Some pairs of multivalued operators

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ABSTRACT. We study the following problem.

Let $(X, d)$ be a metric space and $T_1, T_2 : X \rightarrow P(X)$ two multivalued operators. Determine metric conditions on the pair of multivalued operators $T_1$ and $T_2$, which imply that, for each $x \in X$, there exists a sequence of successive approximations for the pair $(T_1, T_2)$ or for the pair $(T_2, T_1)$, starting from $x$, which converges to a common fixed point or to a common strict fixed point of $T_1$ and $T_2$ and, for each $x \in X$, there exists a sequence of successive approximations of $T_i$, starting from $x$, which converges to a fixed point or to a strict fixed point of $T_i$, for each $i \in \{1, 2\}$.

1. INTRODUCTION

Let $X$ be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of $X$, i.e. $P(X) := \{ Y \mid \emptyset \neq Y \subseteq X \}$. Let $T_1, T_2 : X \rightarrow P(X)$ be two multivalued operators. We denote by $F_{T_1}$ the fixed points set of $T_1$, i.e. $F_{T_1} := \{ x \in X \mid x \in T_1(x) \}$, by $(SF)_{T_1}$ the strict fixed points set of $T_1$, i.e. $(SF)_{T_1} := \{ x \in X \mid T_1(x) = \{x\} \}$ and by $(CF)_{T_1, T_2}$ the common fixed points set of $T_1$ and $T_2$, i.e. $(CF)_{T_1, T_2} := \{ x \in X \mid x \in T_1(x) \cap T_2(x) \}$.

A sequence $(x_n)_{n \in \mathbb{N}}$ is called sequence of successive approximations of $T_1$ if $x_0 \in X$ and $x_{n+1} \in T_1(x_n)$, for each $n \in \mathbb{N}$.

A sequence $(x_n)_{n \in \mathbb{N}}$ is called sequence of successive approximations for the pair $(T_1, T_2)$ if $x_0 \in X$, $x_{2n+1} \in T_1(x_{2n})$ and $x_{2n+2} \in T_2(x_{2n+1})$, for each $n \in \mathbb{N}$.

Definition 1.1. Let $(X, d)$ be a metric space and $T : X \rightarrow P(X)$ a multivalued operator. We say that $T$ is a weakly Picard multivalued operator iff for each $x \in X$ and for every $y \in T(x)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}^*$;

(iii) sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of $T$.

For examples of weakly Picard multivalued operators see for instance [15], [16].

Definition 1.2. Let $(X, d)$ be a metric space and $T_1, T_2 : X \rightarrow P(X)$ two multivalued operators. We say that $(T_1, T_2)$ is a weakly Picard pair of multivalued operators iff for each $x \in X$ and for every $y \in T_1(x) \cup T_2(x)$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;

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Problem 1.1.

Lemma 1.1.

We also recall the functional \( \delta \) on \( D \), defined by

\[
\inf \{ d(a, b) \mid a \in A, b \in B \},
\]

for each \( A, B \in P(X) \), and the generalized functionals \( \delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\} \), defined by

\[
\delta(A, B) = \sup \{ d(a, b) \mid a \in A, b \in B \},
\]

for each \( A, B \in P(X) \), and \( H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\} \), defined by

\[
H(A, B) = \max \{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \},
\]

for each \( A, B \in P(X) \).

The following property of the generalized functional \( H \) is well-known.

Lemma 1.1. Let \((X, d)\) be a metric space, \( A, B \in P(X) \) and \( q \in \mathbb{R}, q > 1 \).

Then for every \( a \in A \), there exists \( b \in B \) such that

\[
d(a, b) \leq q H(A, B).
\]

The purpose of this paper is to study the following problem.

