A note on partially ordered topological spaces and a special type of lower semicontinuous function

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ABSTRACT. $\theta$-closed partial order in a topological space has been studied in details. $\theta^*$-lower semicontinuity of a function to a hyperspace has been introduced and such functions are compared to the multifunctions. Lastly the $\theta^*$-lower semicontinuity of some special types of functions is studied.

1. INTRODUCTION

In [1] Ganguly and Bandyopadhyay introduced the concept of $\theta$-closed partial order in a topological space. In the first section of the paper we have tried to examine this special type of order in details. In the next section the concept of $\theta^*$-lower semicontinuous function has been introduced from a topological space $X$ to the hyperspace of a topological space $Y$ along with Vietoris topology and its usual order relation; such functions have been compared to their analogues in the collection of multifunctions. In the last section we use $\theta$-closed partial order of a topological space $X$ to consider the $\theta^*$-lower semicontinuity of some special type of functions on $X$.

2. PARTIALLY ORDERED TOPOLOGICAL SPACE

Definition 2.1. [2] Let $X$ be a topological space and ‘$\leq$’ be a partial order in it. For each subset $A \subseteq X$ let,
\[ \uparrow A = \{ x \in X : a \leq x, \text{ for some } a \in A \} \] and
\[ \downarrow A = \{ x \in X : x \leq a, \text{ for some } a \in A \}. \]

The sets $\uparrow A$ and $\downarrow A$ are called the increasing hull of $A$ and decreasing hull of $A$ respectively.

It is easy to verify that, for any $A, B \subseteq X$,
(i) $A \subseteq \uparrow A, A \subseteq \downarrow A$;
(ii) $A \subseteq B \Rightarrow \uparrow A \subseteq \uparrow B$ and $\downarrow A \subseteq \downarrow B$;
(iii) $\uparrow (A \cup B) = \uparrow A \cup \uparrow B, \downarrow (A \cup B) = \downarrow A \cup \downarrow B$;
(iv) $\uparrow (A \cap B) \subseteq \uparrow A \cap \uparrow B, \downarrow (A \cap B) \subseteq \downarrow A \cap \downarrow B$.

Definition 2.2. [1] A partial order ‘$\leq$’ on a topological space $X$ is a $\theta$-closed order if its graph $\{(x, y) \in X \times X : x \leq y \}$ is a $\theta$-closed subset of $X \times X$. 

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Definition 2.3. A partial order ‘$\leq$’ on a topological space $X$ is an almost regular order iff for every regularly closed set $A \subseteq X$ and $x \in X$ with $a \nless x, \forall a \in A, \exists$ neighbourhoods (nbds. in short) $V$ and $W$ of $A$ and $x$ respectively in $X$ such that $\uparrow V \cap \downarrow W = \Phi$.

Theorem 2.4. The partial order ‘$\leq$’ on a topological space $X$ is a $\theta$-closed order iff for every $x, y \in X$ with $x \nless y$, there exists nbds. $U, V$ of $x, y$ respectively in $X$ such that $\uparrow (U) \cap \downarrow (V) = \Phi$.

Proof. Let the partial order ‘$\leq$’ on $X$ be $\theta$-closed and $x, y \in X$ with $x \nless y$. Then $(x, y)$ does not belong to the graph $G$ (say) of ‘$\leq$’. Since $G$ is $\theta$-closed, $\exists$ nbds. $U$ of $x$ and $V$ of $y$ in $X$ such that $\overline{U \times V} \cap G = \Phi$ i.e. $\overline{U \times V} \cap G = \Phi$, which means that if $u \in \overline{U}$ and $v \in \overline{V}$ then $u \nless v$. We claim that $\uparrow (\overline{U}) \cap \downarrow (\overline{V}) = \Phi$. If not, $\exists z \in \downarrow (\overline{U}) \cap \downarrow (\overline{V})$. So, $\exists a \in \overline{U}, b \in \overline{V}$ such that $a \leq z$ and $z \leq b$. Then by transitivity of ‘$\leq$’, $a \leq b$ which implies $(a, b) \in G$ – a contradiction.

