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The solution of the thermoelastic equilibrium problem for cylindrical tubes with big torsion angle

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ABSTRACT. In this paper, the coupled problem of the thermoelastic equilibrium for cylindrical tubes with big torsion angle is solved. The mathematical model of isotropic elastic cylinders with big torsion angle is also presented. Using this new mathematical model, we actually solve the thermoelasticity problem for cylindrical tubes.

Consider an isotropic homogeneous cylindrical elastic bar. The lateral surface is external forces free. Body forces are absent. Assume that the region of section is bounded (simply connected or multiply connected). We take the origin of coordinates at the centroid of the end section B_1 ($z = 0$), Oz axis parallel to the generators and Ox, Oy axis arbitrarily directed. The point B_2 is obtained for $z = L$, where L is the length of the bar (sufficiently long). The ends are acted on by distributed forces reducing to twisting moments M of opposite sense $\overrightarrow{M}_3^0(B_1) = -\overrightarrow{M}_3^0(B_2) = M\overrightarrow{k}$. This is the classical model of the torsion [1], [4], [7].

In this paper we investigate the problem of the stationary thermoelasticity of a cylindrical tube with a big torsion angle, the deformations and the state of stress being caused by the torsion moment M (which is known) and by the heating of the boundary or by the change of temperature between the boundary and the environment (see $(I - V)$ below).

1. THE MATHEMATICAL MODEL

Here, we present A. Y. Ishlinski's mathematical model of [3] for the torsion of a cylindrical bar in the case of elastic materials with a big torsion angle.

Consider now an element of length l situated at a distance r of the Oz axis. Let α denote the specific torsion angle. After torsion, the generators of the cylindrical tube take the shape of circular propeller of length

$$ds' = \sqrt{l^2 + r^2\alpha^2}. \tag{1.1}$$

If the expression $\dfrac{r\alpha}{l}$ is sufficiently small, the specific elongation is

$$\frac{1}{l}\left(\sqrt{l^2 + r^2\alpha^2} - l\right) \cong \frac{r^2\alpha^2}{2l^2}. \tag{1.2}$$

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For a generator situated at a distance r of the cylindrical axis, we denote by χ the torsion specific coefficient:

$$\frac{r^2\alpha^2}{2l^2} = \chi r^2, \quad \text{where } \chi = \frac{\alpha^2}{2l^2}. \tag{1.3}$$

We take the expression χr^2 as a measure of the specific elongation to Oz axis, therefore it will represent the component ε_{zz} of the deformation tensor. In cylindrical coordinates (r, φ, z), the not zero deformations are [3]

$$\varepsilon_{zz} = \chi r^2, \quad \varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\varphi\varphi} = \frac{u}{r}. \tag{1.4}$$

In (1.4), $u_r = u$ represents the radial displacement. Due to the condition of uniform torsion, the tangential displacement u_φ will be constant for any constant radius r and will not appear in the components of the deformation tensor.

The constitutive equations in the case of the linear isotropic thermoelasticity, which generalize Cauchy and Hooke equations, are

$$\begin{aligned} T_{ij} &= \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - \beta T\delta_{ij}, \\ \varepsilon_{ij} &= \frac{1+\nu}{E}T_{ij} - \frac{\nu}{E}\Theta\delta_{ij} + \overline{\alpha}T\delta_{ij}, \\ \theta - 3\overline{\alpha} &= \frac{1-2\nu}{E}\Theta, \\ \beta &= \frac{E}{1-2\nu}\overline{\alpha} \end{aligned}$$

and they are due to Duhamel and Neumann [2], [6]. Here, λ, μ are the Lamé's coefficients, E is the Young's modulus, ν is the Poisson's coefficient [1], T is the temperature, $\overline{\alpha}$ is the coefficient of linear dilatation, $\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, $\Theta = T_{11} + T_{22} + T_{33}$.

Due to the axial symmetry to the Oz axis and to the conditions which depends only on r, in cylindrical coordinates we have

$$\begin{aligned} &T_{zz} = \lambda\left(\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{zz} - \beta T\left(r\right), \\ &T_{rr} = \lambda\left(\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{rr} - \beta T\left(r\right), \\ &T_{\varphi\varphi} = \lambda\left(\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi}\right) + 2\mu\varepsilon_{\varphi\varphi} - \beta T\left(r\right), \\ &T_{rz} = T_{r\varphi} = T_{z\varphi} = 0. \end{aligned} \tag{1.5}$$