Problem 1.1. Let \((X, d)\) be a metric space and \( T_1, T_2 : X \to P(X) \) two multivalued operators. Determine metric conditions on the pair of multivalued operators \( T_1 \) and \( T_2 \), which imply that, for each \( x \in X \), there exists a sequence of successive approximations for the pair \((T_1, T_2)\) or for the pair \((T_2, T_1)\), starting from \( x \), which converges to a common fixed point or to a common strict fixed point of \( T_1 \) and \( T_2 \) and, for each \( x \in X \), there exists a sequence of successive approximations of \( T_i \), starting from \( x \), which converges to a fixed point or to a strict fixed point of \( T_i \), for each \( i \in \{1, 2\} \).

For singlevalued operators results of this type are given by Rus [13] and Dien [4] and for multivalued operators results which answer to Problem 1.1 are presented by Dien [4] and Sintămărian [18, 19, 20].

2. Fixed points and common fixed points

Theorem 2.1. Let \((X, d)\) be a complete metric space and \( T_1, T_2 : X \to P(X) \) two multivalued operators for which there exist \( a, b \in \mathbb{R}_+^n \), with \( a + 2b < 1 \), such that

\[
H(T_1(x), T_2(y)) \leq a d(x, y) + b [D(x, T_1(x)) + D(y, T_2(y))],
\]

for each \( x, y \in X \).

Then \( F_{T_1} = F_{T_2} \in P(X) \) and \( \{T_1, T_2\} \) is a weakly Picard pair of multivalued operators.

If in addition we have \( a + 3b + ab < 1 \), then \( T_1 \) and \( T_2 \) are weakly Picard multivalued operators.

Proof. From a result given by Popa in [10] it follows that \( F_{T_1} = F_{T_2} \in P(X) \) and that \( \{T_1, T_2\} \) is a weakly Picard pair of multivalued operators. It is easy to verify that \( F_{T_1} \) and \( F_{T_2} \) are closed sets.
Further on we suppose that \(a + 3b + ab < 1\) and we shall prove that \(T_1\) and \(T_2\) are weakly Picard multivalued operators.

Let \(i, j \in \{1, 2\}\), \(i \neq j\) and \(q \in \mathbb{R}\) such that \(1 < q < \frac{\sqrt{(a + 9b)(a + b)} - (a + 3b)}{2ab}\). Let \(x_0 \in X\) and \(x_1 \in T_i(x_0)\).

There exits \(y_1 \in T_j(x_1)\) such that

\[
d(x_1, y_1) \leq qH(T_i(x_0), T_j(x_1)) \leq \\
q[a d(x_0, x_1) + b D(x_0, T_i(x_0)) + b D(x_1, T_j(x_1))] \leq \\
q[a d(x_0, x_1) + b d(x_0, x_1) + b d(x_1, y_1)]
\]

and so

\[
d(x_1, y_1) \leq q(a + b)/(1 - qb) \leq d(x_0, x_1).
\]

Also there exits \(x_2 \in T_i(x_1)\) such that

\[
d(y_1, x_2) \leq qH(T_j(x_1), T_i(x_1)) \leq \\
q[a d(x_1, x_1) + b D(x_1, T_j(x_1)) + b D(x_1, T_i(x_1))] \leq \\
qb [d(x_1, y_1) + d(x_1, x_2)].
\]

Using the triangle inequality and taking into account the above inequalities we obtain

\[
d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, x_2) \leq \\
d(x_1, y_1) + qb [d(x_1, y_1) + d(x_1, x_2)] = \\
(1 + qb) d(x_1, y_1) + qb d(x_1, x_2) \leq \\
q(a + b)(1 + qb)/(1 - qb) d(x_0, x_1) + q b d(x_1, x_2)
\]

and so

\[
d(x_1, x_2) \leq q(a + b)(1 + qb)/(1 - qb)^2 d(x_0, x_1).
\]

Now, there exists \(y_2 \in T_j(x_2)\) such that

\[
d(x_2, y_2) \leq qH(T_i(x_1), T_j(x_2)) \leq \\
q[a d(x_1, x_2) + b D(x_1, T_i(x_1)) + b D(x_2, T_j(x_2))] \leq \\
q[a d(x_1, x_2) + b d(x_1, x_2) + b d(x_2, y_2)]
\]

and so

\[
d(x_2, y_2) \leq q(a + b)/(1 - qb) d(x_1, x_2).
\]

