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Integral operators on certain classes of analytic functions with negative coefficients

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ABSTRACT. In this paper integral properties of some analytic functions with negative coefficients from certain classes of functions defined using a differential operator are studied. The obtained results are sharp and they are improvement of some known results.

1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, \cdots, n, \cdots\}$, $\mathbb{N}_2 = \mathbb{N} \setminus \{0,\ 1\}$ and let $\mathcal{N}$ be the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0, \quad k \in \mathbb{N}_2, \tag{1.1}$$

that are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Definition 1.1. [3] We define the operator $D^n : \mathcal{N} \to \mathcal{N},\ \ n \in \mathbb{N}$, by
a) $D^0 f(z) = f(z)$;b) $D^1 f(z) = Df(z) = zf'(z)$; c) $D^n f(z) = D(D^{n-1} f(z))$, $z \in U$.

Definition 1.2. [1] Let $\alpha \in [0, 1)$, $\beta \in (0, 1]$, $1 \leq A < B \leq 1$, $B > 0, B/(B - A) \leq \gamma \leq B/[\alpha(B - A)]$ when $\alpha \neq 0$, $B/(B - A) \leq \gamma \leq 1$ when $\alpha = 0$, $n \in \mathbb{N}$, $\lambda \geq 0$. We say that a function $f \in \mathcal{N}$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ if and only if

$$\left| \frac{\frac{zG'_{n,\lambda}(z)}{G_{n,\lambda}(z)} - 1}{(B - A)\gamma \left[\frac{zG'_{n,\lambda}(z)}{G_{n,\lambda}(z)} - \alpha\right] - B\left[\frac{zG'_{n,\lambda}(z)}{G_{n,\lambda}(z)} - 1\right]} \right| < \beta\ , \quad z \in U \tag{1.2}$$

where

$$G_{n,\lambda}(z) = (1 - \lambda)D^n f(z) + \lambda D^{n+1} f(z)\ , \quad f \in \mathcal{N}. \tag{1.3}$$

Remark 1.1. The class $T_{0,0}(-1, 1, 0, 1, 1/2)$ reduces to the class of functions $f \in \mathcal{N}$ that satisfy $|zf'(z)/f(z) - 1| < 1,\ z \in U$, namely these functions are starlike (see [4]). In [1] it is proved that $T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \subset T_{0,0}(-1, 1, 0, 1, 1/2)$, namely all functions in $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ are starlike ($\alpha \in [0, 1),\ \beta \in (0, 1],\ -1 \leq A < B \leq 1,\ B > 0,\ B/(B - A) \leq \gamma \leq B/[\alpha(B - A)]$ when $\alpha \neq 0,\ \ B/(B - A) \leq \gamma \leq 1$ when $\alpha = 0,\ n \in \mathbb{N},\ \lambda \geq 0$).

Remark 1.2. The class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ when $0 < \gamma \leq B/(B - A)$ is studied in [1] and [2].

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In order to prove our main results we need the next characterization theorem of the class $T_j(n,m,\lambda,\alpha)$ ([1]).

Theorem 1.1. *Let* $\alpha\in[0,1),\ \beta\in(0,1],\ n\in\mathbb{N},\ \lambda\geq 0,\ -1\leq A<B\leq 1,\ B>0,$ $B/(B-A)\leq\gamma\leq B/[\alpha(B-A)]$ *when* $\alpha\neq 0,\ B/(B-A)\leq\gamma\leq 1$ *when* $\alpha=0$. *Then a function* $f\in\mathcal{N}$ *of the form* (1.1) *is in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty}\frac{k^n\left[1+\lambda(k-1)\right]\left[(k-1)-\beta B(k-1)+\beta\gamma(B-A)(k-\alpha)\right]}{\beta\gamma(B-A)(1-\alpha)}\ a_k\leq 1. \tag{1.4}$$

The result is sharp.

