example 1.1-b), let $Y = \{[f, g] | f, g \in X, 0 \leq f \leq g\}$, where $[f, g] = \{u \in X | f \leq u \leq g\}, \forall\, f, g \in X$.

Definition 1.1. $\varphi : \mathcal{A} \to \mathcal{P}_k$ is said to be a multimeasure if:
(i) $\varphi(\emptyset) = O$,
(ii) $\varphi(A \cup B) = \varphi(A) + \varphi(B), \forall\, A, B \in \mathcal{A}, A \cap B = \emptyset$.

Definition 1.2. Let $\varphi : \mathcal{A} \to \mathcal{P}_k$. For every $p \in Q$, the p-variation of φ is the non-negative (possibly infinite) set function $v_p(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows:

$$v_p(\varphi, A) = \sup \Big\{ \sum_{i=1}^{n} \|\varphi(E_i)\|_p ; (E_i)_{i=1}^n \subset \mathcal{A}, E_i \cap E_j = \emptyset \text{ for } i \neq j, \bigcup_{i=1}^{n} E_i = A, n \in \mathbb{N}^* \Big\}, \forall\, A \in \mathcal{A}.$$

We denote $v_p(\varphi, \cdot)$ by ν_p if there is no ambiguity.

Remark 1.1. If $\varphi : \mathcal{A} \to \mathcal{P}_k$ is a multimeasure, then ν_p is finitely additive for every $p \in Q$.

Throughout this paper, $\varphi : \mathcal{A} \to Y$ will be a multimeasure and suppose there is $E \in \mathcal{A}$ such that $\varphi(E) \neq O$. We shall assume that $\nu_p(S) < +\infty$ and $(S, \mathcal{A}, \nu_p)$ is complete for every $p \in Q$.

2. Set-valued integral [7]

Definition 2.3. A multifunction $F : S \to Y$ is said to be a simple multifunction if $F = \sum_{i=1}^{n} C_i \cdot \mathcal{X}_{A_i}$, where $C_i \in Y$, $A_i \in \mathcal{A}, i \in \{1, 2, ..., n\}$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} A_i = S$ and $\mathcal{X}_{A_i}$ is the characteristic function of A_i.

The integral of F over $E \in \mathcal{A}$ is:

$$\int_E F d\varphi = \sum_{i=1}^{n} C_i \cdot \varphi(A_i \cap E) \in Y.$$

Definition 2.4. $F : S \to Y$ is said to be φ-totally measurable if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to Y$ satisfying the following condition for every $p \in Q$:

(i) $\qquad h_p(F_n, F) \xrightarrow{\nu_p} 0 \qquad$ (cf. Dunford - Schwartz [10] - III.2.6).

Remarks 2.1.

(a) Every simple multifunction is φ-totally measurable.

(b) If $F : S \to Y$ is φ-totally measurable and $(F_n)_n$ is a sequence of simple multifunctions $F_n : S \to Y$ such that

$$h_p(F_n, F) \xrightarrow{\nu_p} 0, \ \ \forall\, p \in Q,$$

then for every $n \in \mathbb{N}$ and $p \in Q$, $h_p(F_n, F)$ and $\|F\|_p$ are ν_p-measurable (cf. Dunford-Schwartz [10] - III.2.10).

Theorem 2.1. *Let $F, G : S \to Y$ be φ-totally measurable multifunctions and let $\alpha \in \mathbb{R}$. Then it follows:*

(i) $h_p(F, G)$ *is ν_p-measurable,* $\forall\, p \in Q$;

(ii) αF *and* $F + G$ *are φ-totally measurable.*

Definition 2.5. Let $F : S \to Y$ be a φ-totally measurable multifunction. F is said to be φ-integrable (over S) if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to Y$ such that, for every $p \in Q$:

(i) $h_p(F_n, F) \xrightarrow{\nu_p} 0$,

(ii) $\displaystyle\lim_{n,m\to\infty} \int_S h_p(F_n, F_m)\, d\nu_p = 0$.

The sequence $(F_n)_n$ is said to be a defining sequence for F. The integral of F over $E \in \mathcal{A}$ is $\displaystyle\int_E F d\varphi = \lim_{n\to\infty} \left(\int_E F_n d\varphi \right) \in Y$.

Particularly, every simple multifunction is φ-integrable.

