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Commutative modular group algebras

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ABSTRACT. The present article discusses recent progress in the answering of very difficult and long-standing classical conjectures, that are, *The Direct Factor Problem*, *The Splitting Problem* and *The Isomorphism Problem* in the theory of commutative group algebras.

1. INTRODUCTION

As usual, suppose RG is the group ring, regarded as an R-algebra, of an abelian group G over a commutative unitary ring R with prime characteristic (further denoted by p). We define $V(RG)$ as the group of all normalized invertible elements (often called normed units) in RG, and $S(RG) = V_p(RG)$ as its Sylow p-subgroup, i.e. its p-component. All other notations and terminology from the abelian group algebras theory are standard, and those which are not explicitly defined herein are in agreement with the excellent classical monographs of Gregory Karpilovsky [39, 40] and our papers [8–29].

For G an arbitrary abelian group, we let G_0 be the maximal torsion subgroup with p-primary part G_p. The notions and terminology to the abelian groups theory will follow essentially the nice classical books of Laszlo Fuchs [30] and our papers [13, 16, 22, 25, 26, 29].

For R an arbitrary commutative ring with identity $(= 1)$ and for p a prime integer, char$(R) = p$ is said to be the characteristic of this ring, which characteristic plays a key role in our further investigation. Next, we shall let the symbols F, K and $\mathbb{F}_p$ to denote an arbitrary field of $char\,(F) = p$, an algebraically closed field with char $(K) = p$ and a simple field of p-elements, respectively. Following Todor Mollov [48], we shall say that the set $s_q(F) = \big\{i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \mid F(\varepsilon_i) \neq F(\varepsilon_{i+1})\big\}$ is the *spectrum* of F about the prime $q \neq p$ whenever ε_i is a primitive q^i-th root of unity in the algebraic closure $\overline{F}$ of F. Apparently $s_q(K) = \emptyset$ holds.

In the past 15 years, the theory of group algebras has enjoyed a period of vigorous development. The foundations have been strengthened and reorganized from new points of view, especially from the viewpoint of crossed products. During the last two decades the subject has been pursued by a great number of researchers and many interesting results of some importance have been obtained. The purpose of this research expository paper is to give, in a self-contained manner, an up-to-date account of various aspects of this development, in an effort to convey a comprehensive picture of the current state of the subject.

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And so, the history of the best known principle results concerning the commutative modular group algebras (i.e. group algebras RG with $\mathrm{char}\,(RG) = p$ for which $G_p \neq 1$) and more specially, the status of the three very important and difficult problems that almost determine this theory, namely the *Isomorphism*, the *Splitting* and the *Direct Factor Problems*, is the following:

Isomorphism Problem

The isomorphism question asks the following (cf. [39, 40]).

Isomorphism Question. *Let G be an abelian p-group. Then whether $FH \cong FG$ as F-algebras for any group H implies $H \cong G$?*

Now, we are in a position to state the more general variant of the preceding one, namely:

Generalized Isomorphism Question. *Assume G is abelian. Then whether the F-isomorphism $FH \cong FG$ for any group H does imply that $H_p \cong G_p$? In particular, does it follow that $H \cong G$, provided G is p-mixed (i.e. in other words $G_0 = G_p$)?*

But more interesting and significant, however, is the following.

Central Isomorphism Question (INVARIANTS). *What is the complete system of invariants for FG or for KG computed only in the terms of the abelian group G and of the field F or K? We conjecture that when G is p-splitting this set for KG is probably $\{G_p,\ G/G_0,\ |G_0/G_p|\}$, and when G is splitting and G_0/G_p is finite this set for FG is perhaps $\{G_p, G/G_0, |G_0/G_p|, |G_0^{q^n}/G_p|$ for all primes $q \neq p$ and all $n \in s_q(F)\}$.*

The principal known important results in this aspect are these:

1. Primary and Torsion Abelian Groups. In 1950, S. Perlis and G. L. Walker [52] have given a complete set of invariants for a commutative semisimple group algebra provided the group basis is finite (see also Berman [2] and Mollov [48]); a generalization to this fact is obtained by us in [11] when G_0 is finite – actually, we emphasize this result of Perlis-Walker-Berman-Mollov since it is needed for applications in the modular case (but however it is more convenient for us to use the invariants given by Mollov [48]).