The extremal functions are

$$f_{k;\alpha,\beta,\gamma}(z)= \tag{1.5}$$
$$=z-\frac{\beta\gamma(B-A)(1-\alpha)}{k^n\left[1+\lambda\left(k-1\right)\right]\left[k-1-\beta B\left(k-1\right)+\beta\gamma\left(B-A\right)\left(k-\alpha\right)\right]}z^k,\ k\in\mathbb{N}_2.$$

For $c\in(-1,\infty)$ let $I_c:\mathcal{N}\to\mathcal{N}$ be the integral operator defined by $g=I_c(f)$, where $f\in\mathcal{N}$ and

$$g(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt.$$

We note that if $f\in\mathcal{N}$ is a function of the form (1.1), then

$$g(z)=I_c(f)(z)=z-\sum_{k=j+1}^{\infty}\frac{c+1}{c+k}\,a_kz^k,\quad z\in U. \tag{1.6}$$

By using Theorem 1.1 in [1] it is proved that $I_c(T_{n,\lambda}(A,B,\alpha,\beta,\gamma))\subset T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. In this paper the mentioned result is improved.

2. Integral properties of the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ concerning the parameter α

Theorem 2.2. *Let* $\alpha\in[0,1),\ \beta\in(0,1],\ n\in\mathbb{N},\ \lambda\geq 0,\ -1\leq A<B\leq 1,\ B>0,$ $B/(B-A)\leq\gamma\leq B/[\alpha(B-A)]$ *when* $\alpha\neq 0,\ B/(B-A)\leq\gamma\leq 1$ *when* $\alpha=0$ *and let* $c\in(-1,\infty)$. *If* $f\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *and* $g=I_c(f)$, *then* $g\in T_{n,\lambda}(A,B,\alpha^*,\beta,\gamma)$, *where*

$$\alpha^*=1-\frac{(c+1)(1-\alpha)[1-\beta B+\beta\gamma(B-A)]}{(c+1)[1-\beta B+\beta\gamma(B-A)]+1-\beta B+\beta\gamma(B-A)(2-\alpha)} \tag{2.1}$$

and $\alpha<\alpha^*<1$. *The result is sharp.*

Proof. From Theorem 1.1 and from (1.6) we have $g\in T_{n,\lambda}(A,B,\alpha^*,\beta,\gamma)$ if and only if

$$\sum_{k=2}^{\infty}\frac{k^n\left[1+\lambda(k-1)\right]\left[k-1+\beta\gamma(B-A)(k-\alpha^*)-\beta B(k-1)\right]}{\beta\gamma(B-A)(1-\alpha^*)}\frac{c+1}{c+k}\ a_k\leq 1. \tag{2.2}$$

We find the largest α^* such that (2.2) holds. We note that the inequalities

$$\frac{c+1}{c+k}\frac{k-1+\beta\gamma(B-A)(k-\alpha^*)-\beta B(k-1)}{1-\alpha^*}\leq \tag{2.3}$$
$$\leq\frac{(k-1)(1-\beta B)+\beta\gamma(B-A)(k-\alpha)}{1-\alpha},\quad k\in\mathbb{N}_2,$$

imply (2.2), because $f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ and it satisfies (1.4). But the inequalities (2.3) are equivalent to

(2.4) $$[(c+k)\varphi(k)-(c+1)(1-\alpha)\beta\gamma(B-A)]\,\alpha^* \leq$$
$$\leq (c+k)\varphi(k)-(c+1)(1-\alpha)[(k-1)(1-\beta B)+\beta\gamma(B-A)k], \quad k \in \mathbb{N}_2,$$

where

(2.5) $$\varphi(x)=(x-1)(1-\beta B)+\beta\gamma(B-A)(x-\alpha), \quad x \in [2,\infty).$$

We note that

(2.6) $$\varphi(x) \geq x-1+\beta B(1-\alpha) \text{ and } \varphi'(x)=1+\beta[\gamma(B-A)-B] \geq 1, \quad x \in [2,\infty),$$

because $B/(B-A) \leq \gamma$.

Since $B/(B-A) \leq \gamma$ we have

$$(c+k)\varphi(k)-(c+1)(1-\alpha)\beta\gamma(B-A) \geq$$
$$\geq (c+k)[k-1+\beta B(1-\alpha)]-(c+1)(1-\alpha)\beta B =$$
$$= (k-1)[c+k+\beta B(1-\alpha)] \geq (k-1)(c+k), \quad k \in \mathbb{N}_2.$$

Hence

(2.7) $$(c+k)\varphi(k)-(c+1)(1-\alpha)\beta\gamma(B-A)>0, \quad k \in \mathbb{N}_2.$$

From (2.4) and (2.7) we obtain that

$$\alpha^* \leq E(k)=\frac{(c+k)\varphi(k)-(c+1)(1-\alpha)(k-1)[1-\beta B+\beta\gamma(B-A)]}{(c+k)\varphi(k)-(c+1)(1-\alpha)\beta\gamma(B-A)}, \quad k \in \mathbb{N}_2.$$