Theorem 2.2. *Let $F, G : S \to Y$ be φ-integrable multifunctions, $\alpha \in \mathbb{R}$ and $\Gamma(E) = \int_E F d\varphi, \forall E \in \mathcal{A}$. Then we have:*

(i) $h_p(\int_E F d\varphi, \int_E G d\varphi) \le \int_E h_p(F, G) d\nu_p, \ \ \forall\, E \in \mathcal{A}, p \in Q$;

(ii) $\| \int_E F d\varphi \|_p \le \int_E \|F\|_p d\nu_p, \ \ \forall\, E \in \mathcal{A}, p \in Q$;

(iii) Γ *is a multimeasure;*

(iv) $v_p(\Gamma, E) = \int_E \|F\|_p d\nu_p, \ \ \forall\, E \in \mathcal{A}, p \in Q$;

(v) $\Gamma \ll \nu_p, \forall\, p \in Q$ *(i.e.* $\forall\, p \in Q, \forall\, \varepsilon > 0, \exists\, \delta(p, \varepsilon) = \delta > 0$ *such that* $v_p(\Gamma, E) < \varepsilon$ *for all* $E \in \mathcal{A}$ *with* $\nu_p(E) < \delta$*);*

(vi) αF *is φ-integrable and* $\int_E (\alpha F) d\varphi = \alpha \int_E F d\varphi, \forall\, E \in \mathcal{A}$;

(vii) $F + G$ *is φ-integrable and* $\int_E (F + G) d\varphi = \int_E F d\varphi + \int_E G d\varphi, \forall\, E \in \mathcal{A}$.

Definition 2.6. $F : S \to Y$ is said to be strong φ-integrable if there is a sequence $(F_n)_n$ of simple multifunctions such that uniformly in $p \in Q$:

(i) $h_p(F_n, F) \xrightarrow{\nu_p} 0$,

(ii) $\displaystyle\lim_{n,m\to\infty} \int_S h_p(F_n, F_m)\, d\nu_p = 0$.

Definition 2.7. A finite or countable family of pairwise disjoint sets $(E_i)_i \subset \mathcal{A}$ will be called an uniformly exhaustion of S if $\nu_p(E_i) > 0$ for every $i \in I, p \in Q$ and for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\nu_p\left(S \setminus \bigcup_{i=1}^{n_0} E_i\right) < \varepsilon, \forall p \in Q$.

Theorem 2.3. *Let* $(E_n)_{n\in\mathbb{N}^*} \subset \mathcal{A}$ *an uniformly exhaustion of* S *such that* $S = \bigcup_{n=1}^{\infty} E_n$.

Let $F : S \to Y$ *be defined by* $F(s) = C_n \in Y$ *for every* $s \in E_n$, $n \in \mathbb{N}^*$ *(we denote* $F = \sum_{n=1}^{\infty} C_n \cdot \mathcal{X}_{E_n}$*).*

(i) *If for every* $p \in Q$*, there is* $r_p > 0$ *such that* $\|C_n\|_p \le r_p$ *for every* $n \in \mathbb{N}^*$*, then* F *is* $\varphi-$*integrable.*

(ii) *If there exists* $r > 0$ *such that* $\|C_n\|_p \le r$ *for every* $n \in \mathbb{N}^*$ *and* $p \in Q$*, then* F *is strong* $\varphi-$*integrable.*

Theorem 2.4. (Vitali)

Let $F : S \to Y$ *be a multifunction, let* $(F_n)_n$ *be a sequence of strong* $\varphi-$*integrable multifunctions* $F_n : S \to Y$ *and* $\Gamma_n(E) = \int_E F_n d\varphi$, $E \in \mathcal{A}, n \in \mathbb{N}$ *such that, for every* $p \in Q$ *we have:*

(i) $h_p(F_n, F) \xrightarrow{\nu_p} 0$,

(ii) $\Gamma_n << \nu_p$ *uniformly in* $n \in \mathbb{N}$.

Then F *is* $\varphi-$*integrable and* $\int_E F d\varphi = \lim_{n\to\infty} \int_E F_n d\varphi, \forall\, E \in \mathcal{A}$.

Theorem 2.5. *Let* $(F_n)_n$ *be a sequence of strong* $\varphi-$*integrable multifunctions that converges to* F *uniformly with respect to* $s \in S$ *and* $p \in Q$*. Then* F *is strong* $\varphi-$*integrable and* $\int_E F d\varphi = \lim_{n\to\infty} \int_E F_n d\varphi, \forall\, E \in \mathcal{A}$.

Remarks 2.2.

(a) If $X = \mathbb{R}, Y = \{\{x\} | x \in \mathbb{R}\}$, $F = \{f\}$ (f is a function), $\varphi = \{\mu\}$ (μ is finitely additive) and F is $\varphi-$integrable, then $\int_E F d\varphi = \left\{(D)\int_E f d\mu\right\}$, $E \in \mathcal{A}$, where $(D)\int_E f d\mu$ is the Dunford integral [10].