In 1967, S. Berman [2] has established that the countable p-group can be characterized via its group algebra of characteristic p. After this on 1969, Berman and Mollov [3] have proved that the direct sum (= coproduct) of cyclic p-groups can be retrieved from its modular group algebra. Using other methods, W. May (1979) generalizes the last result to the class of all totally projective (= reduced simply presented) abelian p-groups of countable length [43], i.e. to the class of all direct sums of countable abelian p-groups of countable length. Developing the technique, May (1988) establishes a very strong strengthening of this statement for the class of all totally projective abelian p-groups [44]. Recently, on 1990, W. Ullery [53] extends the May's theorem to the so-called λ-elementary p-primary A-groups, introduced by P. Hill (1985) (see, for example, [53]).

2. Mixed Abelian Groups. The first major result in this direction due to Karpilovsky (1982) is when G is p-mixed whose G_0 is algebraically compact [38, 40]. Further on, May (1988) has found an excellent fact which shows that FG determines G when it is p-local Warfield [44]. In 1992, Ullery has shown that G may be gotten from FG provided G is p-mixed of torsion-free rank one for which G_0 is countable [54], a result generalized to G_0 a direct sum of countable groups (or more generally, for totally projective G_0 with length $< \Omega + \omega$) [37] by Hill-Ullery on 1997. Although we deal only with modular group algebras, we emphasize the following result due to Berman-Mollov [4, 5], which treat the semisimple case, because of its importance for the theory: Let G be an abelian group so that G/G_0 is countable and E is an integral domain with 1 and char$(E) = 0$ so that the prime p is not invertible in E whenever p divides the orders of elements in G_0. If $EH \cong EG$ are isomorphic over E, then $H \cong G$. This result was further extended by May [42] and Ullery.

Own advance in these two ways is as follows: In the cited articles of ours we have studied the group algebras of some large classes of abelian groups. Moreover, full systems of invariants for group rings of such kinds of groups are also established. Our results, in the proofs of which was used an original technique, give a positive light on the isomorphism question as well as they enlarge classical facts in this direction (see the foregoing cited bibliography). For instance, in [19], [27], [28] we have theorems that improve in some aspect the quoted above May's and Hill-Ullery's production.

Splitting Problem

The splitting question states the following (cf. [41]).

Splitting Question. *Suppose that G is splitting abelian. Then whether $FH \cong FG$ as F-algebras for some group H yields that H is splitting?*

The splitting problem was first posed by W. May [41, p.149] on 1969. Its advance is the following: The query has a negative solution in general; an example is given by May (1976) in [42]. Nevertheless there exist major situations when this question holds. As example, it is proved by Berman-Mollov (1981) that FG splits if and only if G splits, assuming G is p-mixed so that G/G_0 is countable [4, 5] (see also [45]). An other also important work is that of May [45] where he has first demonstrated that there exists a connection between the direct factor problem and the splitting problem for group algebras of p-mixed abelian groups. Such a transversal the reader can see in [6] as well.

Our merit, however, may be presented thus: In our cited below articles we have developed an original methodology which gives a general strategy and general connection between the splitting and direct factor problems for group algebras of arbitrary abelian groups.

Direct Factor Problem

The direct factor conjecture says the following (cf. [40]).

Direct Factor Conjecture. *The abelian p-group G is a direct factor of $V(FG)$ or more generally, $V(FG)/G$ is totally projective.*

Next, we are in position to formulate (see [40]).

Generalized Direct Factor Conjecture. *Suppose G is p-mixed abelian. Then G is a direct factor of $V(FG)$ or more generally, $V(FG)/G$ is a totally projective p-group.*

Nevertheless, a more significant role plays the following (cf. [22]).

