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Existence of Nash-Bertrand equilibrium in duopoly games with pollution treatment cost

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ABSTRACT. In this paper we will formulate Bertrand duopoly without product differentiation and with pollution treatment cost sharing.

Firm i's profit is given by

$$H_i(x_1,x_2) = g(x_1,x_2)(x_i - c) - \frac{x_i}{\sum\limits_{j=1}^{2} x_j} T\left(\sum_{j=1}^{2} x_j\right), \ i = 1,2,$$

where x_i is firm i's output, c is constant marginal cost.

1. INTRODUCTION

Pollution or industrial waste emerges as a by-product of firms' production activities. Pollution tax is introduced as one of regulatory polices designed for reducing the level of pollution. An alternative regulatory policy is to let firms treat pollution directly and let them share the total cost necessary for pollution treatment.

In this paper we will formulate Bertrand duopoly without product differentiation and whit pollution treatment cost sharing.

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where x_i is firm i's output, c is constant marginal cost.

We assume that each firm has a limited production capacity, so for all i, $0 < x_i \leq L_i$, where L_i is the given capacity limit of firm i.

2. BERTRAND MODEL OF DUOPOLY

The Bertrand's model is based on suggestion that firms actually choose prices, rather than quantities as in Cournot's model. The Bertrand's model is a different game than Cournot's model because: the strategy spaces are different, the payoff functions are different. Thus we obtain other equilibrium point, but the equilibrium concept used is the Nash equilibrium defined in [1].

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We consider the case of differentiated products. If firms 1 and 2 choose prices x_1 and x_2, respectively, the quantity that consumers demand from firm i is

$$q_i(x_i, x_j) = a - x_i + bx_j,$$

where $b > 0$ reflects the extent to which firm i's product is a substitute for firm j's product. This is an unrealistic demand function because demand for firm i's product is positive even when firm i charges an arbitrarily high price, provided firm j also charges a high enough price. We assume that there are no fixed costs of production and that marginal costs are constant at c, where $c < a$, and that the firms act simultaneously (choose their prices). We translate the economic problem into a non-cooperative game. There are again two players. This time, however, the strategies available to each firm are the different prices it might charge, rather than the different quantities it might produce. We will assume that negative prices are not feasible but that any non-negative price can be charged – there is no restriction to prices denominated in pennies. Thus each firm's strategy space can again be represented as $S_i = [0, \infty)$, and a typical strategy s_i is now a price choice, $x_i \geq 0$.

We will again assume that the payoff function for each firm is just its profit. The profit to firm i when it chooses the price x_i and its rival choose the price x_j is

$$H_i(x_i, x_j) = q_i(x_i, x_j)(x_i - c) = (a - x_i + bx_j)(x_i - c).$$

Thus, the price pair (x_1^*, x_2^*) is Nash equilibrium if, for each firm i, x_i^* solves the problem

$$\max_{0 \leq x_i < \infty} H_i(x_i, x_j^*) = \max_{0 \leq x_i < \infty} (a - x_i + bx_j^*)(x_i - c).$$

The solution to firm i's optimization problem is

$$x_i^* = \frac{1}{2}\,(a + bx_j^* + c).$$

Therefore, if the price pair (x_1^*, x_2^*) is to be a Nash equilibrium, the firm's price choices must satisfy

$$x_1^* = \frac{1}{2}\,(a + bx_2^* + c) \;\text{ and }\; x_2^* = \frac{1}{2}\,(a + bx_1^* + c).$$

Solving this pair of equations yields

$$x_1^* = x_2^* = \frac{a + c}{2 - b}\,.$$

3. Existence of unique equilibrium with pollution treatment cost

Let there 2 firms in a Bertrand duopoly. They are assumed to produce homogeneous goods. Pollution emerges as a result of their productive activities. The firms treat jointly the pollution and share among themselves the cost necessary for pollution treatment in proportion to their share in the total output.