Conversely, let the condition holds. Let $(x, y) \in X \times X \setminus G$. Then $x \nless y$. So by hypothesis, $\exists$ nbds. $U$ of $x$ and $V$ of $y$ in $X$ such that $\uparrow (U) \cap \downarrow (V) = \Phi$. We claim that $\overline{U \times V} \cap G = \Phi$. If not, $\exists (a, b) \in \overline{U \times V} \cap G \Rightarrow a \in U, b \in V$ and $a \leq b$. Thus $b \in \downarrow (\overline{U})$. Also $b \in \downarrow (\overline{V})$ [since $\overline{U} \subseteq \downarrow (\overline{V})$] – contradicts that $\uparrow (\overline{U}) \cap \downarrow (\overline{V}) = \Phi$. This proves that $(x, y)$ is not a $\theta$-contact point [6] of $G$.

Consequently, $G$ is $\theta$-closed. $hd$

Corollary 2.5. Let ‘$\leq$’ be a $\theta$-closed order in a topological space $X$. Then $\uparrow (a)$ and $\downarrow (a)$ are $\theta$-closed for each $a \in X$.

Proof. Let $a \in X$ and $b \in X \setminus \uparrow (a)$. Then $a \nless b$. Since ‘$\leq$’ is a $\theta$-closed order, $\exists$ nbds. $U, V$ of $a, b$ respectively in $X$ such that $\uparrow (U) \cap \downarrow (V) = \Phi$, [by theorem 2.4]. Now $\overline{V} \cap \uparrow (a) \subseteq \uparrow (U) \cap \downarrow (V) = \Phi$. Consequently, $b$ cannot be a $\theta$-contact point of $\uparrow (a)$. So $\uparrow (a)$ is $\theta$-closed.

Similarly $\downarrow (a)$ is $\theta$-closed. $hd$

Corollary 2.6. Every topological space $X$, equipped with a $\theta$-closed order ‘$\leq$’ is a Urysohn space.

Proof. Let $a, b \in X$ with $a \neq b$. Then either $a \nless b$ or $b \nless a$. Let us assume that $a \nless b$.

Since ‘$\leq$’ is a $\theta$-closed order, $\exists$ nbds. $U, V$ of $a, b$ respectively in $X$ such that $\uparrow (U) \cap \downarrow (V) = \Phi$, [by theorem 2.4]. Now $\overline{U} \cap \overline{V} \subseteq \uparrow (U) \cap \downarrow (V) = \Phi$. Therefore, $b$ cannot be a $\theta$-contact point of $\uparrow (a)$. So $\uparrow (a)$ is $\theta$-closed.

Similarly $\downarrow (a)$ is $\theta$-closed. $hd$

Corollary 2.7. Let $X$ be a topological space equipped with a $\theta$-closed order ‘$\leq$’. Let $H \subseteq X$ be an $H$-set [6] in $X$. Then both $\uparrow H$ and $\downarrow H$ are $\theta$-closed.

Proof. Let $a \in X \setminus \uparrow H$. Then $h \nless a, \forall h \in H$. Since ‘$\leq$’ is $\theta$-closed, for each $h \in H, \exists$ open nbds. $U_h, V_h$ of $h$ and $a$ respectively in $X$ such that $\uparrow (U_h) \cap \downarrow (V_h) = \Phi$. [by theorem 2.4]. Now, $\{U_h : h \in H\}$ is an open cover of $H$. Since $H$ is an $H$-set in $X, \exists$ a finite subset $H_0 \subseteq H$ such that $\bigcup_{h \in H_0} \overline{U_h} \supseteq H$. Let $V = \bigcap_{h \in H_0} V_h$. Then $V$ is an open nbd. of $a$ in $X$. Now $\overline{V} \cap \uparrow H \subseteq \overline{V} \cap \uparrow (\bigcap_{h \in H_0} U_h) \subseteq (\bigcap_{h \in H_0} \overline{V_h}) \cap (\bigcup_{h \in H_0} \uparrow U_h) = \Phi$. [since $\uparrow (U_h) \cap \downarrow (V_h) = \Phi, \forall h \in H_0$] Thus, $a$ is not a
Proposition 3.3. Let \( \mathbb{A} \) be a regular closed set and \( x \in X \) be such that \( y \not\leq x, \forall y \in A \). Then for each \( y \in A \), \( \exists \) open nbds. \( U_y \) and \( V_y \) of \( y \) and \( x \) respectively in \( X \) such that, \( \uparrow (U_y) \cap \uparrow (V_y) = \Phi \). [by theorem 2.4]. \( A \) being a regular closed set in an \( H \)-closed space \( X \), it is an \( H \)-closed subspace [7] and hence an \( H \)-set. Now \( \{U_y : y \in A\} \) is an open cover of \( A \) and \( A \) is an \( H \)-set. So \( \exists \) a finite subset \( A_0 \subseteq A \) such that \( \bigcup_{y \in A_0} U_y \supseteq A \). Let \( V = \bigcap_{y \in A_0} V_y \). Then \( V \) is an open nbhd. of \( x \) in \( X \). Now, \( \downarrow (V) \cap A \subseteq (\bigcup_{y \in A_0} \downarrow V_y) \cap (\bigcup_{y \in A_0} \uparrow V_y) = \Phi \). [since \( \uparrow (U_y) \cap \uparrow (V_y) = \Phi, \forall y \in A \) \( \Rightarrow A \subseteq \downarrow (V) \). Again \( \downarrow (V) \) is \( \theta \)-closed [by corollary 2.7] since, \( \overline{V} \) is an \( H \)-set [7]. So \( X \setminus \overline{V} \) is an open nbhd. of \( A \). We claim that, \( \uparrow (X \setminus \overline{V}) \cap \downarrow V = \Phi \). If not, \( \exists \ z \in X \setminus \overline{V} \cap \downarrow (X \setminus \overline{V}) \). So \( \exists \ w \in X \setminus \overline{V} \) such that \( w \leq z \Rightarrow w \in \overline{V} \) a contradiction. Therefore \( \uparrow (X \setminus \overline{V}) \cap \downarrow V = \Phi \). This completes the proof. \( \square \)