Also there exits \(x_3 \in T_i(x_2)\) such that

\[
d(y_2, x_3) \leq qH(T_j(x_2), T_i(x_2)) \leq \\
q[a d(x_2, x_2) + b D(x_2, T_j(x_2)) + b D(x_2, T_i(x_2))] \leq \\
qb [d(x_2, y_2) + d(x_2, x_3)]
\]

Using again the triangle inequality and taking into account the above two inequalities we get

\[
d(x_2, x_3) \leq d(x_2, y_2) + d(y_2, x_3) \leq d(x_2, y_2) + qb [d(x_2, y_2) + d(x_2, x_3)] = \\
= (1 + qb) d(x_2, y_2) + qb d(x_2, x_3) \leq \\
q(a + b)(1 + qb)/(1 - qb) d(x_1, x_2) + q b d(x_2, x_3)
\]
and so

\[ d(x_2, x_3) \leq q(a + b)(1 + qb)/(1 - qb)^2 d(x_1, x_2). \]

By induction we obtain that there exists a sequence \( (x_n)_{n \in \mathbb{N}} \) of successive approximations of \( T_i \), starting from \( (x_0, x_1) \), with the property that

\[ d(x_n, x_{n+1}) \leq q(a + b)(1 + qb)/(1 - qb)^2 d(x_{n-1}, x_n), \]

for each \( n \in \mathbb{N}^* \).

It follows that \( (x_n)_{n \in \mathbb{N}} \) is a convergent sequence, because \((X, d)\) is a complete metric space and \( q(a + b)(1 + qb)/(1 - qb)^2 < 1 \). Let \( x^* = \lim_{n \to \infty} x_n \). We shall prove that

\[ \text{there exist } u \in \{T_i(x_n) \} \text{ for each } n \in \mathbb{N}. \]

Further on we suppose that \( x^* \in T_j(x^*) \). Taking into account the fact that \( T_j(x^*) \) is a closed set, we are able to write that \( x^* \in T_j(x^*) \), which means that \( x^* \in F_{T_j} \). But \( F_{T_1} = F_{T_2} \) and therefore \( x^* \in F_{T_1} \).

\[ \square \]

**Remark 2.1.** If we take \( a = 0 \) in Theorem 2.1, then we obtain a result presented in Theorem 2.2 from [18].

**Theorem 2.2.** Let \((X, d)\) be a complete metric space and \( T_1, T_2 : X \to P_d(X) \) two multivalued operators. We suppose that:

(i) there exist \( a_1, b_1 \in \mathbb{R}_+ \), with \( a_1 + 2b_1 < 1 \), such that for each \( x \in X \), any \( u_x \in T_1(x) \) and for all \( y \in X \), there exists \( u_y \in T_2(y) \) so that

\[ d(u_x, u_y) \leq a_1 d(x, y) + b_1 [d(x, u_x) + d(y, u_y)]; \]

(ii) there exist \( a_2, b_2 \in \mathbb{R}_+ \), with \( a_2 + 2b_2 < 1 \), such that for each \( x \in X \), any \( u_x \in T_2(x) \) and for all \( y \in X \), there exists \( u_y \in T_1(y) \) so that

\[ d(u_x, u_y) \leq a_2 d(x, y) + b_2 [d(x, u_x) + d(y, u_y)]. \]

Then \( F_{T_1} = F_{T_2} \subset P_d(X) \) and \( \{T_1, T_2\} \) is a weakly Picard pair of multivalued operators.

If in addition we have that \( b_1 + b_2 + \max \{a_1 + b_1 + a_1 b_2, a_2 + b_2 + a_2 b_1\} < 1 \), then \( T_1 \) and \( T_2 \) are weakly Picard multivalued operators.

**Proof.** From Theorem 2.2 in [17] it follows that \( F_{T_1} = F_{T_2} \subset P_d(X) \) and that \( \{T_1, T_2\} \) is a weakly Picard pair of multivalued operators.