(b) If $X = \mathbb{R}, Y = \{\{x\} | x \in \mathbb{R}\}$, $F = \{f\}$ (f is a function) and F is $\varphi-$integrable, then f is Brooks - integrable with respect to φ and $\int_E F d\varphi = (B)\int_E f d\varphi$, $E \in \mathcal{A}$, where $(B)\int_E f d\varphi$ is the Brooks integral [3].

(c) If $X = \mathbb{R}$ and $\varphi = \{\mu\}$ (μ is finitely additive), then we get the integral defined by Martellotti - Sambucini [13] for F with respect to μ.

(d) If X is a real Banach algebra, then we obtain the integral defined in [5].

3. A MULTIVALUED VERSION OF RADON - NIKODYM THEOREM

Definition 3.8. Let $T \neq \emptyset$. A multifunction $U : T \to \mathcal{P}_k$ is called uniformly bounded if there exists $r > 0$ such that $\|U(t)\|_p \leq r$, for every $t \in T, p \in Q$.

Definition 3.9. For a multifunction $\Gamma : \mathcal{A} \to Y$, $p \in Q, \varepsilon > 0$ and $E \in \mathcal{A}$, let:

$D_p(\Gamma, E, \varepsilon) = \{C \in Y \mid h_p(\Gamma(B), \nu_p(B)C) \leq \varepsilon\nu_p(B), \forall B \in \mathcal{A}, B \subset E\}$,

$\widetilde{D}_p(\Gamma, E, \varepsilon) = \{C \in Y \mid h_p(\Gamma(B), \varphi(B) \cdot C) \leq \varepsilon\nu_p(B), \forall B \in \mathcal{A}, B \subset E\}$,

$D(\Gamma, E, \varepsilon) = \bigcap_{p\in Q} D_p(\Gamma, E, \varepsilon)$, $\widetilde{D}(\Gamma, E, \varepsilon) = \bigcap_{p\in Q} \widetilde{D}_p(\Gamma, E, \varepsilon)$.

Definition 3.10.

(a) A set property P is said to be uniformly exhaustive on $E \in \mathcal{A}$ if there exists an uniformly exhaustion $(E_i)_i$ of E, such that every E_i has P.

(b) A set property P is called uniformly null difference (shortly, UND) if whenever $A, B \in \mathcal{A}$ with $\nu_p(A) > 0$ and $\nu_p(B) > 0$ for every $p \in Q$, from $\nu_p(A\Delta B) = 0, \forall p \in Q$, it follows that either A and B both have P or neither does.

Theorem 3.6. *Let $\Gamma : \mathcal{A} \to Y$ be an uniformly bounded multimeasure such that $\Gamma \ll \nu_p$, uniformly in $p \in Q$. Then, for every $\gamma > 0$, the properties:*

(i) $D(\Gamma, E, \gamma) \neq \emptyset$,

(ii) $\widetilde{D}(\Gamma, E, \gamma) \neq \emptyset$,

(iii) $D(\Gamma, E, \gamma) \cap \widetilde{D}(\Gamma, E, \gamma) \neq \emptyset$

are uniformly null difference.

Proof. (i) Let $\gamma > 0$. Since $\Gamma \ll \nu_p$ uniformly in $p \in Q$, we have: $\forall \varepsilon > 0, \exists\, \delta(\varepsilon) = \delta > 0$ such that $\forall\, E \in \mathcal{A}$ with $\nu_p(E) < \delta$, it follows

(3.1) $\quad \|\Gamma(E)\|_p \leq \upsilon_p(\Gamma, E) < \varepsilon, \ \ \forall\, p \in Q.$

Let $A, B \in \mathcal{A}$ with $\nu_p(A) > 0$, $\nu_p(B) > 0$, $\nu_p(A\triangle B) = 0, \forall\, p \in Q$.

We shall prove that $D(\Gamma, A, \gamma) = D(\Gamma, B, \gamma)$.