3. Functions into Hyperspaces

In this article we shall discuss about a hyperspace [2] and the functions into a hyperspace.

Let \( X \) be a topological space and \( 2^X \) be the collection of all nonempty closed subsets of \( X \). There have been various endeavors to topologize \( 2^X \). The most commonly used topology is the Vietoris topology [3]. This topology is constructed as follows:

For each subset \( S \subseteq X \) we denote, \( S^+ = \{ A \in 2^X : A \subseteq S \} \) and \( S^- = \{ A \in 2^X : A \cap S \neq \Phi \} \). The Vietoris topology on \( 2^X \) is one generated by the subbase \( \{W^+ : W \text{ is open in } X\} \cup \{W^- : W \text{ is open in } X\} \). Now, the usual inclusion relation \( \subseteq \) induces a partial order on \( 2^X \).

Since \( \bigvee V_k^+ \cap \bigvee V_k^- \cap \cdots \cap \bigvee V_n^+ = (V_1 \cap V_2 \cap \cdots \cap V_n)^+ \), a basic open set of the Vietoris topology is of the form, \( V_1^+ \cap \cdots \cap V_n^- \cap V_0^+ \), where \( V_0 \) is open in \( X \) for \( i = 0, 1, \ldots, n \). The space \( 2^X \) with the Vietoris topology is usually known as a \( \text{`hyperspace'} \).

Proposition 3.1. \( \uparrow (V_1^+ \cap \cdots \cap V_n^-) = V_1^- \cap \cdots \cap V_n^- \).

Proof. \( A \in \{V_1^- \cap \cdots \cap V_n^-\} \Rightarrow \exists B \in V_1^+ \cap \cdots \cap V_n^- \) such that \( B \subseteq A \).
\( \Rightarrow A \cap V_i \neq \Phi, \forall i = 1, \ldots, n \) [since \( B \cap V_i \neq \Phi, \forall i = 1, \ldots, n \)] \( \Rightarrow A \in V_1^- \cap \cdots \cap V_n^- \). The reverse inclusion follows from definition 2.1. \( \square \)

Proposition 3.2. \( \downarrow (V_1^- \cap \cdots \cap V_n^-) = 2^X \)

Proof. \( A \in 2^X \). Since \( A \subseteq X \) and \( X \in (V_1^- \cap \cdots \cap V_n^-) \) so it follows that \( A \in \downarrow (V_1^- \cap \cdots \cap V_n^-) \). Thus \( 2^X \subseteq \downarrow (V_1^- \cap \cdots \cap V_n^-) \). Reverse inclusion is obvious. \( \square \)

Proposition 3.3. If \( X \) be a \( T_1 \)-space and \( V_i \subseteq V_0 \), for \( i = 1, \ldots, n \) then \( \uparrow (V_1^- \cap \cdots \cap V_n^-) = V_1^- \cap \cdots \cap V_n^- \).
Proposition 3.4. Let $V_i \subseteq V$, $i = 1, \ldots, n$ [since $A \cap V_i \neq \emptyset$, $i = 1, \ldots, n$]. Now $\{x_1, \ldots, x_n\} \subseteq A \cap V_0$ [since $V_i \subseteq V_0$, $i = 1, \ldots, n$] and $\{x_1, \ldots, x_n\}$ is closed in $X$, since $X$ is $T_1$. Therefore, $\{x_1, \ldots, x_n\} \in V^{-} \cap \cdots \cap V^{-} \cap V^{+}$. Consequently $A \in (V^{-} \cap \cdots \cap V^{-} \cap V^{+})$. Thus $V^{-} \cap \cdots \cap V^{-} \subseteq (V^{-} \cap \cdots \cap V^{-} \cap V^{+})$.