We show that $E(k) \geq E(2), \ k \in \mathbb{N}_2$. Let

(2.8) $$F(x)=\frac{(c+x)\varphi(x)-(c+1)(1-\alpha)\beta\gamma(B-A)}{(c+1)(1-\alpha)[1-\beta B+\beta\gamma(B-A)](x-1)}, \quad x \in [2,\infty).$$

We have

(2.9) $$F(x)=\frac{c+x}{(c+1)(1-\alpha)}+\frac{\beta\gamma(B-A)(1-\alpha)}{(c+1)(1-\alpha)[1-\beta B+\beta\gamma(B-A)]}$$

and since

$$F'(x)=\frac{1}{(c+1)(1-\alpha)}>0$$

we deduce that F is an increasing function in $[2,\infty)$. But $E(k)=1-1/F(k)$, so $E(k) \geq E(2), \ k \in \mathbb{N}_2$. Thus $\alpha^*=E(2)$ is given by (2.1) and it satisfies (2.4), (2.3) and (2.2). Consequently, $g \in T_{n,\lambda}(A,B,\alpha^*,\beta,\gamma)$.

From (2.9) we have $F(2)>0$ and since $\alpha^*=1-1/F(2)$ we deduce that $\alpha^*<1$. We also have

(2.10) $$\alpha^*-\alpha=\frac{(1-\alpha)[1-\beta B+\beta\gamma(B-A)(1-\alpha)]}{(c+1)[1-\beta B+\beta\gamma(B-A)]+1-\beta B+\beta\gamma(B-A)(2-\alpha)}>0.$$

The result is sharp because

(2.11) $$I_c(f_{2;\alpha,\beta,\gamma})=f_{2;\alpha^*,\beta,\gamma},$$

where the function

$$f_{2;\alpha,\beta,\gamma}(z)=z-\frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\varphi(2)}z^2, \quad z \in U,$$

is an extremal function of $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ (see (1.5)) and the function

$$f_{2;\alpha^*,\beta,\gamma}(z) = z - \frac{\beta\gamma(B-A)(1-\alpha^*)}{2^n(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha^*)]}z^2, \quad z \in U,$$

is an extremal function of $T_{n,\lambda}(A,B,\alpha^*,\beta,\gamma)$.

Indeed,

$$I_c(f_{2;\alpha,\beta,\gamma})(z) = z - \frac{(c+1)\beta\gamma(B-A)(1-\alpha)}{(c+2)2^n(1+\lambda)\varphi(2)}z^2, \quad z \in U, \tag{2.12}$$

and we have to prove that

$$\frac{1-\alpha^*}{1-\beta B+\beta\gamma(B-A)(2-\alpha^*)} = \frac{(c+1)(1-\alpha)}{(c+2)\varphi(2)}, \tag{2.13}$$

where $\varphi(2) = 1-\beta B+\beta\gamma(B-A)(2-\alpha)$. By simple computations we obtain

$$1-\alpha^* = \frac{(c+1)(1-\alpha)[1-\beta B+\beta\gamma(B-A)]}{(c+2)\varphi(2)-(c+1)(1-\alpha)\beta\gamma(B-A)},$$

$$2-\alpha^* = \frac{(c+2)\varphi(2)+(c+1)(1-\alpha)(1-\beta B)}{(c+2)\varphi(2)-(c+1)(1-\alpha)\beta\gamma(B-A)},$$

$$1-\beta B+\beta\gamma(B-A)(2-\alpha^*) =$$

$$= \frac{(c+2)\varphi(2)(1-\beta B)+(c+2)\varphi(2)+\beta\gamma(B-A)}{(c+2)\varphi(2)-(c+1)(1-\alpha)\beta\gamma(B-A)} =$$

$$= \frac{(c+2)\varphi(2)[1-\beta B+\beta\gamma(B-A)]}{(c+2)\varphi(2)-(c+1)(1-\alpha)\beta\gamma(B-A)}$$

