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A general common fixed point theorem of Meir and Keeler type for weakly compatible mappings

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ABSTRACT. In this paper, using a combination method used in [3], [20] and [22], the result of [16, Theorem 1] is improved by removing the assumption of continuity, relaxing compatibility to weakly compatibility property and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation.

1. INTRODUCTION

Let S and T be two self mappings of a metric space (X,d). Jungck [6] defines S and T to be compatible if $\lim d(STx_n,TSx_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n=\lim Tx_n=x$ for some $x\in X$.

In 1993, Jungck, Murthy and Cho [8] define S and T to be compatible of type (A) if $\lim d(TSx_n,S^2x_n)=0$ and $\lim d(STx_n,T^2x_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n=\lim Tx_n=x$ for some $x\in X$.

Recently, Pathak and Khan [18] introduced a new concept of compatible of type (B) as a generalization of compatible mappings of type (A). S and T is said to be compatible of type (B) if

$$\lim d(STx_n,T^2x_n)\le \frac{1}{2}[\lim d(STx_n,St)+\lim d(St,S^2x_n)],$$

$$\lim d(TSx_n,S^2x_n)\le \frac{1}{2}[\lim d(TSx_n,Tt)+\lim d(Tt,T^2x_n)],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n=\lim Tx_n=t$ for some $t\in X$.

Clearly compatible mappings of type (A), are compatible mappings of type (B). By [18, Ex. 2.4] it follows that "the reverse implication is not true".

In [19] the concept of compatible mappings of type (P) was introduced and compared with compatible mappings and compatible mappings of type (A).
S and T are compatible of type (P) if $\lim d(S^2x_n,T^2x_n)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n=\lim Tx_n=t$ for some $t\in X$.

Lemma 1.1. [6] (resp. [8], [18], [19]). *Let S and T be compatible (resp. compatible of type (A), compatible of type (B), compatible of type (P)) self mappings of a metric space (X,d). If $Sx=Tx$ for some $x\in X$, then $STx=TSx$.*

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In 1994, Pant [13] introduced the notion of R-weakly commuting mappings. Two self mappings A and S of a metric space (X, d) are called R-weakly commuting at a point $x \in X$ if $d(ASx, SAx) \leq Rd(Ax, Sx)$ for some $R > 0$. The mappings A and S are called pointwise R-weakly commuting if given x in X, there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. It is proved in [14] that the notion of pointwise R-weakly commutativity is equivalent to commutativity in coincidence points.

Recently, Jungck [7] (resp. Dhage [2]) defines S and T to be weakly compatible (resp. coincidentally commuting) if $Sx = Tx$ implies $STx = TSx$.
Thus S and T are weakly compatible and coincidentally commuting if and only if S and T are pointwise R-weakly commuting mappings.

Remark 1.1. By Lemma 1.1 it follows that every compatible (compatible of type (A), compatible of type (B), compatible of type (P)) pair of mappings are weakly compatible.

2. Preliminaries

In 1969, Meir and Keeler [10] established a fixed point theorem for a self mappings f of a metric space (X, d) satisfying the following condition:

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2.1) $$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(fx, fy) < \varepsilon.$$

There exists a vast literature which generalize the result of Meir and Keeler (see "References" of [4], [5], [1]). In [9] Maiti and Pal proved a fixed point theorem for a self mapping f of a metric space (X, d) satisfying the following condition, which is a generalization of (2.1):

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2.2) $$\varepsilon \leq \max\{d(x, y), d(x, fx), d(y, fy)\} < \varepsilon + \delta \quad \text{implies} \quad d(fx, fy) < \varepsilon.$$

In [17] and [21], Park-Rhodes and Rao-Rao extend this result for two mappings and proved some fixed point theorems for self mappings f and g of a metric space (X, d) satisfying the following condition:

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2.3) $$\varepsilon \leq \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{1}{2}[d(fx, gy) + d(fy, gx)]\} < \varepsilon + \delta$$

implies $d(fx, gy) < \varepsilon$.

In 1986, Jungck [6] and Pant [11] extend these results for four mappings.

It is known from Jungck [6], Cho, Murthy and Jungck [1], Pant [12]-[15] and other papers the fact that in case of theorems for four mappings $A, B, S, T : (X, d) \to (X, d)$, a condition of type Meir-Keeler didn't assure the existence of a fixed point.

Recently, Pant [16] proved the following theorem.

Theorem 2.1. [16] *Let S, I and T, J be compatible pairs of self mappings of a complete metric space (X, d) such that $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

(2.4) $$\varepsilon \leq \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\} < \varepsilon + \delta$$

implies $d(Sx,Ty) < \varepsilon$,

(2.5) $\quad d(Sx,Ty) \le k[d(Ix,Jy) + d(Ix,Sx) + d(Jy,Ty)]$,

where $0 \le k < 1$.

If at least one of the mappings S, T, I is continuous and J is continuous too then S, T, I and J have a unique common fixed point.

3. Implicit relations

Let $\mathcal{F}_4$ be the set of all real continuous functions $F : \mathbb{R}_+^4 \to \mathbb{R}$ satisfying the following conditions:

$(F_1) : F(u,0,u,0) \le 0$ implies $u = 0$,

$(F_2) : F(u,0,0,u) \le 0$ implies $u = 0$.

The function $F : \mathbb{R}_+^4 \to \mathbb{R} : \mathbb{R}_+^4 \to \mathbb{R}$ satisfies condition (F_u) if

$(F_u) : F(u,u,0,0) \le 0$ implies $u = 0$.

Example 3.1. $F(t_1,...,t_4) = t_1 - k(t_2 + t_3 + t_4)$, where $0 \le k < 1$.