Conversely let $A \subseteq (V^{-} \cap \cdots \cap V^{-} \cap V^{+})$. Then $\exists B \in V^{-} \cap \cdots \cap V^{-} \cap V^{+}$ such that $B \subseteq A$. Therefore $B \cap V_i \neq \emptyset$, $i = 1, \ldots, n$. So $A \cap V_i \neq \emptyset$, $i = 1, \ldots, n$. Consequently $A \subseteq B \subseteq V^{-} \cap \cdots \cap V^{-} \cap V^{+}$. Therefore $\exists (V^{-} \cap \cdots \cap V^{-} \cap V^{+}) \subseteq V^{-} \cap \cdots \cap V^{-} \cap V^{+}$.

\[ \square \]

Proposition 3.5. If $X$ be a $T_1$-space and $V_i \subseteq V_0$, $i = 1, \ldots, n$ then $\exists (V^{-} \cap \cdots \cap V^{-} \cap V^{+}) = V^{+}$.

\[ \square \]

Definition 3.5. A topological space $X$ equipped with a $\theta$-closed partial order '<' is said to be a $\theta$-partially ordered space($\theta$-PO space in short) if $\subseteq$ is $\theta$-open for every $\theta$-open set $V$ of $X$.

Theorem 3.6. If $X$ is a $T_3$-space then the space $2^X$ equipped with the Vietoris topology and the usual set-inclusion as the partial order, is a $\theta$-PO space.

\[ \square \]

Definition 3.7. A function $f : X \rightarrow Y$, $Y$ being equipped with a partial order '<', is called $\theta$-lower semicontinuous with respect to '<' at $x \in X$ if for every open nbd $V$ of $f(x)$ in $Y$, $\exists$ an open nbd $U$ of $x$ in $X$ such that $f(U) \subseteq \subseteq V$. 

\[ \square \]
\( f \) is \( \theta^* \)-lower semicontinuous with respect to \( \leq' \) iff it is \( \theta^* \)-lower semicontinuous at each point of \( X \).

**Theorem 3.8.** Let \( Y \) be a \( T_1 \)-space and \( 2^Y \) have the Vietoris topology. Then a function \( \Phi : X \to 2^Y \) is \( \theta^* \)-lower semicontinuous with respect to \( \leq' \) iff \( \Phi^{-1}(V^-) \) is \( \theta \)-open in \( X \) whenever \( V \) is an open subset of \( Y \).

**Proof.** Let \( \Phi \) be \( \theta^* \)-lower semicontinuous with respect to \( \leq' \) and \( V \) be any open subset of \( Y \).

Let \( a \in \Phi^{-1}(V^-) \). Then \( \Phi(a) \in V^- \). Since \( \Phi \) is \( \theta^* \)-lower semicontinuous so \( \exists \) an open \( U \) of \( a \) in \( X \) such that \( \Phi(U) \subseteq \uparrow (V^-) = V^- \) [by proposition 3.1]

\[ \Rightarrow a \in U \subseteq U \subseteq \Phi^{-1}(V^-). \]

This shows that \( \Phi^{-1}(V^-) \) is \( \theta \)-open.

Conversely, let the condition holds. Let \( a \in X \) and \( G \) be any open nbd. of \( \Phi(a) \) in \( 2^Y \). Then \( \exists \) open sets \( V_0, V_1, \ldots, V_n \) in \( Y \) such that \( \Phi(a) \in V_1^{-} \cap \cdots \cap V_{n+} \cap V_0^- \subseteq G \).

We define, \( U = \Phi^{-1}(V_1^-) \cap \cdots \cap \Phi^{-1}(V_{n+}^-) \).