First, we show that $D(\Gamma, A, \gamma) \subset D(\Gamma, B, \gamma)$. Let $C \in D(\Gamma, A, \gamma)$ and let $H \in \mathcal{A}, H \subset B$. Since $B\backslash A \subset A\triangle B$, we have $0 \leq \nu_p(B\backslash A) \leq \nu_p(A\triangle B) = 0$, $\forall\, p \in Q$, so

(3.2) $\quad \nu_p(B\backslash A) = 0, \ \ \forall\, p \in Q.$

From (3.2) it results:

$$\nu_p(H) = \nu_p(H \cap A) + \nu_p(H\backslash A) \leq \nu_p(H \cap A) + \nu_p(B\backslash A) = \nu_p(H \cap A) \leq \nu_p(H),$$

which implies

(3.3) $\quad \nu_p(H) = \nu_p(H \cap A), \ \ \forall\, p \in Q.$

Since

$$H \subset B \Rightarrow H\backslash A \subset A\triangle B \Rightarrow \nu_p(H\backslash A) \leq \nu_p(A\triangle B) = 0 < \delta,$$

from (3.1) we have $\|\Gamma(H\backslash A)\|_p < \varepsilon$. Since arbitrary of $\varepsilon > 0$, we obtain $\|\Gamma(H\backslash A)\|_p = 0, \forall p \in Q$. So $\Gamma(H\backslash A) = O$, which implies

(3.4) $\quad \Gamma(H) = \Gamma((H \cap A) \cup (H\backslash A)) = \Gamma(H \cap A) + \Gamma(H\backslash A) = \Gamma(H \cap A).$

Since $C \in D(\Gamma, A, \gamma)$ and $H \cap A \in \mathcal{A}, H \cap A \subset A$, it follows

(3.5) $\quad h_p(\Gamma(H \cap A), \nu_p(H \cap A)C) \leq \gamma\nu_p(H \cap A).$

Now, from (3.3), (3.4) and (3.5), it results:

$$h_p(\Gamma(H), \nu_p(H)C) = h_p(\Gamma(H \cap A), \nu_p(H \cap A)C) \leq \gamma\nu_p(H \cap A) = \gamma\nu_p(H),$$

which proves that $C \in D(\Gamma, B, \gamma)$.

The inverse inclusion, $D(\Gamma, B, \gamma) \subset D(\Gamma, A, \gamma)$, results analogously. Thus, $D(\Gamma, A, \gamma) = D(\Gamma, B, \gamma)$ and the assertion (i) is proved.

(ii) We shall prove that $\widetilde{D}(\Gamma, A, \gamma) = \widetilde{D}(\Gamma, B, \gamma)$, by double inclusion.

First, let $C \in \widetilde{D}(\Gamma, A, \gamma), H \in \mathcal{A}, H \subset B$. Like in the proof of (i), we have (3.3) and (3.4). By the relations:

$$H \subset B \Rightarrow H \backslash A \subset A \triangle B \Rightarrow 0 \leq \|\varphi(H \backslash A)\|_p \leq \nu_p(H \backslash A) \leq \nu_p(A \triangle B) = 0,$$

it follows that $\|\varphi(H \backslash A)\|_p = 0, \forall p \in Q$, which implies

$$\varphi(H \backslash A) = O. \tag{3.6}$$

From (3.6) it results

$$\varphi(H) = \varphi((H \cap A) \cup (H \backslash A)) = \varphi(H \cap A) + \varphi(H \backslash A) = \varphi(H \cap A). \tag{3.7}$$

Since $C \in \widetilde{D}(\Gamma, A, \gamma)$ and $H \cap A \in \mathcal{A}, H \cap A \subset A$, we have

$$h_p(\Gamma(H \cap A), \varphi(H \cap A) \cdot C) \leq \gamma\nu_p(H \cap A). \tag{3.8}$$

Finally, from (3.3), (3.4), (3.7) and (3.8), we obtain:

$$h_p(\Gamma(H), \varphi(H) \cdot C) = h_p(\Gamma(H \cap A), \varphi(H \cap A) \cdot C) \leq \gamma\nu_p(H \cap A) = \gamma\nu_p(H),$$

that is $C \in \widetilde{D}(\Gamma, B, \gamma)$. The inverse inclusion, $\widetilde{D}(\Gamma, B, \gamma) \subset \widetilde{D}(\Gamma, A, \gamma)$, follows in the same way. So, $\widetilde{D}(\Gamma, A, \gamma) = \widetilde{D}(\Gamma, B, \gamma)$ and the statement is proved.

(iii) It results immediately from (i) and (ii). □

Theorem 3.7. *Let P be an UND property such that P is uniformly exhaustive on S. Then there exists $(E_i)_i$ an uniformly exhaustion of S, such that every E_i has P and $S = \bigcup\limits_i E_i$.*

Proof. Since P is uniformly exhaustive on S, there exists $(E_i)_{i \in I}$ an uniformly exhaustion of S, such that every E_i has P. Thus, we have

$$\forall \varepsilon > 0, \exists n_0(\varepsilon) = n_0 \in \mathbb{N}^* \text{ such that } \nu_p(S \backslash \bigcup_{i=1}^{n_0} E_i) < \varepsilon, \quad \forall p \in Q. \tag{3.9}$$

Let $E_0 = S \backslash \bigcup\limits_{i \in I} E_i$. By the inclusion $E_0 \subset S \backslash \bigcup\limits_{i=1}^{n_0} E_i$ and from (3.9), it results that $\nu_p(E_0) < \varepsilon, \forall \varepsilon > 0$. So, $\nu_p(E_0) = 0, \forall p \in Q$, which implies that $E_0 \in \mathcal{A}$.