Central Direct Factor Conjecture. *G_p is a direct factor of $S(FG)$ or more generally, $S(FG)/G_p$ is totally projective.*

The principal major facts of this theme state thus:

1. Primary and Torsion Abelian Groups. The first more important research exploration belonging to May (1979) is when G is a direct sum of countable p-groups [43]. In 1988, May supersedes his assertion for the class of all simply presented abelian p-groups [44] by making use of a development of the same technique. Recently, Ullery (1992) extends this result for all p-primary A-groups [55]. An interesting fact is this of May (1989) when the p-group has cardinality not exceeding $\aleph_1$ and length not exceeding Ω [45]; it is well-documented by Hill-Ullery (1990) that the length restriction can be dropped [35]. Moreover, they also have established such type a theorem for direct sums of p-groups of powers less than or equal to $\aleph_1$, thus extending in general the classical May's result in [45].

2. Mixed Abelian Groups. The first major claim is that of May (1989) when the torsion part of the group basis is a direct sum of p-cyclics [45] and of Ullery (1992) when the group is p-mixed with countable torsions [54]. More recently, Hill and Ullery (1997) have improved in [37] these theorems when the torsion subgroup of the whole group is a direct sum of countable p-groups and either it has countable length (see [10], too) or the whole group has countable torsion-free rank.

Our achievement in this theme may be demonstrated like this: In our cited publications we develop an original and modern algebraic technique via a construction of so-named "*height-finite subgroups*" in the group of all normed units in modular abelian group rings. Such contemporary ideas guarantee the affirmative answer of very important in this directory problems and they also insure simple approaches of the proofs.

On the other hand, in a part of these research investigations, we have developed a method which gives a connection between the decompositions of G and $V(FG)$. Thereby, the direct factor conjecture may be attached successfully once again by another way.

In present paper we shall show that there is a general transversal between these three queries. More precisely, we will demonstrate a global study of these problems. And so, we shall begin with

2. COMMUTATIVE GROUP ALGEBRAS WITH PRIME CHARACTERISTIC

Here, in this paragraph, we state the central results that motivate this article and which are selected in the next three sections. So, we start with

Direct Factor Conjecture

The first main result, however, is the following strong generalization of a result due to Hill-Ullery (1997) proved in [37].

Theorem 2.1. (Direct factor). *Suppose G is an abelian group whose G_p is of length $< \Omega$ and F is perfect. Then $S(FG)/G_p$ is totally projective and so G_p is a direct factor of $S(FG)$ with a totally projective complement. In particular, if $G_0 = G_p$ is with countable length and F is perfect, then G is a direct factor of $V(FG)$ with complementary factor which is a totally projective p-group.*

An immediate consequence is the following:

Corollary 2.1. *Suppose G is an abelian p-group of length $< \Omega$ and F is perfect. Then G is a direct factor of $V(FG)$ with a totally projective complement.*

Hint of the method for proving. In our complete proof, which will be given more formally here, and in all details elsewhere, we shall use the convenient for us criterion of Hill-Ullery [37] regarding any abelian p-group of countable length to be totally projective, namely:

Criterion. (Hill-Ullery, 1997). Let G be an abelian p-group of countable length. Then G is totally projective iff it is a countable union of an ascending sequence of height-finite subgroups.

And so, using the foregoing necessary and sufficient condition, we are ready to construct an ascending countable chain of height-finite subgroups $\{S_n\}_{n=1}^{\infty}$ of $S = S(FG)/G_p$ such that $S = \bigcup\limits_{n=1}^{\infty} S_n$. These subgroups can be selected as follows: first of all we may harmlessly presume that length(G_p) is limit. Since G_p and $S(FG)$, along with $S(FG)/G_p$, have equal countable lengths, the heights can be ordered in the following manner:

$$\sigma_1, \dots, \sigma_n, \dots < \text{length}\,(G_p) = \lambda \quad (n < \omega)$$