Further on we suppose that \( b_1 + b_2 + \max \{a_1 + b_1 + a_1 b_2, a_2 + b_2 + a_2 b_1\} < 1 \). We shall prove that \( T_1 \) and \( T_2 \) are weakly Picard multivalued operators.

Let \( i, j \in \{1, 2\}, i \neq j \). Let \( x_0 \in X \) and \( x_1 \in T_i(x_0) \).

It follows that there exists \( y_1 \in T_j(x_1) \) such that

\[ d(x_1, y_1) \leq a_i d(x_0, x_1) + b_i [d(x_0, x_1) + d(x_1, y_1)], \]
and so
\[ d(x_1, y_1) \leq (a_i + b_i)/(1 - b_i) \, d(x_0, x_1). \]

Also there exists \( x_2 \in T_i(x_1) \) such that
\[ d(y_1, x_2) \leq a_j \, d(x_1, x_1) + b_j \, [d(x_1, y_1) + d(x_1, x_2)] = b_j \, [d(x_1, y_1) + d(x_1, x_2)]. \]

Using the triangle inequality we obtain
\[
  d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, x_2) \leq d(x_1, y_1) + b_j \, [d(x_1, y_1) + d(x_1, x_2)] = \\
  = (1 + b_j) \, d(x_1, y_1) + b_j \, d(x_1, x_2) \leq \\
  \leq (1 + b_j)(a_i + b_i)/(1 - b_i) \, d(x_0, x_1) + b_j \, d(x_1, x_2)
\]

and therefore
\[
  d(x_1, x_2) \leq (1 + b_j)(a_i + b_i)/(1 - b_i) \, d(x_0, x_1).
\]

Now, there exists \( y_2 \in T_j(x_2) \) such that
\[
  d(x_2, y_2) \leq a_i \, d(x_1, x_2) + b_i \, [d(x_1, x_2) + d(x_2, y_2)]
\]

and so
\[
  d(x_2, y_2) \leq (a_i + b_i)/(1 - b_i) \, d(x_1, x_2).
\]

Also there exists \( x_3 \in T_i(x_2) \) such that
\[
  d(y_2, x_3) \leq a_j \, d(x_2, x_2) + b_j \, [d(x_2, y_2) + d(x_2, x_3)] = \\
  = b_j \, [d(x_2, y_2) + d(x_2, x_3)].
\]

Using again the triangle inequality and taking into account the above two inequalities we get
\[
  d(x_2, x_3) \leq d(x_2, y_2) + d(y_2, x_3) \leq d(x_2, y_2) + b_j \, [d(x_2, y_2) + d(x_2, x_3)] = \\
  = (1 + b_j) \, d(x_2, y_2) + b_j \, d(x_2, x_3) \leq \\
  \leq (1 + b_j)(a_i + b_i)/(1 - b_i) \, d(x_1, x_2) + b_j \, d(x_2, x_3)
\]

and therefore
\[
  d(x_2, x_3) \leq (1 + b_j)(a_i + b_i)/(1 - b_i) \, d(x_1, x_2).
\]

By induction we obtain that there exists a sequence \((x_n) \in \mathbb{N}\) of successive approximations of \( T_i \), starting from \((x_0, x_1)\), with the property that
\[
  d(x_n, x_{n+1}) \leq (1 + b_j)(a_i + b_i)/(1 - b_i) \, d(x_{n-1}, x_n),
\]

for each \( n \in \mathbb{N} \).

It follows that \((x_n) \in \mathbb{N}\) is a convergent sequence, because \((X, d)\) is a complete metric space and \((1 + b_j)(a_i + b_i)/(1 - b_i) < 1\). Let \( x^* = \lim_{n \to \infty} x_n \).

From \( x_n \in T_i(x_{n-1}) \) we have that there exists \( u_n \in T_j(x^*) \) such that
\[
  d(x_n, u_n) \leq a_i \, d(x_{n-1}, x^*) + b_i \, [d(x_{n-1}, x_n) + d(x^*, u_n)],
\]

for all \( n \in \mathbb{N} \).