By hypothesis \( U \) is \( \theta \)-open [since finite intersection of \( \theta \)-open sets is again \( \theta \)-open] and \( a \in U \). So \( \exists \) an open \( nbd. W \) of \( a \) in \( X \) such that \( a \in W \subseteq U \subseteq U \Rightarrow \Phi(a) \in \Phi(W) \subseteq \Phi(U) \subseteq V_1^- \cap \cdots \cap V_{n+}^- = \uparrow (V_1^- \cap \cdots \cap V_{n+}^- \cap V_0^-) \subseteq \uparrow G \) [by proposition 3.3]. This shows that, \( \Phi \) is \( \theta^* \)-lower semicontinuous. \( \square \)

### 4. Multifunctions

In the previous article, we have studied about functions into a hyperspace. These functions are nothing but set-valued functions or multifunctions. In this article we shall treat them as the ordinary multifunction and compare the two different aspects.

Mukherjee, Raychaudhuri and Sinha introduced lower-\( \theta^* \)-continuous multifunctions in [4]; in the same way the concept of lower-\( \theta^* \)-semicontinuous multifunction can also be introduced.

**Definition 4.1.** A multifunction \( F : X \to Y \), where \( X,Y \) are topological spaces, is called lower-\( \theta^* \)-semicontinuous function iff for each \( x_0 \in X \) and each open set \( V \in Y \) with \( F(x_0) \cap V \neq \emptyset \), there is an open \( nbd. U \) of \( x_0 \) such that \( F(x) \cap V \neq \emptyset \) for each \( x \in U \).

**Definition 4.2.** [4] A multifunction \( F : X \to Y \) is called \( \theta^* \)-closed if whenever \( x \in X, y \in Y \) and \( y \notin F(x) \), there exists open nbds. \( U,V \) of \( x,y \) in \( X \) and \( Y \) respectively such that \( p \in U \Rightarrow F(p) \cap V \neq \emptyset \).

**Theorem 4.3.** [4] If \( F : X \to Y \) be a multifunction which is \( \theta^* \)-closed, then \( F(x) \) is closed in \( Y \), for each \( x \in X \).

**Theorem 4.4.** Let \( F : X \to Y \) be a multifunction, where \( X,Y \) are topological spaces and \( Y \) is a \( T_1 \)-space. If \( F \) be lower-\( \theta^* \)-semicontinuous and \( \theta^* \)-closed then

\[
\begin{align*}
  f : X & \to 2^Y \\
  x & \mapsto F(x)
\end{align*}
\]

is \( \theta^* \)-lower semicontinuous, when \( 2^Y \) is endowed with Vietoris topology.

**Proof.** The function \( f \) is well-defined by theorem 4.3. Let \( V \) be any open set in \( Y \) and \( a \in f^{-1}(V^-) \). Then \( f(a) \in V^- \) i.e. \( F(a) \cap V \neq \emptyset \).
Theorem 5.2. Proof. (i) Let \( f \) be \( \theta^* \)-lower semicontinuous. Therefore \( \exists \) an open nbd. \( U \) of \( a \) in \( X \) such that
\[
F(x) \cap V \neq \emptyset, \forall x \in U
\]
\[
\Rightarrow f(x) \in V, \forall x \in U
\]
\[
\Rightarrow U \subseteq f^{-1}(V).
\]
Thus \( f^{-1}(V) \) is \( \theta \)-open for each open set \( V \) in \( Y \).
Consequently, \( f \) is \( \theta^* \)-lower semicontinuous [by theorem 3.8].

Theorem 4.5. Let \( X \) be a topological space and \( Y \) be a \( T_1 \)-space. Let \( f : X \to 2^Y \) be a \( \theta^* \)-lower semicontinuous function, where \( 2^Y \) is endowed with Vietoris topology. Then the multifunction,
\[
F : X \to Y \quad x \mapsto f(x)
\]
is lower-\( \theta^* \)-semicontinuous.
Proof. Let \( x_0 \in X \) and \( V \) be open in \( Y \) such that \( F(x_0) \cap V \neq \emptyset \) i.e. \( f(x_0) \in V \) i.e. \( x_0 \in f^{-1}(V) \). Since \( f \) is \( \theta^* \)-lower semicontinuous function, \( f^{-1}(V) \) is \( \theta \)-open in \( X \) [by theorem 3.8]. So \( \exists \) an open nbd. \( U \) of \( x_0 \) in \( X \) such that, \( x_0 \in U \subseteq \overline{U} \subseteq f^{-1}(V) \)
\[
\Rightarrow f(U) \subseteq V \quad \text{i.e.} \quad f(x) \in V, \forall x \in U \quad \text{i.e.} \quad F(x) \cap V \neq \emptyset, \forall x \in U.
\]
Thus \( F \) is lower-\( \theta^* \)-semicontinuous.