Substituting (1.4) in the above relations we get

$$\begin{aligned} &T_{zz} = (\lambda + 2\mu)\chi r^2 + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \beta T\left(r\right), \\ &T_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\left(\chi r^2 + \frac{u}{r}\right) - \beta T\left(r\right), \\ &T_{\varphi\varphi} = (\lambda + 2\mu)\frac{u}{r} + \lambda\left(\chi r^2 + \tfrac{\partial u}{\partial r}\right) - \beta T\left(r\right). \end{aligned} \tag{1.6}$$

The equilibrium equation is:

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\varphi\varphi}}{r} = 0. \tag{1.7}$$

Substituting (1.6) into (1.7) we get the linear differential equation of the radial displacement $u_r = u(r)$

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}u = -\frac{2\nu}{1-\nu}\chi r + \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\beta T'(r)\,. \tag{1.8}$$

The torsion problem for the cylinder, respectively for the cylindrical tube, without heating, are developed in the papers [3] and [5] respectively.

2. THE SOLUTION OF THE THERMOELASTIC EQUILIBRIUM PROBLEM

First, we solve the problem of the temperature distribution $T(r)$ for the crown $a \leq r \leq b$. Due to the axial symmetry and to the boundary conditions (independent on φ), the heat equation will be

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0, \quad a \leq r \leq b. \tag{2.9}$$

Theorem 2.1. *If we attach to equation* (2.9) *the boundary conditions for* $T(r)$ *compatible in an exclusive manner, without heat sources, we get the situations (*$I - V$*) with the corresponding solutions:*

$$I.\ \begin{cases} T(a) = 0 \\ T(b) = T^* \end{cases} \to T(r) = \frac{T^*}{\ln\dfrac{b}{a}}\ln r - \frac{T^*\ln a}{\ln\dfrac{b}{a}},$$

$$II.\ \begin{cases} T(a) = T^* \\ T(b) = 0 \end{cases} \to T(r) = \frac{T^*}{\ln\dfrac{a}{b}}\ln r - \frac{T^*\ln b}{\ln\dfrac{a}{b}},$$

$$III.\ \begin{cases} T(a) = 0 \\ \dfrac{\partial T}{\partial r}(b) = T^* \end{cases} \to T(r) = T^* b\ln r - T^* b\ln a\,,$$

$$IV.\ \begin{cases} \dfrac{\partial T}{\partial r}(a) = T^* \\ T(b) = 0 \end{cases} \to T(r) = T^* a\ln r - T^* a\ln b\,,$$

$$V.\ \begin{cases} T(a) = T_1 \\ T(b) = T_2 \end{cases} \to T(r) = k\ln r + k_1,$$

where $k = \dfrac{T_2 - T_1}{\ln(b/a)}$, $k_1 = \dfrac{T_1\ln b - T_2\ln a}{\ln(b/a)}$. *Moreover, considering the change of temperature between the boundary and the environment, we have the general conditions*

$$VI.\ m_i T(r_i) + n_i\frac{\partial T}{\partial r}(r_i) = p_i, \quad i = 1, 2,$$

where $r_1 = a$, $r_2 = b$, *with the same type of solution as* $I - V$.

We denote by $T(r) = k\ln r + k_1$, the solution of $T(r)$ corresponding to any case $I - V$, therefore $T'(r) = \dfrac{k}{r}$ in all these cases. Consequently, equation (1.8) becomes

$$r^2u'' + ru' - u = -\frac{2\nu}{1-\nu}\chi r^3 + \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\beta kr$$

and its general solution will be

$$u(r) = c_1 r + c_2\frac{1}{r} - \frac{\nu}{4(1-\nu)}\chi r^3 + \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)}\beta kr\ln r. \tag{2.10}$$

Theorem 2.2. *The solution of the thermoelastic equilibrium problem for cylindrical tubes with big torsion angle is:*

$$\begin{aligned} u(r) = &-\frac{\nu\chi}{4(1-\nu)}\left[r^3 + (1-2\nu)\left(a^2+b^2\right)r + \frac{a^2b^2}{r}\right] \\ &+ \frac{(1+\nu)(1-2\nu)}{2E}\beta r\left[\frac{1-2\nu}{1-\nu}k\frac{b^2\ln b - a^2\ln a}{b^2-a^2} - k + 2k_1\right] \\ &+ \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)}\beta k\frac{a^2b^2(\ln b - \ln a)}{b^2-a^2}\frac{1}{r} + \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)}\beta kr\ln r, \end{aligned}$$

$$\begin{aligned} T_{rr} = &\frac{E\nu\chi}{4(1-\nu^2)}\left[r^2 - \left(a^2+b^2\right) + \frac{a^2b^2}{r^2}\right] \\ &+ \frac{1-2\nu}{2(1-\nu)}\beta k\left[\frac{b^2\ln b - a^2\ln a}{b^2-a^2} - \frac{a^2b^2(\ln b - \ln a)}{b^2-a^2}\frac{1}{r^2} - \ln r\right] \end{aligned}$$