Using the triangle inequality we obtain
\[
  d(x^*, u_n) \leq d(x^*, x_n) + d(x_n, u_n) \leq \\
  \leq d(x^*, x_n) + a_i \, d(x_{n-1}, x^*) + b_i \, [d(x_{n-1}, x_n) + d(x^*, u_n)]
\]

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Remark 2.3. If we take $d(x^*, u_n) = 0$, this implies that $d(x^*, u_n) \rightarrow 0$, as $n \rightarrow \infty$. Since $u_n \in T_j(x^*)$, for all $n \in \mathbb{N}^*$ and $T_j(x^*)$ is a closed set, it follows that $x^* \in T_j(x^*)$. Therefore $x^* \in F_{T_j} = F_{T_i}$. □

Remark 2.2. It is not difficult to verify that the sequence $(x_n)_{n \in \mathbb{N}}$ from the proof of Theorem 2.2 has the property that

$$d(x_n, x^*) \leq \left(1 + b_j \right) \left(1 - b_j \right) \left[a_i \left(1 - b_j \right) + b_j \right] d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

Remark 2.3. If we take $a = 0$ in Theorem 2.2, then we obtain a result presented in Theorem 2 from [19].

3. Strict fixed point and common strict fixed point

There are many strict fixed point and common strict fixed point theorems for multivalued operators which satisfy metric conditions in which functional $\delta$ appears (see, for example, Reich [11], Ćirić [2], [3], Rus [12], Avram [1], Fisher [5], Khan-Khan-Kubiaczyk [6], Dien [4], Kubiaczyk [8], Khan-Kubiaczyk [7]).

The following result gives an answer to Problem 1.1 for two multivalued operators which satisfy a metric condition in which functional $\delta$ appears.

Theorem 3.1. Let $(X, d)$ be a complete metric space and $T_1, T_2 : X \rightarrow P_b(X)$ two multivalued operators for which there exist $a, b \in \mathbb{R}_+$, with $a + 2b < 1$, such that

$$\delta(T_1(x), T_2(y)) \leq a d(x, y) + b \left[\delta(x, T_1(x)) + \delta(y, T_2(y))\right],$$

for each $x, y \in X$.

Then $F_{T_1} = F_{T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}$ and, for each $i, j \in \{1, 2\},$ with $i \neq j$, any sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for the pair $(T_i, T_j)$ converges to $x^*$ and

$$d(x_n, x^*) \leq \left(\frac{a + b}{1 - b}\right)^n \frac{1 - b}{1 - (a + 2b)} \delta(x_0, T_i(x_0)),$$

for every $n \in \mathbb{N}$.

Also, for each $i \in \{1, 2\}$, any sequence $(y_n)_{n \in \mathbb{N}}$ of successive approximations of $T_i$ converges to $x^*$ and

$$d(y_n, x^*) \leq \left(\frac{a + b}{1 - b}\right)^{n-1} \left[a d(y_0, x^*) + b \delta(y_0, T_i(y_0))\right],$$

for every $n \in \mathbb{N}^*$.

Proof. The fact that $T_1$ and $T_2$ have a unique common fixed point, which is a strict fixed point both of $T_1$ and of $T_2$, is a known result. In order to prove some other parts of the conclusion we shall take again the proof.

Let $i, j \in \{1, 2\}, i \neq j$. Let $x_0 \in X$, $x_{2n-1} \in T_i(x_{2n-2})$ and $x_{2n} \in T_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$.
We have
\[
\delta(T_i(x_0), T_j(x_1)) \leq a \delta(x_0, x_1) + b [\delta(x_0, T_i(x_0)) + \delta(x_1, T_j(x_1))] \leq a \delta(x_0, T_i(x_0)) + b [\delta(x_0, T_i(x_0)) + \delta(T_i(x_0), T_j(x_1))] = (a + b) \delta(x_0, T_i(x_0)) + b \delta(T_i(x_0), T_j(x_1))
\]
and so
\[
d(x_1, x_2) \leq \delta(T_i(x_0), T_j(x_1)) \leq (a + b)/(1 - b) \delta(x_0, T_i(x_0)).
\]