Let $(B_i)_{i \in I}$ be the family of sets defined by: $B_1 = E_0 \cup E_1 \in \mathcal{A}, B_i = E_i \in \mathcal{A}$ for $i \geq 2$. We have $\nu_p(B_1) \geq \nu_p(E_1) > 0$ and $\nu_p(B_i) = \nu_p(E_i) > 0, \forall i \geq 2, p \in Q$. Evidently, $S = \bigcup\limits_{i \in I} B_i$.

Let $\varepsilon > 0$. For n_0 of (3.9) we have $\bigcup\limits_{i=1}^{n_0} B_i = E_0 \cup \bigcup\limits_{i=1}^{n_0} E_i$.

By the inclusion $S\backslash \bigcup\limits_{i=1}^{n_0} B_i \subset S\backslash \bigcup\limits_{i=1}^{n_0} E_i$ and from (3.9), it follows

$$\nu_p\left(S\backslash \bigcup_{i=1}^{n_0} B_i\right) \leq \nu_p\left(S\backslash \bigcup_{i=1}^{n_0} E_i\right) < \varepsilon, \quad \forall\, p \in Q$$

which assures the fact that $(B_i)_{i\in I}$ is an uniformly exhaustion of S. Now, for every $i \geq 2, B_i = E_i$ has P. So, we have only to prove that B_1 has P. By the relations

$$\begin{aligned} B_1 \triangle E_1 &= (E_0 \cup E_1) \triangle E_1 = E_0 \backslash E_1 \subset E_0 \Rightarrow \\ &\Rightarrow 0 \leq \nu_p(B_1 \triangle E_1) \leq \nu_p(E_0) = 0, \quad \forall p \in Q, \end{aligned}$$

it follows that $\nu_p(B_1 \triangle E_1) = 0, \forall p \in Q$. Since P is UND and E_1 has P, we obtain that B_1 has P. □

Theorem 3.8. *Let $F : S \to Y$ be an uniformly bounded φ-integrable multifunction which is the limit, uniformly with respect to $s \in S$ and $p \in Q$, of strong φ-integrable multifunctions $F_n : S \to Y$, $n \in \mathbb{N}$ and let $\Gamma(E) = \int_E F d\varphi, \forall E \in \mathcal{A}$. Then we have:*

(i) there exists $r > 0$ such that

$$\frac{1}{\nu_p(E)} \|\Gamma(E)\|_p \leq r, \quad \forall E \in \mathcal{A} \text{ with } \nu_p(E) > 0, \;\; \forall p \in Q;$$

(ii) for every $p \in Q, \varepsilon > 0$ and $E \in \mathcal{A}$ with $\nu_p(E) > 0$, there exists $B \in \mathcal{A}$, $B \subset E$ with $\nu_p(B) > 0$ such that $\widetilde{D}_p(\Gamma, B, \varepsilon) \neq \emptyset$.

Proof. (i) Since F is uniformly bounded, we have:

(3.10) $\exists r > 0$ such that $\|F(s)\|_p \leq r, \;\; \forall\, s \in S, p \in Q$.

From (3.10) and Theorem 2.2 - (ii), it follows for every $E \in \mathcal{A}$ with $\nu_p(E) > 0$, $\forall\, p \in Q$:

$$\begin{aligned} \frac{1}{\nu_p(E)} \|\Gamma(E)\|_p &= \frac{1}{\nu_p(E)} \left\| \int_E F d\varphi \right\|_p \leq \frac{1}{\nu_p(E)} \int_E \|F\|_p d\nu_p \leq \\ &\leq \frac{1}{\nu_p(E)} \int_E r d\nu_p = \frac{1}{\nu_p(E)} \cdot r\nu_p(E) = r. \end{aligned}$$

This proves (i).

(ii) Let $p \in Q$ and $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} F_n(s) = F(s)$ uniformly in $s \in S$ and $p \in Q$, there exists $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that for every natural $n \geq n_0$,

(3.11) $h_p(F_n(s), F(s)) < \varepsilon, \quad \forall s \in S, p \in Q$.