That is why we take $S_n = M_n G_p/G_p$, where $M_n = \langle x_n = \alpha_1^{(n)} + \alpha_2^{(n)} g_2^{(n)} + \cdots + \alpha_{s_n}^{(n)} g_{s_n}^{(n)}$ is a p-element $|\, 0 \neq \alpha_i^{(n)} \in F$, $\sum_i \alpha_i^{(n)} = 1$, $1 \leq i \leq s_n$; $1 \neq g_i^{(n)} \in G$ and all possible products height ${}_G^{(p)}(g_i^{(n)^{\pm\varepsilon_i}} g_j^{(n)^{\pm\varepsilon_j}} \dots g_k^{(n)^{\pm\varepsilon_k}}) \in \{\sigma_1, \dots, \sigma_n\}$ or $\geq \lambda$ for all $1 < i \neq j \neq \dots \neq k \leq s_n$, where $0 < \varepsilon_i, \varepsilon_j, \dots, \varepsilon_k \leq \text{order}\,(x_n)\rangle$.

Our aim relies on this to be shown that M_n are correctly defined almost height-finite subgroups in $S(FG)$ that is, the elements of the type x_n^{ε} have special canonical record and height in the set $\{\sigma_1, \dots, \sigma_n\}$ or $\geq \lambda$ whenever $0 < \varepsilon < \text{order}\,(x_n)$. Consequently applying standard arguments, each S_n will be with finite height spectrum, thus finishing the proof after all. □

Further, by making use of the last theorem, we become to extend the claim for p-groups of length Ω.

Theorem 2.2. (Direct Factor). *Suppose that G is a coproduct of p-groups with a countable lengths. Then G is a direct factor of $V(FG)$ with a totally projective complement of $length \leq \Omega$.*

In accordance with the first Direct Factor Theorem, we have:
Hint 1. We can use the idea given by us in [20, 25].
Hint 2. The method for proof in [35, Theorem 3] may be employed successfully.□

Remark 2.1. This assertion is a modern generalization of [43].

Of some interest and importance is also the following.
Main Proposition. *The group $S(FG)/G_p$ is a C_Ω-group provided F is perfect.*

Hint. Since G_p is nice in $S(FG)$ (e.g. [45]; [25, 29]), for each $\alpha < \Omega$ we derive that

$$S(FG)/G_p/(S(FG)/G_p)^{p^\alpha} \cong S(FG)/G_p S(FG^{p^\alpha}) = \\ = \bigcup_{n=1}^{\infty} [T_n G_p S(FG^{p^\alpha})/G_p S(FG^{p^\alpha})],$$

where T_n are subgroups of $S(FG)$ constructed in the same manner as M_n that have heights $\geq \alpha$ or a finite number of different countable heights of the special elements, even if in their canonical form there exist elements of G with uncountable heights. After this, the arguments are similar as to the above given. So, the prescribed technique of the central direct factor theorem plus some folklore facts on the height functions lead us to the desired claim. □

Important series of affirmations, about the structure of $S(FG)$ and $V(FG)$, may also be formulated. In the next statement we will provide a priory that F is perfect.

Criteria. (a) $S(FG)$ *is totally projective of countable length iff* G_p *is totally projective of countable length* [10].

(b) $S(FG)$ *is a* $C_\lambda - group$ *(*$\lambda \leq \Omega$*) iff* G_p *is a* $C_\lambda - group$ *(*$\lambda \leq \Omega$*)* [21].

(c) $S(FG)$ *is summable of countable length iff* G_p *is summable of countable length* [29].

(d) $S(FG)$ *is* $p^{\omega+n}$*-projective iff* G_p *is* $p^{\omega+n}$*-projective* [26].

(e) $S(FG)$ *is a coproduct of torsion-complete groups iff* G_p *is a coproduct of torsion-complete groups. In particular,* $S(FG)$ *is semi-complete iff* G_p *is semi-complete.*

(f) $S(FG)$ *is pure-complete iff* G_p *is pure-complete.*

(g) $V(FG)$ *is a coproduct of* p*-local algebraically compact groups iff* G *is a coproduct of* p*-local algebraically compact groups.*

The length restriction on G_p can be omitted for the following two group classes.

Criteria. *Assume that* R *is perfect with no nilpotents. Then*

(h) $S(RG)$ *is a* Σ*-group iff* G_p *is a* Σ*-group* [16].