and now (2.13) follows obviously. □

3. Integral properties concerning the parameter β

Theorem 3.3. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $n \in \mathbb{N}$, $\lambda \geq 0$, $-1 \leq A < B \leq 1$, $B > 0$, $B/(B-A) \leq \gamma \leq B/[\alpha(B-A)]$ *when* $\alpha \neq 0$, $B/(B-A) \leq \gamma \leq 1$ *when* $\alpha = 0$ *and let* $c \in (-1,\infty)$. *If* $f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *and* $g = I_c(f)$, *then* $g \in T_{n,\lambda}(A,B,\alpha,\beta^*,\gamma)$, *where*

$$\beta^* = \frac{(c+1)\beta}{c+2+\beta\gamma(B-A)(2-\alpha)-\beta B} \tag{3.14}$$

and $0 < \beta^* < \beta$. *The result is sharp.*

Proof. From Theorem 1.1 and from (1.6) we have $g \in T_{n,\lambda}(A,B,\alpha,\beta^*,\gamma)$ if and only if

$$\sum_{k=2}^{\infty} \frac{k^n\,[1+\lambda(k-1)]\,[(k-1)(1-\beta^* B)+\beta^*\gamma(B-A)(k-\alpha)]}{\beta^*\gamma(B-A)(1-\alpha)}\,\frac{c+1}{c+k}\;a_k \leq 1. \tag{3.15}$$

We find the smallest β^* such that (3.15) holds. We note that the inequalities

$$\frac{c+1}{c+k}\frac{(k-1)(1-\beta^* B)+\beta^*\gamma(B-A)(k-\alpha)}{\beta^*} \le \tag{3.16}$$
$$\le \frac{(k-1)(1-\beta B)+\beta\gamma(B-A)(k-\alpha)}{\beta}, \quad k \in \mathbb{N}_2$$

imply (3.15), because $f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ and it satisfies (1.4).

But (3.16) can be rewritten (for all $k \in \mathbb{N}_2$) as

$$(c+1)\beta(k-1) \le (k-1)[(c+k)\beta\gamma(B-A)(k-\alpha)-\beta B(k-1)]\,\beta^*, \tag{3.17}$$

or, equivalently,

$$\beta^* \ge G(k) = \frac{(c+1)\beta}{c+k+\beta\gamma(B-A)(k-\alpha)-\beta B(k-1)}, \quad k \in \mathbb{N}_2.$$

Since $c+k+\beta\gamma(B-A)(k-\alpha)-\beta B(k-1) = [1+\beta\gamma(B-A)-\beta B]k+c-\alpha\beta\gamma(B-A)+\beta B$ and $1+\beta\gamma(B-A)-\beta B \ge 1$, we deduce that $G(k)$ is a decreasing function of k and $G(2) \ge G(k),\ k \in \mathbb{N}_2$. Thus $\beta^* = G(2)$ is given by (3.14) and it satisfies (3.16) and (3.15). Consequently $g \in T_{n,\lambda}(A,B,\alpha,\beta^*,\gamma)$.

The relation $\beta^* < \beta$ is equivalent to $1+\beta\gamma(B-A)(2-\alpha)-\beta B > 0$ and this is true when $\gamma \ge B/(B-A)$ (see (2.10)).

The result is sharp because

$$I_c(f_{2;\alpha,\beta,\gamma}) = f_{2;\alpha,\beta^*,\gamma}. \tag{3.18}$$

Indeed,

$$f_{2;\alpha,\beta^*,\gamma}(z) = z - \frac{\beta^*\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)[1+\beta^*[\gamma(B-A)(2-\alpha)-B]}\, z^2, \quad z \in U,$$

and we have to prove that (see (3.18) and (2.12))

$$\frac{\beta^*}{1+\beta^*[\gamma(B-A)(2-\alpha)-B]} = \frac{(c+1)\beta}{(c+2)\varphi(2)}. \tag{3.19}$$

By simple computation we obtain

$$1+\beta^*[\gamma(B-A)(2-\alpha)-B] =$$
$$= \frac{c+2+\beta\gamma(B-A)(2-\alpha)-\beta B+(c+1)\beta\gamma(B-A)(2-\alpha)-(c+1)\beta B}{c+2+\beta\gamma(B-A)(2-\alpha)-\beta B} =$$
$$= \frac{(c+2)[1+\beta\gamma(B-A)(2-\alpha)-\beta B}{c+2+\beta\gamma(B-A)(2-\alpha)-\beta B} = \frac{(c+2)\varphi(2)}{c+2+\beta\gamma(B-A)(2-\alpha)-\beta B}$$