Let $F_{n_0} = \sum\limits_{i=1}^{k} C_i \cdot \mathcal{X}_{A_i}$ and let $E \in \mathcal{A}$ with $\nu_p(E) > 0$. Thus,

$$0 < \nu_p(E) = \nu_p(E \cap S) = \nu_p\left(E \cap \bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \nu_p(E \cap A_i)$$

and so, there is $j_0 \in \{1, 2, \ldots, k\}$ such that $\nu_p(E \cap A_{j_0}) > \frac{1}{2k}\nu_p(E) > 0$. Let $B = E \cap A_{j_0}$. So $B \in \mathcal{A}$ and $\nu_p(B) > 0$. Let $H \in \mathcal{A}, H \subset B$. From Theorem 2.2-(i) and (3.11) we have:

$$h_p(\Gamma(H), \varphi(H) \cdot C_{j_0}) = h_p\left(\int_H F d\varphi, \int_H F_{n_0} d\varphi\right) \leq \int_H h_p(F, F_{n_0}) d\nu_p \leq$$
$$\leq \int_H \varepsilon d\nu_p = \varepsilon \nu_p(H),$$

which shows that $C_j \in \widetilde{D}_p(\Gamma, B, \varepsilon)$. □

Theorem 3.9. (Radon-Nikodym)

Let $\Gamma : \mathcal{A} \to Y$ be an uniformly bounded multimeasure such that:

(i) $\Gamma \ll \nu_p$, *uniformly in $p \in Q$,*

(ii) $\exists r > 0$ *such that* $\frac{1}{\nu_p(E)}\|\Gamma(E)\|_p \leq r, \ \forall E \in \mathcal{A}$ *with* $\nu_p(E) > 0$ *for each* $p \in Q$,

(iii) *for every $\varepsilon > 0$, the set property $D(\Gamma, E, \varepsilon) \cap \widetilde{D}(\Gamma, E, \varepsilon) \neq \emptyset$ is uniformly exhaustive on every $E \in \mathcal{A}$ with $\nu_p(E) > 0$ for each $p \in Q$.*

Then there exists a strong φ-integrable uniformly bounded multifunction $F : S \to Y$ such that $\Gamma(E) = \int_E F d\varphi, \forall E \in \mathcal{A}$.

Proof. Since (iii), Theorem 3.6 - (iii) and Theorem 3.7, there exists $(E_i)_{i \in I}$ an uniformly exhaustion of each $E \in \mathcal{A}$ with $\nu_p(E) > 0$ for every $p \in Q$, such that $E = \bigcup_{i \in I} E_i$ and $D(\Gamma, E_i, \varepsilon) \cap \widetilde{D}(\Gamma, E_i, \varepsilon) \neq \emptyset, \forall i \in I$. Following the same way as in [12], we can obtain a sequence $(E^n_\alpha)_n, \alpha \in \mathbb{N}^n$, of uniformly exhaustions of S such that:

(3.12) $\quad D(\Gamma, E^n_\alpha, 2^{-n}) \cap \widetilde{D}(\Gamma, E^n_\alpha, 2^{-n}) \neq \emptyset, \ \ \forall \alpha \in \mathbb{N}^n, n \in \mathbb{N},$

(3.13) $\quad E^n_\alpha = \bigcup_i E^{n+1}_{\alpha,i}$, where $(E^{n+1}_{\alpha,i})_i$ is an uniformly exhaustion of $E^n_\alpha, \ \ \forall \alpha \in \mathbb{N}^n, n \in \mathbb{N},$

(3.14) $\quad$ for every $n \in \mathbb{N}$, we have $S = \bigcup_\alpha E^n_\alpha$ and $(E^n_\alpha)_\alpha$ is an uniformly exhaustion of S.

Let $F_n = \sum_\alpha C^n_\alpha \cdot \mathcal{X}_{E^n_\alpha}$, where $C^n_\alpha \in D(\Gamma, E^n_\alpha, 2^{-n}) \cap \widetilde{D}(\Gamma, E^n_\alpha, 2^{-n}), \forall \alpha \in \mathbb{N}^n$, $n \in \mathbb{N}$.