For each \( n \in \mathbb{N}^* \) we have
\[
d(T_j(x_{2n-1}), T_j(x_{2n})) \leq a d(x_{2n-1}, x_{2n}) + b [\delta(x_{2n-1}, T_j(x_{2n-1}))+\delta(x_{2n}, T_j(x_{2n}))] \leq a \delta(T_j(x_{2n-2}), T_j(x_{2n-1}))+b [\delta(T_j(x_{2n-2}), T_j(x_{2n-1}))+\delta(T_j(x_{2n-1}), T_j(x_{2n}))] = (a + b) \delta(T_j(x_{2n-1}), T_j(x_{2n-1}))+b \delta(T_j(x_{2n-1}), T_j(x_{2n}))
\]
and from here we get that
\[
d(x_{2n}, x_{2n+1}) \leq \delta(T_j(x_{2n-1}), T_j(x_{2n})) \leq (a + b)/(1 - b) \delta(T_j(x_{2n-2}), T_j(x_{2n-1})).
\]

Also, for each \( n \in \mathbb{N}^* \) we have
\[
d(T_i(x_{2n}), T_i(x_{2n+1})) \leq a d(x_{2n}, x_{2n+1}) + b [\delta(x_{2n}, T_i(x_{2n}))+\delta(x_{2n+1}, T_i(x_{2n+1}))] \leq a \delta(T_i(x_{2n-1}), T_i(x_{2n}))+b [\delta(T_i(x_{2n-1}), T_i(x_{2n}))+\delta(T_i(x_{2n}), T_i(x_{2n+1}))] = (a + b) \delta(T_i(x_{2n-1}), T_i(x_{2n}))+b \delta(T_i(x_{2n}), T_i(x_{2n+1}))
\]
and so
\[
d(x_{2n+1}, x_{2n+2}) \leq \delta(T_i(x_{2n}), T_i(x_{2n+1})) \leq (a + b)/(1 - b) \delta(T_i(x_{2n-1}), T_i(x_{2n})).
\]

Now, we are able to write that
\[
d(x_n, x_{n+1}) \leq [(a + b)/(1 - b)]^n \delta(x_0, T_i(x_0)),
\]
for each \( n \in \mathbb{N} \).

Let \( p \in \mathbb{N}^* \). Using the triangle inequality we obtain
\[
d(x_n, x_{n+p}) \leq [(a + b)/(1 - b)]^n (1 - b)/(1 - (a + 2b)) \delta(x_0, T_i(x_0)),
\]
for each \( n \in \mathbb{N} \). It follows that \( (x_n)_{n \in \mathbb{N}} \) is a Cauchy sequence and so a convergent sequence, because \((X, d)\) is a complete metric space and \((a + b)/(1 - b) < 1\). Let \( x^* = \lim_{n \to \infty} x_n \).

Letting \( p \) tend to infinity in the above inequality we get that
\[
d(x_n, x^*) \leq [(a + b)/(1 - b)]^n (1 - b)/(1 - (a + 2b)) \delta(x_0, T_i(x_0)),
\]
for every \( n \in \mathbb{N} \).

We have
\[
\delta(x^*, T_i(x^*)) \leq d(x^*, x_{2n+2}) + \delta(x_{2n+2}, T_i(x^*)) \leq \delta(x^*, x_{2n+2}) + \delta(T_i(x_{2n+1}), T_i(x^*)) \leq d(x^*, x_{2n+2}) + ad(x_{2n+1}, x^*) + b [\delta(x_{2n+1}, T_i(x_{2n+1}))+\delta(x^*, T_i(x^*))] \leq d(x^*, x_{2n+2}) + ad(x_{2n+1}, x^*) + b [\delta(T_i(x_{2n}), T_i(x_{2n+1}))+\delta(x^*, T_i(x^*))] \leq d(x^*, x_{2n+2}) + ad(x_{2n+1}, x^*) +
\]
\[+ \quad b \{[(a + b)/(1 - b)]^{2n+1} \delta(x_0, T_i(x_0)) + \delta(x^*, T_i(x^*))\},\]
for all \( n \in \mathbb{N} \).