and now (3.19) follows obviously. □

4. INTEGRAL PROPERTIES CONCERNING THE PARAMETER γ

Theorem 4.4. *Let* $\alpha \in [0,1)$, $\beta \in (0,1]$, $n \in \mathbb{N}$, $\lambda \ge 0$, $-1 \le A < B \le 1$, $B > 0$, $B/(B-A) \le \gamma \le B/[\alpha(B-A)]$ *when* $\alpha \ne 0$, $B/(B-A) \le \gamma \le 1$ *when* $\alpha = 0$ *and let* $c \in (-1,\infty)$. *If* $f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *and* $g = I_c(f)$, *then* $g \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma^*)$,

$$\gamma^* = \max\left\{\frac{(c+1)(1-\beta B)\gamma}{(c+2)(1-\beta B)+\beta\gamma(B-A)(2-\alpha)}; \frac{B}{B-A}\right\}, \tag{4.20}$$

and $0 < \gamma^* \leq \gamma$. *The result is sharp.*

Proof. From Theorem 1.1 and from (1.6) we have $g \in T_{n,\lambda}(A, B, \alpha, \beta^*, \gamma)$ if and only if

$$\text{(4.21)} \quad \sum_{k=2}^{\infty} \frac{k^n [1 + \lambda(k-1)] [(k-1)(1-\beta B) + \beta\gamma^*(B-A)(k-\alpha)]}{\beta\gamma^*(B-A)(1-\alpha)} \frac{c+1}{c+k} a_k \leq 1.$$

We find the smallest γ^* such that (4.21) holds. We note that the inequalities

$$\text{(4.22)} \quad \frac{c+1}{c+k} \frac{(k-1)(1-\beta B) + \beta(B-A)(k-\alpha)\gamma^*}{\gamma^*} \leq$$
$$\leq \frac{(k-1)(1-\beta B) + \beta(B-A)(k-\alpha)\gamma}{\gamma}, \quad k \in \mathbb{N}_2,$$

imply (4.21), because $f \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ and it satisfies (1.4). But the inequalities (4.22) can be rewritten (for all $k \in \mathbb{N}_2$) as

$$\gamma^* \geq H(A, B, \alpha, \beta, \gamma, c; k) = \frac{(c+1)(1-\beta B)\gamma}{(c+k)(1-\beta B) + \beta\gamma(B-A)(k-\alpha)}, \quad k \in \mathbb{N}_2.$$

We have $(c+k)(1-\beta B) + \beta\gamma(B-A)(k-\alpha) = [1 - \beta B + \beta\gamma(B-A)]k + c(1 - \beta B) - \alpha, \beta\gamma(B-A)$ and since $1 - \beta B + \beta\gamma(B-A) \geq 1$ we obtain that

$$\max\{H(A, B, \alpha, \beta, \gamma, c; k);\ k \in \mathbb{N}_2\} = H(A, B, \alpha, \beta, \gamma, c; 2).$$

The inequalities (4.22) hold for each γ^* belonging to the interval $[H(A, B, \alpha, \beta, \gamma, c; 2); \gamma]$. It is easy to see that $H(A, B, \alpha, \beta, \gamma, c; 2) < \gamma$). For appropriate values of parameters, $H(A, B, \alpha, \beta, \gamma, c; 2)$ can be less or greater then $B/(B-a)$. We obtain that the inequalities (4.22) hold for $\gamma^* = \max\{B/(B - a); H(A, B, \alpha, \beta, \gamma, c; 2)\}$. In order to prove that the result is sharp we show that, for $\gamma^* = H(A, B, \alpha, \beta, \gamma, c; 2)$

$$\text{(4.23)} \quad I_c(f_{2;\alpha,\beta,\gamma}) = f_{2;\alpha,\beta,\gamma^*}.$$

We have

$$\text{(4.24)} \quad f_{2;\alpha,\beta,\gamma^*}(z) = z - \frac{\beta\gamma^*(B-A)(1-\alpha)}{2^n(1+\lambda)[1 + \beta[\gamma^*(B-A)(2-\alpha) - B]} z^2, \quad z \in U.$$

By a simple computation we obtain

$$\text{(4.25)} \quad \frac{\gamma^*}{1 - \beta B + \beta\gamma^*(B-A)(2-\alpha)} = \frac{(c+1)\gamma}{(c+2)\varphi(2)}.$$

Combining (4.24), (2.12) and (4.25) we obtain (4.23). □

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