Since $C^n_\alpha \in D(\Gamma, E^n_\alpha, 2^{-n})$, it results:

(3.15) $\quad h_p(\Gamma(B), \nu_p(B) C^n_\alpha) \leq \frac{1}{2^n} \nu_p(B), \ \ \forall B \in \mathcal{A}, \ B \subset E^n_\alpha, \ p \in Q.$

From (3.15) and (ii), it follows:

$$\|C^n_\alpha\|_p = h_p(0, C^n_\alpha) = \frac{1}{\nu_p(E^n_\alpha)} h_p(0, \nu_p(E^n_\alpha) C^n_\alpha) \leq$$
$$\leq \frac{1}{\nu_p(E^n_\alpha)}\left[\|\Gamma(E^n_\alpha)\|_p + h_p(\Gamma(E^n_\alpha), \nu_p(E^n_\alpha) C^n_\alpha)\right] \leq r + 2^{-n} \leq r + 1.$$

So,

(3.16) $\|C^n_\alpha\|_p \leq r+1, \ \ \forall \alpha \in \mathbb{N}^n, n \in \mathbb{N}, p \in Q$

and by Theorem 2.3 - (ii), F_n is strong φ-integrable for every $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$, $m < n$. Following the same way as in [12], we can write $F_n = \sum\limits_{(\alpha,\beta)} C^m_{\alpha,\beta} \cdot \mathcal{X}_{E^m_{\alpha,\beta}}$ and $F_n = \sum\limits_{(\alpha,\beta)} C^n_{\alpha,\beta} \cdot \mathcal{X}_{E^n_{\alpha,\beta}}, \forall \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^{n-m}$. Since $(E^n_{\alpha,\beta})_\beta$ is an uniformly exhaustion of E^n_α, we have:

(3.17) $C^m_\alpha, C^n_{\alpha,\beta} \in D(\Gamma, E^m_{\alpha,\beta}, 2^{-m})$.

Thus, from (3.9) it results:

$$\begin{aligned} h_p(C^m_\alpha, C^n_{\alpha,\beta}) &\leq \frac{1}{\nu_p(E^n_\alpha)}\Big[h_p(\Gamma(E^n_\alpha), \nu_p(E^n_\alpha)C^m_\alpha) + h_p(\Gamma(E^n_\alpha), \nu_p(E^n_\alpha)C^n_{\alpha,\beta})\Big] \leq \\ &\leq 2^{-m} + 2^{-m} = 2^{1-m} \ \text{ and} \\ h_p(F_m, F_n) &= h_p\Bigg(\sum_{(\alpha,\beta)} C^m_{\alpha,\beta} \cdot \mathcal{X}_{E^m_{\alpha,\beta}}, \sum_{(\alpha,\beta)} C^n_{\alpha,\beta} \cdot \mathcal{X}_{E^n_{\alpha,\beta}}\Bigg) \leq \\ &\leq \sum_{(\alpha,\beta)} h_p(C^m_{\alpha,\beta}, C^n_{\alpha,\beta}) \cdot \mathcal{X}_{E^m_{\alpha,\beta}} \leq 2^{1-m}. \end{aligned}$$

So, for every $\varepsilon > 0$, there exists $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that

$$h_p(F_m(s), F_n(s)) \leq 2^{1-m} < \varepsilon, \ \ \forall m, n \geq n_0, s \in S, p \in Q,$$

which shows that the sequence $(F_n(s))_n$ is Cauchy in Y, uniformly in $s \in S$ and $p \in Q$. Since Y is complete, there exists the limit $F(s) = \lim\limits_{n\to\infty} F_n(s), \forall s \in S$.

From (3.16) and the definition of F it results:

$$\|F(s)\|_p = h_p(F(s), 0) \leq h_p(F(s), F_n(s)) + h_p(F_n(s), 0) \leq r+2, \ \ \forall s \in S, p \in Q,$$

thus F is uniformly bounded. Since Theorem 2.5, F is strong φ-integrable and

(3.18) $\lim\limits_{n\to\infty} h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) = 0, \ \ \forall p \in Q, E \in \mathcal{A}$.

Now we prove that $h_p(\Gamma(E), \int_E F d\varphi) = 0, \forall p \in Q, E \in \mathcal{A}$. Let $E \in \mathcal{A}$, $p \in Q$, $\varepsilon > 0$. From (i), there is $\delta\left(\frac{\varepsilon}{4}\right) = \delta > 0$, such that for every $A \in \mathcal{A}$ with $\nu_p(A) < \delta$, we have

(3.19) $\|\Gamma(A)\|_p \leq v_p(\Gamma, A) < \frac{\varepsilon}{4}$.

Since the family $(E \cap E^n_\alpha)_{\alpha \in \mathbb{N}^n}$ is an uniformly exhaustion of E, for $\delta > 0$, there is $q \in \mathbb{N}$ such that

(3.20) $\nu_p\Bigg(E \backslash \bigcup\limits_{\alpha \in \mathbb{N}^n, \ \alpha < \underbrace{(q,\dots,q)}_{n}} (E \cap E^n_\alpha)\Bigg) < \delta$.

