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On the monotonicity of Schurer type polynomials

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ABSTRACT. The paper is a study of a sequence of Schurer type operators. It deduces a representation for the difference of two consecutive terms of the sequence of this type operators, by divided differences. Using this representation we enounces some sufficient conditions for the monotonicity of the sequence of Schurer operators.

1. PRELIMINARIES

In 1962 Schurer [5] considered and studied the following operators named Schurer operators:

$$\widetilde{B}_{m,p} : C([0, 1 + p]) \to C([0, 1])$$

defined by:

$$(\widetilde{B}_{m,p}f)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$ are the fundamental Schurer polynomials (see [1], [6]), $m \in \mathbb{N}$ and $p \in \mathbb{N}_0$ a given integer.

Note that for $p = 0$, the operator at (1.1) reduces to the well known Bernstein operators B_m.

The aims's paper is to study the monotonicity of the sequence $(\widetilde{B}_{m,p}f)$. So we shall establish a useful formula for the difference of two consecutive terms of the Schurer polynomials, formula based on divided differences of order 1 and 2.

We will use the properties of the Schurer operator, contained in the following

Theorem 1.1. *Let* $e_i(x) = x^i$ $(i = 0, 1, 2)$ *be test function. The following equalities:*
(i) $(\widetilde{B}_{m,p}e_0)(x) = 1$;
(ii) $(\widetilde{B}_{m,p}e_1)(x) = \left(1 + \frac{p}{m}\right)$;
(iii) $(\widetilde{B}_{m,p}e_1)(x) = (m+p)m^{-2}((m+p)x^2 + x(1-x))$,
hold, for any $x \in [0, 1+p]$.

Conform to Korovkin's first theorem it was proved that for $f \in C([0, 1+p])$ the sequence of the polynomial (1.1) converges uniformly to f on $[0, 1]$.

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For this type of operators let's remind the corresponding order of approximation by using the modulus of continuity of f by:

$$\left|(\widetilde{B}_{m,p}f)(x)-f(x)\right|\le 2\omega_f(\delta_{m,p,x}),$$

where $\delta_{m,p,x}=\sqrt{p^2x^2+(m+p)x(1-x)/m}$. We have the following result:

Theorem 1.2. *The difference between the polynomials* $(\widetilde{B}_{m+1,p}f)(x)$ *and* $(\widetilde{B}_{m,p}f)(x)$ *can be expressed under the form:*

$$\begin{aligned}&(\widetilde{B}_{m+1,p}f)(x)-(\widetilde{B}_{m,p}f)(x)=\\&=-\frac{(m+p)x(1-x)}{m(m+1)}\sum_{k=0}^{m+p-1}\binom{m+p-1}{k}\left(\frac{1}{m}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]+\right.\\&\left.+\frac{p}{m+p-k}\left[\frac{k}{m},\frac{k+1}{m+1};f\right]\right)\cdot\\&\cdot x^k(1-x)^{m+p+1-k}+x^{m+p+1}\left(f\left(\frac{m+p+1}{m+1}\right)-f\left(\frac{m+p}{m}\right)\right).\end{aligned}$$

Proof. Let's note $n=m+p$ and we have $m=n-p$.

Now we have

$$\begin{aligned}(\widetilde{B}_{n-p,p}f)(x)&=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n-p}\right)\\(\widetilde{B}_{n-p+1,p}f)(x)&=\sum_{k=0}^{n+1}\binom{n+1}{k}x^k(1-x)^{n+1-k}f\left(\frac{k}{n+1-p}\right)\\(\widetilde{B}_{n-p+1,p}f)(x)&=\sum_{k=1}^{n}\binom{n+1}{k}x^k(1-x)^{n+1-k}f\left(\frac{k}{n+1-p}\right)+\\&+(1-x)^{n+1}f(0)+x^{n+1}f\left(\frac{n+1}{n+1-p}\right)\\(\widetilde{B}_{n-p,p}f)(x)&=\sum_{k=0}^{n}x\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n-p}\right)+\\&+\sum_{k=0}^{n}(1-x)\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n-p}\right)\end{aligned}$$