(i) $S(RG)$ *is factor-summable iff* G_p *is factor-summable.*

Now, we shall establish two schemas that are expansions of [34] and under which the Central Direct Factor Problem reduces to the Central Direct Factor Problem for the class of all σ-summable groups.

Proposition 2.1. *The following two statements are equivalent.*

(i) *For the abelian group* A *it is true that* $S(RA)/A_p$ *is totally projective.*

(ii) *For the* σ*-summable abelian group* G_p *it is true that* $S(RG)/G_p$ *is totally projective.*

Ideea for proof. "$\Uparrow$" First of all, we presume that A is arbitrary abelian. Owing to a result of Hill [34], $G_p = A_p \times B$ where both G_p and B are σ-summable groups so that $A \subseteq G$. But we have constructed a subgroup $T \supseteq B$ such that $S(RG) = S(RA) \times T$ and hence $S(RG)/G_p \cong S(RA)/A_p \times T/B$. Therefore, by [30], it is self-evident that the claim is verified. □

Next, we shall restate the last theorem to the following more weak variant.

Proposition 2.2. *The following are equivalent.*

(i) *Any abelian* p*-group* A_p *is a direct factor of* $S(RA)$.

(ii) *Any* σ*-summable abelian* p*-group* G_p *is a direct factor of* $S(RG)$.

Ideea for proof. "$\Uparrow$" Following Hill [34], we write as above, $G_p = A_p \times B$ where G_p is σ-summable and $A \subseteq G$ is arbitrary. Thus A_p will be a direct factor of $S(RG)$ whence of $S(RA) \subseteq S(RG)$, thus completing the proof. □

Remark 2.2. Of a global interest is to find a criterion when two σ-summable abelian p-groups are isomorphic only in the terms of numerical invariants. If this is the case, then the Direct Factor Problem will perhaps be completely solved.

In fact, let us assume there exist relationships between the mentioned numerical invariants in question of $G_p \times S(F(\coprod_{i=1}^{\infty} G))$ and $S(F(\coprod_{i=1}^{\infty} G))$. But taking G_p to be σ-summable, by application of [13] and the previous preliminaries, we will deduce that $G_p \times S(F(\coprod_{i=1}^{\infty} G)) \cong S(F(\coprod_{i=1}^{\infty} G))$. Therefore by virtue of a lemma due to May [43, 40], G_p is a direct factor of $S(FG)$, as claimed. Finally, we can apply the last proposition to settle the question in general after all.

Splitting Conjecture

This section is a natural sequel and a supplement to the listed above. It examines under what circumstances the torsion part of a given group separates as its direct factor and when it is an invariant property for the group basis. The results obtained here play a key role pertaining to the isomorphism of group algebras, a question which will be commented in the sequel and which is in the focus of our interests.

For completeness of the exposition, recall that the abelian group G is p-splitting if G_p is its direct factor. And so, we claim that the following is valid.

Theorem 2.3. (Splitting). *Let G be a p-splitting abelian group whose G_p is with countable length. Then the R-isomorphism $RH \cong RG$ for any group H implies that H is p-splitting. In particular, if G is splitting for which G_0/G_p is finite, and RH and RG are R-isomorphic, then H is splitting, i.e., in other words, RG splits iff G splits.*

Hint. Referring to our theorem on the direct factor, the idea for proof given by us in [15, 22] may be coped successfully. □

We close the sections with the

Isomorphism Conjecture

Undoubtedly, the isomorphism of group algebras, which is now in the focus of our studies, plays a paramount role in the contemporary theory of group algebras. Our theorems of this type stated below shed a positive light on the present conjecture.

So, we start with a strong expansion of a result due to Hill-Ullery (1997) argued in [37].

Theorem 2.4. (Isomorphism). *Suppose that G is p-mixed abelian whose G_0 is of countable length. Then the R-isomorphism $RH \cong RG$ for some group H implies that there is a totally projective p-group T with $length < \Omega$ so that $H \times T \cong G \times T$.*

Hint. The first direct factor theorem stated above is applicable. □

For p-primary groups, the preceding fact can be extended to the following.