From this we get that

\[
\delta(x^*, T_i(x^*)) \leq (1 - b)^{-1} \{ d(x^*, x_{2n+2}) + a d(x_{2n+1}, x^*) + b[(a + b)/(1 - b)]^{2n+1} \delta(x, T_i(x_0)) \},
\]

for each \( n \in \mathbb{N} \).

Letting \( n \) tend to infinity it follows that \( \delta(x^*, T_i(x^*)) = 0 \), so \( T_i(x^*) = \{ x^* \} \).

It is easy to verify that \((CF)_{T_1, T_2} = (SF)_{T_1} = (SF)_{T_2} = \{ x^* \} \).

In order to prove that \( F_{T_i} = \{ x^* \} \), let \( x \in F_{T_i} \). Then we have

\[
d(x,x^*) \leq \delta(T_i(x), T_j(x^*)) \leq a d(x, x^*) + b [\delta(x, T_i(x)) + \delta(x^*, T_j(x^*))] =
\]

\[
= a d(x, x^*) + b \delta(x, T_i(x))
\]

and therefore

\[
d(x,x^*) \leq b/(1 - a) \delta(x, T_i(x)).
\]

We also have

\[
\delta(x, T_i(x)) \leq \delta(T_i(x), T_j(x^*)) \leq \delta(T_i(x), T_j(x^*)) + \delta(T_j(x^*), T_i(x)) =
\]

\[
= 2 \delta(T_i(x), T_j(x^*)) \leq 2 \{ a d(x, x^*) + b \delta(x, T_i(x)) \} \leq
\]

\[
\leq 2(1 - b)(1 - a)\delta(x, T_i(x)) + b\delta(x, T_i(x)) = 2b/(1 - a)\delta(x, T_i(x)).
\]

From this we get that \( \delta(x, T_i(x)) = 0 \), so \( T_i(x) = \{ x \} \), i.e. \( x \in (SF)_{T_i} \).

Let \( y_0 \in X \) and \( y_{n+1} \in T_i(y_n) \), for each \( n \in \mathbb{N} \). We have

\[
d(y_1, x^*) \leq \delta(T_i(y_0), T_j(x^*)) \leq
\]

\[
\leq a d(y_0, x^*) + b [\delta(y_0, T_i(y_0)) + \delta(x^*, T_j(x^*))] =
\]

\[
= a d(y_0, x^*) + b \delta(y_0, T_i(y_0)).
\]

Taking into account the above inequality we are able to write

\[
\delta(T_i(y_0), T_j(y_1)) \leq \delta(T_i(y_0), T_j(x^*)) + \delta(T_j(x^*), T_i(y_1)) \leq
\]

\[
\leq a d(y_0, x^*) + b [\delta(y_0, T_i(y_0)) + \delta(x^*, T_j(x^*))] +
\]

\[
+ a d(x^*, y_1) + b \delta(x^*, T_j(x^*)) + \delta(y_1, T_i(y_1))] =
\]

\[
= a d(y_0, x^*) + b \delta(y_0, T_i(y_1)) + a d(x^*, y_1) + b \delta(y_1, T_i(y_1)) \leq
\]

\[
\leq a d(y_0, x^*) + b \delta(y_0, T_i(y_0)) + a d(x^*, y_1) + b \delta(T_i(y_0), T_i(y_1)) \leq
\]

\[
\leq (1 + a) \{ a d(y_0, x^*) + b \delta(y_0, T_i(y_0)) \} + b \delta(T_i(y_0), T_i(y_1))
\]

and from here we get

\[
\delta(T_i(y_0), T_i(y_1)) \leq 1 + a \frac{1}{1 - b} \{ a d(y_0, x^*) + b \delta(y_0, T_i(y_0)) \}.
\]