we make the change $i=k+1$ and then keep index by k we have:

$$\begin{aligned}(\widetilde{B}_{n-p,p}f)(x)&=\sum_{k=1}^{n+1}x\binom{n}{k-1}x^{k-1}(1-x)^{n+1-k}f\left(\frac{k-1}{n-p}\right)+\\&+\sum_{k=0}^{n}(1-x)\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n-p}\right)=\\&=\sum_{k=1}^{n}x\binom{n}{k-1}x^{k-1}(1-x)^{n+1-k}f\left(\frac{k-1}{n-p}\right)+x^{n+1}f\left(\frac{n}{n-p}\right)+\end{aligned}\tag{1.2}$$

$$+\sum_{k=1}^{n}(1-x)\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n-p}\right)+(1-x)^{n+1}f(0)=$$
$$=\sum_{k=1}^{n}\left(\binom{n}{k-1}x^k(1-x)^{n-k+1}f\left(\frac{k-1}{n-p}\right)+\binom{n}{k}x^k(1-x)^{n-k+1}f\left(\frac{k}{n-p}\right)\right)+$$
$$+x^{n+1}f\left(\frac{n}{n-p}\right)+(1-x)^{n+1}f(0).$$

(1.3)
$$(\widetilde{B}_{n-p+1,p}f)(x)-(\widetilde{B}_{n-p,p}f)(x)=$$
$$=\sum_{k=1}^{n}\left(\binom{n+1}{k}x^k(1-x)^{n+1-k}f\left(\frac{k}{n+1-p}\right)-\right.$$
$$\left.-\binom{n}{k-1}x^k(1-x)^{n-k+1}f\left(\frac{k-1}{n-p}\right)\right)+$$
$$+\sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{n-k+1}f\left(\frac{k}{n-p}\right)+(1-x)^{n+1}f(0)+$$
$$+x^{n+1}f\left(\frac{n+1}{n+1-p}\right)-x^{n+1}f\left(\frac{n}{n-p}\right)-(1-x)^{n+1}f(0)=$$
$$=\sum_{k=1}^{n}\left(\binom{n+1}{k}x^k(1-x)^{n+1-k}f\left(\frac{k}{n+1-p}\right)-\right.$$
$$\left.-\binom{n}{k-1}x^k(1-x)^{n-k+1}f\left(\frac{k-1}{n-p}\right)\right)-$$
$$-\sum_{k=1}^{n}\binom{n}{k}x^k(1-x)^{n-k+1}f\left(\frac{k}{n-p}\right)+x^{n+1}\left(f\left(\frac{n+1}{n+1-p}\right)-f\left(\frac{n}{n-p}\right)\right)=$$
$$=\sum_{k=1}^{n}\left(\frac{n+1}{n+1-k}\binom{n}{k}f\left(\frac{k}{n+1-p}\right)-\frac{k}{n+1-k}\binom{n}{k}f\left(\frac{k-1}{n-p}\right)-\right.$$
$$\left.-\binom{n}{k}f\left(\frac{k}{n-p}\right)\right)x^k(1-x)^{n-k+1}+x^{n+1}\left(f\left(\frac{n+1}{n+1-p}\right)-f\left(\frac{n}{n-p}\right)\right)=$$
$$=\sum_{k=1}^{n}\binom{n}{k}\left(f\left(\frac{k}{n-p}\right)+\frac{k}{n+1-k}f\left(\frac{k-1}{n-p}\right)-\right.$$
$$\left.-\frac{n+1}{n+1-k}f\left(\frac{k}{n+1-p}\right)\right)x^k(1-x)^{n-k+1}+x^{n+1}\left(f\left(\frac{n+1}{n+1-p}\right)-f\left(\frac{n}{n-p}\right)\right).$$

We used the identities:

$$\binom{n}{k-1}=\frac{k}{n+1-k}\binom{n}{k},\ \binom{n+1}{k}=\frac{n+1}{n+1-k}\binom{n}{k}.$$