Theorem 2.5. (Isomorphism). *Assume that G is a coproduct of p-groups each of which has countable length. Then $RH \cong RG$ as R-algebras for an arbitrary group H implies $H \times T \cong G \times T$ for some totally projective p-group T of $length < \Omega$.*

Hint. The second direct factor theorem demonstrated above may be applied. □

Valuable applications to the quoted result Criteria are also established.

Theorem 2.6. *Suppose G is abelian such that G_p is torsion-complete. Then $RH \cong RG$ as R-algebras for some group H yields $H_p \cong G_p$. In particular when G is p-primary, $H \cong G$.*

Hint. It is no harm in assuming that R is a field. By an appeal to the criterion (e), H_p must be semi-complete, say $H_p = T \times C$, where T is torsion-complete and C is a coproduct of cyclics groups. Thus we can write $C = \bigcup_{n=1}^{\infty} C_n$, where $C_n \subseteq C_{n+1}$ are bounded at p^n pure subgroups. Therefore $H_p = \bigcup_{n=1}^{\infty} (T \times C_n)$ and so by virtue of [13], $S(FH) = S(FH; H_p) = \bigcup_{n=1}^{\infty} S(FH; T \times C_n) = S(FG)$, whence $G_p = \bigcup_{n=1}^{\infty} [S(FH; T \times C_n) \cap G_p]$. In order to prove that H_p is torsion-complete, it suffices to show only that C is bounded by exploiting [30]. In fact, to make this, we concentrate on the selection of a special bounded Cauchy sequence in G_p. Thus, using the well-known and documented topological Kulikov's criterion for torsion-complete groups, archived in [30], this sequence must be convergent. After extra routine arguments we will yield that C_n are bounded in general, say there is a positive integer k with the property $C_n^{p^k} = 1$. Finally, it is plain that $C^{p^k} = 1$, concluding the checking for torsion-completeness of H_p. Because the Ulm-Kaplansky cardinal functions of G_p and H_p can be deduced from RG ([41]; [38, 40]), we then establish that $G_p \cong H_p$ [30], as required. □

Remark 2.3. This attainment completely settles a generalized version of question raised by May [43, p.34] and refines a similar one from [24].

Theorem 2.7. *Let G be abelian for which G_p is a coproduct of torsion-complete groups. Then $F_pH \cong F_pG$ as F_p-algebras for any group H implies $H_p \cong G_p$. In particular when G is p-torsion, $H \cong G$.*

Hint. We need only apply the criterion (e) together with the corresponding result in [1]. □

Remark 2.4. This assertion extends a result obtained by Beers-Richman-Walker (1983) in [1] and other given by us in [17].

Further, we have

Theorem 2.8. *Let G be a coproduct of p-local algebraically compact groups. Then $F_pH \cong F_pG$ as F_p-algebras for some group H iff $H \cong G$.*

Hint. To get that H is a coproduct of p-local algebraically compact groups we apply (g). After this, the corresponding fact in [1] works. □

Remark 2.5. This claim generalizes a result proved by Beers-Richman-Walker (1983) in [1].

Theorem 2.9. *Suppose G is a p-mixed algebraically compact group and H is a group such that $RH \cong RG$ as R-algebras. Then $H \cong G$.*

Hint. The algorithm of the corresponding theorem for torsion-complete p-groups may be employed successfully. □

Next, we shall formulate our goals listed in the following subsection entitled

2.1. **Group Algebras of Splitting Mixed Abelian Groups.** Now, we proceed the central results in this subsection, starting with

Theorem 2.10. (Invariants). *Assume that G is p-splitting abelian whose G_p is torsion-complete. Then $KH \cong KG$ as K-algebras for an arbitrary group H iff the following are fulfilled:*

(1) *H is p-splitting abelian;*
(2) *$H_p \cong G_p$;*
(3) *$H/H_0 \cong G/G_0$;*
(4) *$|H_0/H_p| = |G_0/G_p|$.*