From (3.18), there is $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$,

$$h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) < \frac{\varepsilon}{4}. \tag{3.21}$$

Because $F_n = \sum_\alpha C_\alpha^n \cdot \mathcal{X}_{E_\alpha^n}$, we obtain

$$\int_E F_n d\varphi = \lim_{l\to\infty} \sum_{\alpha<(\underbrace{l,\ldots,l}_{n})} C_\alpha^n \cdot \varphi(E\cap E_\alpha^n)$$

and thus, for $n \geq n_1$, we have:

$$h_p\left(\sum_{\alpha<(\underbrace{q,\ldots,q}_{n})} C_\alpha^n \cdot \varphi(E\cap E_\alpha^n), \int_E F_n d\varphi\right) < \frac{\varepsilon}{4}. \tag{3.22}$$

Let us denote $\beta = (\underbrace{q,\ldots,q}_{n})$ and let $n \geq n_1$. From (3.22) and (3.21) it follows:

$$\begin{aligned}
h_p\left(\Gamma(E), \int_E F d\varphi\right) &\leq h_p\left(\Gamma(E), \Gamma\left(\bigcup_{\alpha<\beta}(E\cap E_\alpha^n)\right)\right) + \\
&+ h_p\left(\Gamma\left(\bigcup_{\alpha<\beta}(E\cap E_\alpha^n)\right), \sum_{\alpha<\beta} C_\alpha^n \cdot \varphi(E\cap E_\alpha^n)\right) + \\
&+ h_p\left(\sum_{\alpha<\beta} C_\alpha^n \cdot \varphi(E\cap E_\alpha^n), \int_E F_n d\varphi\right) + h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) = \\
&= T_1 + T_2 + h_p\left(\sum_{\alpha<\beta} C_\alpha^n \cdot \varphi(E\cap E_\alpha^n), \int_E F_n d\varphi\right) + \\
&+ h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) < T_1 + T_2 + \frac{\varepsilon}{4} + \frac{\varepsilon}{4},
\end{aligned} \tag{3.23}$$

where we denoted $T_1 = h_p\left(\Gamma(E), \Gamma(\bigcup_{\alpha<\beta}(E\cap E_\alpha^n))\right)$ and

$$T_2 = h_p\left(\Gamma(\bigcup_{\alpha<\beta}(E\cap E_\alpha^n)), \sum_{\alpha<\beta} C_\alpha^n \cdot \varphi(E\cap E_\alpha^n)\right).$$

Now, from (3.19) and (3.20) we obtain

$$\begin{aligned}
T_1 &= h_p\left(\Gamma\left(\bigcup_{\alpha<\beta}(E\cap E_\alpha^n)\right) + \Gamma\left(E\setminus \bigcup_{\alpha<\beta}(E\cap E_\alpha^n)\right), \Gamma\left(\bigcup_{\alpha<\beta}(E\cap E_\alpha^n)\right)\right) \leq \\
&\leq \left\|\Gamma\left(E\setminus \bigcup_{\alpha<\beta}(E\cap E_\alpha^n)\right)\right\|_p < \frac{\varepsilon}{4}.
\end{aligned} \tag{3.24}$$

Since the sets $(E \cap E^n_\alpha)_\alpha$ are pairwise disjoint and $C^n_\alpha \in \widetilde{D}(\Gamma, E^n_\alpha, 2^{-n})$, we have:

$$(3.25)\quad T_2 = h_p\left(\sum_{\alpha<\beta}\Gamma(E\cap E^n_\alpha), \sum_{\alpha<\beta}\varphi(E\cap E^n_\alpha)\cdot C^n_\alpha\right) \leq$$
$$\leq \sum_{\alpha<\beta} h_p(\Gamma(E\cap E^n_\alpha), \varphi(E\cap E^n_\alpha)\cdot C^n_\alpha) \leq \sum_{\alpha<\beta} 2^{-n}\cdot \nu_p(E\cap E^n_\alpha) =$$
$$= 2^{-n}\cdot\nu_p\left(\bigcup_{\alpha<\beta}(E\cap E^n_\alpha)\right) \leq 2^{-n}\cdot\nu_p(E) < \frac{\varepsilon}{4},$$

for n suitable large.

From (3.23), (3.24) and (3.25) it results:

$$h_p(\Gamma(E), \int_E F d\varphi) < \varepsilon, \quad \forall\, \varepsilon > 0,$$

which implies $h_p(\Gamma(E), \int_E F d\varphi) = 0, \forall p \in Q$. So, $\Gamma(E) = \displaystyle\int_E F d\varphi$ for every $E \in \mathcal{A}$.

□

Acknowledgements. The author is grateful to Prof. dr. Anca-Maria Precupanu and Prof. dr. Christiane Godet-Thobie for helpful ideas and interesting conversations during the preparation of this article.

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