Example 3.2. $F(t_1,...,t_4) = t_1 - h\max\{t_2,t_3,t_4\}$, where $0 \le h < 1$.

Example 3.3. $F(t_1,...,t_4) = t_1^2 - a(t_2^2 + t_3^2 + t_4^2)$, where $0 \le a < 1$.

Example 3.4. $F(t_1,...,t_4) = t_1^2 - a[t_2^2 + t_3t_4]$, where $0 \le a < 1$.

Example 3.5. $F(t_1,...,t_4) = t_1 - at_2 - \dfrac{bt_3t_4}{1+t_2+t_3}$, where $0 \le a < 1$ and $b > 0$.

Theorem 3.2. *Let (X,d) be a metric space and* $S,T,I,J : (X,d) \to (X,d)$ *four mappings satisfying the inequality*

(3.6) $\quad F(d(Sx,Ty), d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)) < 0$

for all x, y *in* X*, where* F *satisfies property* (F_u)*. Then* S,T,I,J *have at most one common fixed point.*

Proof. Suppose that S,T,I,J have two common fixed points z and u with $z \ne u$.
Then by (3.6) we have successively

$F(d(Sz,Tu), d(Iz,Ju), d(Iz,Sz), d(Ju,Tu)) \le 0$

$F(d(z,u), d(z,u), 0, 0) \le 0$ which implies by (F_u) that $z = u$.

In this paper, using a combination of methods used in [3], [20] and [22] the result of [16, Theorem 1] is improved by removing the assumption of continuity, relaxing compatibility to weakly compatibility property and replacing the completeness of the space with a set of four alternative conditions for four functions satisfying an implicit relation. □

4. Main results

Theorem 4.3. *Let S, T, I and J be the self mappings of a metric space (X, d) such that*

(4.7) $S(X) \subset J(X)$ *and* $T(X) \subset I(X)$,

given $\varepsilon > 0$ there exists $\delta > 0$ such that

(4.8) $\varepsilon \leq \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\} < \varepsilon + \delta$

implies $d(Sx, Ty) < \varepsilon$,

(4.9) $there\ exists\ F \in \mathcal{F}_4\ such\ that\ (3.6)\ holds\ for\ every\ x, y \in X.$

If one of $S(X)$, $T(X)$, $I(X)$ or $J(X)$ is a complete subspace of X, then

(4.10) $S\ and\ I\ have\ a\ coincidence\ point,$

(4.11) $T\ and\ J\ have\ a\ coincidence\ point.$

Moreover, if the pairs (S, I) and (T, J) are weakly compatible, then S, T, I and J have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{y_n\}$ in X as follows:

$$y_{2n} = Sx_{2n} = Jx_{2n+1},\ y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}$$

for $n = 0, 1, 2, ...$ This can be done since $S(X) \subset J(X)$ and $T(X) \subset I(X)$. As in [16, Theorem 1] it follows that $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that $J(X)$ is a complete subspace of X, then the subsequence $Jx_{2n+1} = Sx_{2n}$ is a Cauchy sequence in $J(X)$ and hence has a limit v.

Let $v \in J^{-1}u$, then $Jv = u$. Since y_{2n} is convergent then y_n is convergent and $Ix_{2n+2} = Tx_{2n+1}$ also converges to u.

Setting $x = x_{2n}$ and $y = v$ in (3.6) we have

$$F(d(Sx_{2n}, Tv), d(Ix_{2n}, Jv), d(Ix_{2n}, Sx_{2n}), d(Jv, Tv)) \leq 0.$$

Letting n tend to infinity we obtain

$$F(d(u, Tv), 0, 0, d(u, Tv)) \leq 0.$$

By (F_2) we have $u = Tv$. Hence J and T have a coincidence point.

Since $T(X) \subset I(X)$, $u = Tv$ implies $u \in I(X)$.

Let $w \in I^{-1}u$, then $Iw = u$. Now using a similar argument one can show using property (F_1) that $Sw = u$. Thus S and I have a coincidence point.

If one assumes that $I(X)$ is complete then by an analogous argument the same conclusion as before follows. The remaining two ones are essentially the same as the previsious ones. Indeed, if $S(X)$ is complete then by (4.7) $u \in S(X) \subset J(X)$.

Similarly, if $T(X)$ is complete, $u \in T(X) \subset I(X)$.

Then (4.10) and (4.11) are completely established.

By $u = Jv = Tv$ and weak-compatibility of (J, T) we have

$$Tu = TJv = JTv = Ju.$$

By $Iw = Sw = u$ and weak-compatibility of (I, S) we have

$$Su = SIw = ISw = Iu.$$

On the other hand we have successively by (3.6)

$$F(d(Sw,Tu),d(Iw,Ju),d(Iw,Sw),d(Ju,Tu))\leq 0$$
$$F(d(u,Tu),d(u,Tu),0,0)\leq 0$$

which implies by (F_u) that $u=Tu$.

Similarly one can show that $u=Su$. Then

$$u=Tu=Ju=Su=Iu.$$

The uniqueness of the common fixed point follows from Theorem 3.2. □

Corollary 4.1. *Let S,T,I and J be the self mappings of a complete metric spaces satisfying conditions (4.7), (4.8), (4.9) of Theorem 4.3. Then (4.10) and (4.11) hold. Moreover, if the pair (S,I) and (T,J) are compatible (compatible of type (A), compatible of type (B), compatible of type (P)) then S,T,I and J have a unique common fixed point.*

Proof. It follows by Theorem 4.3 and Remark 1.1. □

Corollary 4.2. *Theorem* 2.1.

Proof. It follows by Corollary 4.1 and Example 3.1. □

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