We make the change $k=j+1$ and then denote again index by k:

$$(\widetilde{B}_{n-p+1,p}f)(x)-(\widetilde{B}_{n-p,p}f)(x)= \tag{1.4}$$

$$=\sum_{k=0}^{n}\binom{n}{k+1}\left(f\left(\frac{k+1}{n-p}\right)+\frac{k+1}{n-k}f\left(\frac{k}{n-p}\right)-\frac{n+1}{n-k}f\left(\frac{k+1}{n+1-p}\right)\right)\cdot$$

$$\cdot x^{k+1}(1-x)^{n-k}+x^{n+1}\left(f\left(\frac{n+1}{n+1-p}\right)-f\left(\frac{n}{n-p}\right)\right)=$$

$$=-x(1-x)\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{n}{k+1}f\left(\frac{k+1}{n-p}\right)+\frac{n}{n-k}f\left(\frac{k}{n-p}\right)-\right.$$

$$\left.-\frac{n(n+1)}{(k+1)(n-k)}f\left(\frac{k+1}{n+1-p}\right)\right)x^k(1-x)^{n-k-1}+$$

$$+x^{n+1}\left(f\left(\frac{n+1}{n+1-p}\right)-f\left(\frac{n}{n-p}\right)\right)$$

since

$$\binom{n}{k+1}=\frac{n}{k+1}\binom{n-1}{k}.$$

But

$$\frac{n(n+1)}{(k+1)(n-k)}f(\frac{k+1}{n+1-p})=n\left(\frac{1}{k+1}+\frac{1}{n-k}\right)f\left(\frac{k+1}{n+1-p}\right)$$

so we can write now:

$$\frac{n}{k+1}f\left(\frac{k+1}{n-p}\right)+\frac{n}{n-k}f\left(\frac{k}{n-p}\right)-\frac{n(n+1)}{(k+1)(n-k)}f\left(\frac{k+1}{n+1-p}\right)=$$

$$=\frac{n}{k+1}f\left(\frac{k+1}{n-p}\right)+\frac{n}{n-k}f\left(\frac{k}{n-p}\right)-\frac{n}{k+1}f\left(\frac{k+1}{n+1-p}\right)-$$

$$-\frac{n}{n-k}f\left(\frac{k+1}{n+1-p}\right)=\frac{n}{k+1}\left(f\left(\frac{k+1}{n-p}\right)-f\left(\frac{k+1}{n+1-p}\right)\right)+$$

$$+\frac{n}{n-k}\left(f\left(\frac{k}{n-p}\right)-f\left(\frac{k+1}{n+1-p}\right)\right)=$$

$$=\frac{n}{k+1}\cdot\frac{k+1}{(n-p)(n+1-p)}\left[\frac{k+1}{n+1-p},\frac{k+1}{n-p};f\right]+$$

$$+\frac{k-n+p}{(n-p)(n+1-p)}\cdot\frac{n}{n-k}\left[\frac{k}{n-p},\frac{k+1}{n+1-p};f\right]=$$

$$=\frac{n}{(n-p)(n+1-p)}\left[\frac{k+1}{n+1-p},\frac{k+1}{n-p};f\right]+$$

$$+\frac{k-n+p}{(n-p)(n+1-p)}\cdot\frac{n}{n-k}\left[\frac{k}{n-p},\frac{k+1}{n+1-p};f\right]=$$

$$=\frac{n}{(n-p)(n+1-p)}\left(\left[\frac{k+1}{n+1-p},\frac{k+1}{n-p};f\right]+\frac{n-k-p}{n-k}\left[\frac{k}{n-p},\frac{k+1}{n+1-p};f\right]\right)=$$

$$= \frac{n}{(n-p)(n+1-p)} \left(\left[\frac{k+1}{n+1-p}, \frac{k+1}{n-p}; f \right] - \right.$$
$$\left. - \left[\frac{k}{n-p}, \frac{k+1}{n+1-p}; f \right] + \frac{p}{n-k} \left[\frac{k}{n-p}, \frac{k+1}{n+1-p}; f \right] \right) =$$
$$= \frac{n}{(n-p)(n+1-p)} \left(\frac{1}{n-p} \left[\frac{k}{n-p}, \frac{k+1}{n+1-p}, \frac{k+1}{n-p}; f \right] + \right.$$
$$\left. + \frac{p}{n-k} \left[\frac{k}{n-p}, \frac{k+1}{n+1-p}; f \right] \right).$$