5. SOME SPECIAL MULTIFUNCTIONS

In this article, we discuss the \( \theta^* \)-lower semicontinuity of a very special type of multifunction. Since the consideration of either a hyperspace or an ordinary space as the codomain of a multifunction is immaterial, as seen from the previous article, we discuss the \( \theta^* \)-lower semicontinuity of the multifunction in the hyperspace-setting.

Definition 5.1. We define a pair of functions \( i, d : X \to 2^X \), where \( X \) is a topological space equipped with a partial order \( \leq' \) which is assumed to be a \( \theta \)-closed order, as follows:
\[
i(x) = \uparrow (x) \quad \text{and} \quad d(x) = \downarrow (x)
\]
Since \( \leq' \) is a \( \theta \)-closed order, \( \uparrow (x) \) & \( \downarrow (x) \) are \( \theta \)-closed [by corollary 2.5]. So the functions \( 'i ' \) and \( 'd ' \) are well-defined.

Theorem 5.2. (i) The function \( i : X \to 2^X \) is \( \theta^* \)-lower semicontinuous with respect to \( \leq' \) iff \( \downarrow V \) is \( \theta \)-open in \( X \) for every open set \( V \) of \( X \).
(ii) The function \( d : X \to 2^X \) is \( \theta^* \)-lower semicontinuous with respect to \( \leq' \) iff \( \uparrow V \) is \( \theta \)-open in \( X \) for every open set \( V \) of \( X \).

Proof. (i) Let \( V \) be any open set in \( X \). Now, \( i^{-1}(V) = \{ x \in X : i(x) \in V \} = \{ x \in X : x \leq y, \text{ for some } y \in V \} = \downarrow V \)
It now clearly follows from theorem 3.8 that, \( 'i ' \) is \( \theta^* \)-lower semicontinuous with respect to \( \leq' \) iff \( \downarrow V \) is \( \theta \)-open in \( X \).
(ii) The result follows from the following fact.
Let \( V \) be any open set in \( X \). Now, \( d^{-1}(V) = \{ x \in X : d(x) \in V^{-} \} = \{ x \in X : \downarrow (x) \cap V \neq \emptyset \} = \{ x \in X : y \leq x, \text{ for some } y \in V \} = \uparrow V. \]
Theorem 5.3. If $F : X \to Y$, $Y$ being equipped with a $\theta$-closed order $\leq$ be a set-valued mapping such that $F(x)$ is an H-set in $Y$ and if $F$ is lower-$\theta^*$-semitopoperatorion and $\downarrow V$ is open for each open $V$ of $Y$, then

$$f : X \to 2^Y \quad \{ \quad x \mapsto \uparrow F(x) \}$$

is $\theta^*$-lower semicontinuous.

Proof. Since $F(x)$ is an H-set in $Y$ and $\leq$ is a $\theta$-closed order, $\uparrow F(x)$ is $\theta$-closed [by corollary 2.7]. So $f$ is well-defined.

Let $V$ be open in $Y$. Now, $f^{-1}(V^-) = \{ x \in X : \uparrow F(x) \subseteq V^- \} = \{ x \in X : \uparrow F(x) \cap V \neq \emptyset \} = \{ x \in X : F(x) \cap \downarrow V \neq \emptyset \}$. Let, $x_0 \in f^{-1}(V^-)$. Then $F(x_0) \cap \downarrow V \neq \emptyset$. Since $\downarrow V$ is open [by hypothesis] and $F$ is lower-$\theta^*$-semitopoperatorion exists an open nbd. $U$ of $x_0$ in $X$ such that $F(x) \cap \downarrow V \neq \emptyset, \forall x \in U \Rightarrow \overline{U} \subseteq f^{-1}(V^-)$ i.e. $x_0 \in U \subseteq \overline{U} \subseteq f^{-1}(V^-)$. Thus $f^{-1}(V^-)$ is $\theta$-open in $X$. Consequently, $f$ is $\theta^*$-lower semicontinuous [by theorem 3.8].

We can get a similar result if we take $\downarrow F(x)$ instead of $\uparrow F(x)$ in the above theorem with only changing $\downarrow V$ instead of $\uparrow V$ in the hypothesis. \qed

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