Theorem 2.11. (Invariants). *Suppose that G is splitting abelian whose G_p is torsion-complete and G_0/G_p is finite. Then $FH \cong FG$ as F-algebras for an arbitrary group H iff the following hold true:*

(1) *H is splitting abelian;*
(2) *$H_p \cong G_p$;*
(3) *$H/H_0 \cong G/G_0$;*
(4) *$|H_0/H_p| = |G_0/G_p|$;*
(5) *$|H_0^{q^i}/H_p| = |G_0^{q^i}/G_p|$ for all primes $q \neq p$ and $i \in s_q(F)$.*

Theorem 2.12. (Invariants). *Let G be splitting abelian for which G_p is a coproduct of torsion-complete groups and G_0/G_p is finite. Then $\mathbb{F}_pH \cong \mathbb{F}_pG$ as $\mathbb{F}_p$-algebras for an arbitrary group H iff $(1)-(5)$ are valid for all $i \in s_q(\mathbb{F}_p)$.*

The next consequence is immediate.

Corollary 2.2. *Suppose G is splitting p-mixed with torsion-complete periodical part. Then $RH \cong RG$ as R-algebras for an arbitrary group H implies $H \cong G$.*

Remark 2.6. In view of the preliminary theorems announced above, a general algorithm for the isomorphism of group algebras of such groups is given by us in [15] (cf. [8, 12, 14, 18, 19, 23, 26, 29] as well).

Another kind of results are stated in the next subsection under title

2.2. **Group Algebras of Rank One Mixed Abelian Groups.** Now, we come to the significant result in the present context.

Theorem 2.13. (Isomorphism). *Suppose that G is of torsion-free rank one whose G_0 is torsion-complete p-primary. Then $RH \cong RG$ as R-algebras for a group H iff $H \cong G$.*

General method for proof. Foremost, we shall illustrate an algorithm of proof in the general case. Indeed, the theorem of the corresponding isomorphism type alluded to above ensures $H_p \cong G_p$. Moreover, it is no loss of generality in presuming that $RH = RG$ and that R is a field. Hence, by a result of May [44, 45] combined with [13], we derive $V(RG) = GS(RG) = HS(RH) = V(RH)$. Whence it is a routine matter to observe that the p-Ulm height matrices (for example, cf. [46], [30]) of G and H are equivalent [19]. Besides, referring to [41], G/G_0 is a structural invariant of RG. Finally, we may deduce that $G \cong H$ ([46], [30]), thus finishing the proof. $\square$

Remark 2.7. Results of such a type was obtained by us in [19, 26] as well.

Next, we shall formulate one useful assertion which gives a contemporary advance in the isomorphism problem.

Proposition 2.3. *The following are equivalent:*
(i) FG *determines a* p*-group* G *when it is arbitrary of countable length;*
(ii) FG *determines a* p*-group* G *when it is* σ*-summable of countable length.*

Proof. "$\Uparrow$". Following Hill [34] for an arbitrary p-primary group A with countable length we may write $G = A \times B$ where G is σ-summable p-torsion and B can be chosen to be countable of limit length and with finite Ulm-Kaplansky invariants. By the same token we consider the σ-summable p-group G' so that $G' = A' \times B$ for an arbitrary p-group A' with countable length. Consequently $FA \cong FA'$ implies $FG \cong FG'$ and thus by hypothesis $G \cong G'$, i.e. $A \times B \cong A' \times B$. By application of a classical result of P. Crawley [7] on the cancellation property of abelian groups (see also cf. [31]), we infer that $A \cong A'$, as stated. $\square$

We terminate the paper with

3. Epilogue

According to our central theorem about the direct factor and to the important results of Mollov-Nachev [47, 49, 50, 51] concerning the calculation of the Ulm-Kaplansky cardinal functions for $S(FG)$, we see at once that the structure of $S(FG)$ will be completely described provided G_p is of length $< \Omega$ and F is perfect. As a final discussion which immediately arises, it is well to note that our main results stated here encompass those obtained by the cited below classical authors as well as other facts given by us in [8–29].

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