$$\begin{aligned} T_{zz} = &\frac{E\chi}{1-\nu^2}\left[r^2 - \frac{a^2+b^2}{2}\nu^2\right] - \frac{1-2\nu}{1-\nu}\beta k\ln r \\ &+ \frac{\nu(1-2\nu)}{1-\nu}\beta k\frac{b^2\ln b - a^2\ln a}{b^2-a^2} - \frac{\nu(1-2\nu)}{2(1-\nu)}\beta k - (1-2\nu)\beta k_1, \end{aligned}$$

$$\begin{aligned} T_{\varphi\varphi} = &\frac{E\nu\chi}{4(1-\nu^2)}\left[3r^2 - \left(a^2+b^2\right) - \frac{a^2b^2}{r^2}\right] \\ &+ \frac{1-2\nu}{2(1-\nu)}\beta k\left[\frac{b^2\ln b - a^2\ln a}{b^2-a^2} + \frac{a^2b^2(\ln b - \ln a)}{b^2-a^2}\frac{1}{r^2} - \ln r - 1\right]. \end{aligned}$$

Proof. Replacing $T(r) = k\ln r + k_1$ and $u(r)$ from (2.10) into T_{rr} from (1.6) and using the boundary conditions $T_{rr}(a) = 0$, $T_{rr}(b) = 0$ (the lateral surface is stresses free) we can find the constants c_1, c_2 and afterward the formulas for $u(r)$, $T_{rr}, T_{zz}, T_{\varphi\varphi}$. □

Remark 2.1. For a given moment M, the constant χ can be found from the relation below

$$M = \int\int_D T_{\varphi\varphi} r d\sigma, \quad D : a \le r \le b,\ 0 \le \varphi \le 2\pi.$$

We obtain

$$\chi=\frac{1-\nu^2}{E\nu}\frac{15}{b^5-a^5-5a^2b^2\left(b-a\right)}$$
$$\cdot\left\{\frac{M}{2\pi}-\frac{1-2\nu}{2\left(1-\nu\right)}\beta k\left[\frac{4a^2b^2\left(\ln b-\ln a\right)}{3\left(a+b\right)}-\frac{2\left(b^3-a^3\right)}{9}\right]\right\}.$$

Particularly, in absence of heat, the solutions for the cylindrical tube can be found by making $\beta=0$ [5]:

$$u\left(r\right)=-\frac{\nu\chi}{4\left(1-\nu\right)}\left[r^3+\left(1-2\nu\right)\left(a^2+b^2\right)r+\frac{a^2b^2}{r}\right],$$
$$T_{rr}=\frac{E\nu\chi}{4\left(1-\nu^2\right)}\left[r^2-\left(a^2+b^2\right)+\frac{a^2b^2}{r^2}\right],$$
$$T_{zz}=\frac{E\chi}{1-\nu^2}\left[r^2-\frac{a^2+b^2}{2}\nu^2\right],$$
$$T_{\varphi\varphi}=\frac{E\nu\chi}{4\left(1-\nu^2\right)}\left[3r^2-\left(a^2+b^2\right)-\frac{a^2b^2}{r^2}\right].$$

Let us observe that, due to the heat, the deformations, the displacements and the stresses are different in comparison with the case of a simple torsion (by the presence of β). Various studies and diagrams can also be done for stresses and deformations in the thermoelastic case.

In the case of a simple torsion, the new deformed radius of the tube become

$$a'=a+u\left(a\right)=a\left[1-\frac{\nu\chi}{2}\left(a^2+b^2\right)\right],$$
$$b'=b+u\left(b\right)=b\left[1-\frac{\nu\chi}{2}\left(a^2+b^2\right)\right].$$

By fixing the distance L between the ends of the tube, the inner radius a and the outer radius b, then the thickness $d=b-a$ of the tube becomes after torsion

$$d'=b'-a'=d\left[1-\frac{\nu\chi}{2}\left(a^2+b^2\right)\right],$$

with $d'<d$.

In the thermoelastic coupled case, we have

$$d'=d\left[1-\frac{\nu\chi}{2}\left(a^2+b^2\right)+\frac{\left(1+\nu\right)\left(1-2\nu\right)}{2E}\beta\left(2k\frac{b^2\ln b-a^2\ln a}{b^2-a^2}-k+2k_1\right)\right].$$

The variation of the thickness in the thermoelastic coupled case will depend on the terms with β and on the inequality between T_1, T_2 (if $T(r)$ is taken from (V)).

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