The difference (1.4) become now :

(1.5) $$(\widetilde{B}_{n-p+1,p}f)(x) - (\widetilde{B}_{n-p,p}f)(x) =$$
$$= -\frac{nx(1-x)}{(n-p)(n+1-p)} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{1}{n-p} \left[\frac{k}{n-p}, \frac{k+1}{n+1-p}, \frac{k+1}{n-p}; f \right] + \right.$$
$$\left. + \frac{p}{n-k} \left[\frac{k}{n-p}, \frac{k+1}{n+1-p}; f \right] \right) x^k (1-x)^{n-k+1} +$$
$$+ x^{n+1} \left(f\left(\frac{n+1}{n+1-p} \right) - f\left(\frac{n}{n-p} \right) \right).$$

In final, replacing in (1.5) $m = n - p$ we have

(1.6) $$(\widetilde{B}_{m+1,p}f)(x) - (\widetilde{B}_{m,p}f)(x) =$$
$$= -\frac{(m+p)x(1-x)}{m(m+1)} \sum_{k=0}^{m+p-1} \binom{m+p-1}{k} \left(\frac{1}{m} \left[\frac{k}{m}, \frac{k+1}{m+1}, \frac{k+1}{m}; f \right] + \right.$$
$$\left. + \frac{p}{m+p-k} \left[\frac{k}{m}, \frac{k+1}{m+1}; f \right] \right) x^k (1-x)^{m+p-k+1} +$$
$$+ x^{m+p+1} \left(f\left(\frac{m+p+1}{m+1} \right) - f\left(\frac{m+p}{m} \right) \right).$$

□

2. Conditions to ensure the monotonicity

We give the following definition of higher-order convex function:

Definition 2.1. [6] A real-valued function on an interval E is called convex of order n on E if all its differences of order $n+1$, on $n+2$ distinct points of E, are positive. The function f is said to be non-concave of order n on the interval E is all divided differences of order $n+1$, on any $n+2$ points of E, are non-negative.

We reconsidered the difference (1.6) we have:

(2.7) $$(\widetilde{B}_{m+1,p}f)(x) - (\widetilde{B}_{m,p}f)(x) =$$
$$= -\frac{(m+p)x(1-x)}{m(m+1)} \sum_{k=0}^{m+p-1} \binom{m+p-1}{k} \left(\frac{1}{m} \left[\frac{k}{m}, \frac{k+1}{m+1}, \frac{k+1}{m}; f \right] + \right.$$

$$+\frac{p}{m+p-k}\left[\frac{k}{m},\frac{k+1}{m+1};f\right]\Bigg)x^k(1-x)^{m+p-k+1}-$$
$$-x^{m+p+1}\left(f\left(\frac{m+p}{m}\right)-f\left(\frac{m+p+1}{m+1}\right)\right)$$

Since on the interval $[0,1]$ we have all the terms of difference (1.6) positive, from the Definition 2.1 there follows:

Theorem 2.3. *If the function is convex of first order on the interval* $[0,1+p]$ *and increasing on* $(0,1+p)$ *then the sequence* $(\widetilde{B}_{m,p}f)(x)$ *is decreasing on* $(0,1+p)$*, that is* $(\widetilde{B}_{m+1,p}f)(x) < (\widetilde{B}_{m,p}f)(x)$ *on* $(0,1+p)$ *for* $n=1,2,..$

We notice that for $p=0$ our result becomes:

$$(\widetilde{B}_{m+1,0}f)(x)-(\widetilde{B}_{m,0}f)(x)=$$
$$=-\frac{x(x-1)}{m(m+1)}\sum_{k=0}^{m-1}\binom{m-1}{k}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]x^k(1-x)^{m-k+1}$$

formula who was established by D. D. Stancu in the paper [7].

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