Firm i's profit can be write, in some suitable conditions, as

$$H_i(x_1, x_2) = f(x_1 + x_2)x_i - C_i(x_i) - \frac{x_i}{x_1 + x_2}T(x_1 + x_2), \quad i = 1, 2,$$

where x_i is firm i's output, C_i is the production cost function, f is the inverse demand function for the goods, and T is the pollution treatment total cost function. In the following analysis we assume differentiability of all functions up to the order as necessary. Furthermore we let

$$x = x_1 + x_2, \quad F(x) = f(x) - \frac{T(x)}{x}.$$

Assume furthermore that each firm has a limited production capacity, so we have, for all i, $0 \leq x_i \leq L_i$ where L_i is the given capacity limit of firm i.
Let finally

$$L = \sum_{i=1}^{2} L_i$$

be the total capacity limit of the industry.

An 2-person non-cooperative game is obtained with the 2 firms being the players with strategy sets $[0, L_i]$ and payoff functions H_i.

The following fundamental assumptions we consider fulfilled:
for all $i = 1, 2,\ x_i \in [0, L_i],\ x \in [0, L]$,

$$(A) \quad F'(x) + x_i F"(x) < 0,$$

$$(B) \quad F'(x) < C_i"(x_i).$$

Since

$$F'(x) = f'(x) - \frac{T'(x)x - T(x)}{x^2},$$

it follows that

$$F'(x) < 0 \; if \; f'(x) < 0 \quad \text{and} \quad T'(x)x \geq T(x).$$

By introducing the notation

$$G_i(x_i, x) = F(x) + x_i F'(x) - C_i'(x_i)$$

from conditions (A) and (B) it follows that G_i strictly decreases in x_i with any fixed x, and also strictly decreases in x with any fixed value of x_i. An $x_i \in [0, L_i]$ is the profit maximizing output of firm i with a fixed $x \in [0, L]$ if and only if

$$x_i = \varphi_i(x)$$

where

$$\varphi_i(x) = 0 \qquad \text{if} \quad G_i(0, x) \leq 0,$$

$$\varphi_i(x) \in (0, L_i) \quad \text{if} \quad G_i(x_i, x) = 0,$$

and

$$\varphi_i(x) = L_i \quad \text{if} \quad G_i(L_i, x) \geq 0.$$

It is easy to see that $\varphi_i(x)$ is unique for all $x \in [0, L]$, since $G_i(x_i, x)$ strictly decreases in x_i as a consequence of assumption (B). This monotonicity and the continuity of G_i imply that $\varphi_i(x)$ is continuous in x and is non-increasing. We obtain the Nash-Cournot equilibrium industry output as the unique solution of equation

$$x = \sum_{i=1}^{2} \varphi_i(x).$$

The existence and uniqueness of the solution of this equation follows from the facts that φ_i is continuous for all i, the left hand side strictly increases but the right hand side is non-increasing. Furthermore at $x = 0$ we have

$$x - \sum_{i=1}^{2} \varphi_i(x) \leq 0$$

and at $x = L$ we have

$$L - \sum_{i=1}^{2} \varphi_i(L) \geq L - \sum_{i=1}^{2} L_i = 0.$$

We can find sufficient and necessary conditions to ensure that the equilibrium point is interior. These conditions are

$$x_{\min} < \sum_{i=1}^{2} \varphi_i(x_{\min}), \quad x_{\max} > \sum_{i=1}^{2} \varphi_i(x_{\max})$$

M is nonempty and contains at least two points,

where

$$x_{\min} = \inf\{x/x \in M\}, \quad x_{\max} = \sup\{x/x \in M\}$$
$$M = M_1 \cap M_{2,}\ M_i = \{x/x \in [0, L], G_i(0, x) > 0, G_i(L_i, x) < 0\}, i = 1, 2.$$

Remark 3.1. There are possibilities to consider effects on the equilibrium of an endogenous change in pollution treatment technology. The pollution treatment cost is now expressed as $T(x, \theta)$, where θ denotes a parameter related to pollution treatment technology. If $\dfrac{\partial T}{\partial \theta} < 0$ the total cost pollution treatment decreases as a result of technological improvement. So, if $T(x, \theta) = \dfrac{T(x)}{\theta}$, or $T(x, \theta) = T(x) - \theta x$, the equilibrium industry output increases as a result of technological improvement in pollution treatment.

Remark 3.2. If x^{o*} and x^* denote the equilibrium industry output without and with pollution treatment we can show that $x^{o*} < x^*$, hence the equilibrium in absence of pollution treatment is larger than that for pollution treatment cost sharing.

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