Now we have

\[
d(y_2, x^*) \leq \delta(T_i(y_1), T_j(x^*)) \leq a d(y_1, x^*) + b [\delta(y_1, T_i(y_1)) + \delta(x^*, T_j(x^*))] =
\]

\[
= a d(y_1, x^*) + b \delta(y_1, T_i(y_1)) \leq a d(y_1, x^*) + b \delta(T_i(y_0), T_i(y_1)) \leq
\]

\[
\leq \frac{a + b}{1 - b} \{ a d(y_0, x^*) + b \delta(y_0, T_i(y_0)) \}.
\]
Using this result we obtain
\[ \delta(T_i(y_1), T_i(y_2)) \leq \delta(T_i(y_1), T_j(x^*)) + \delta(T_j(x^*), T_i(y_2)) \leq \]
\[ \leq a \ d(y_1, x^*) + b \left[ \delta(y_1, T_i(y_1)) + \delta(x^*, T_j(x^*)) \right] + \]
\[ + a \ d(x^*, y_2) + b \left[ \delta(x^*, T_j(x^*)) + \delta(y_2, T_i(y_2)) \right] = \]
\[ = a \ d(y_1, x^*) + b \delta(y_1, T_i(y_1)) + a \ d(x^*, y_2) + b \delta(y_2, T_i(y_2)) \leq \]
\[ \leq a \ d(y_1, x^*) + b \delta(T_i(y_0), T_i(y_1)) + a \ d(x^*, y_2) + b \delta(T_i(y_1), T_i(y_2)) \leq \]
\[ \leq \frac{a + b}{1 - b} \ (1 + a) \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))] + b \delta(T_i(y_1), T_i(y_2)) \]
and so
\[ \delta(T_i(y_1), T_i(y_2)) \leq \frac{a + b}{1 - b} \cdot \frac{1 + a}{1 - b} \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))]. \]

We have
\[ d(y_3, x^*) \leq \delta(T_i(y_2), T_i(y_3)) \leq a \ d(y_2, x^*) + b \left[ \delta(y_2, T_i(y_2)) + \delta(x^*, T_j(x^*)) \right] = \]
\[ = a \ d(y_2, x^*) + b \delta(y_2, T_i(y_2)) \leq a \ d(y_2, x^*) + b \delta(T_i(y_1), T_i(y_2)) \leq \]
\[ \leq \left( \frac{a + b}{1 - b} \right)^2 \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))]. \]

Using this result we obtain
\[ \delta(T_i(y_2), T_i(y_3)) \leq \delta(T_i(y_2), T_j(x^*)) + \delta(T_j(x^*), T_i(y_3)) \leq \]
\[ \leq a \ d(y_2, x^*) + b \left[ \delta(y_2, T_i(y_2)) + \delta(x^*, T_j(x^*)) \right] + \]
\[ + a \ d(x^*, y_3) + b \left[ \delta(x^*, T_j(x^*)) + \delta(y_3, T_i(y_1)) \right] = \]
\[ = a \ d(y_2, x^*) + b \delta(y_2, T_i(y_2)) + a \ d(x^*, y_3) + b \delta(y_3, T_i(y_3)) \leq \]
\[ \leq a \ d(y_2, x^*) + b \delta(T_i(y_1), T_i(y_2)) + a \ d(x^*, y_3) + b \delta(T_i(y_2), T_i(y_3)) \leq \]
\[ \leq \left( \frac{a + b}{1 - b} \right)^2 \ (1 + a) \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))] + b \delta(T_i(y_2), T_i(y_3)). \]

which implies
\[ \delta(T_i(y_2), T_i(y_3)) \leq \left( \frac{a + b}{1 - b} \right)^2 \frac{1 + a}{1 - b} \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))]. \]

By induction can be proved that the sequence \((y_n)_{n \in \mathbb{N}}\) has the following properties:
\[ d(y_n, x^*) \leq \left( \frac{a + b}{1 - b} \right)^{n-1} \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))] \]
and
\[ \delta(T_i(y_{n-1}), T_i(y_n)) \leq \left( \frac{a + b}{1 - b} \right)^{n-1} \frac{1 + a}{1 - b} \ [a \ d(y_0, x^*) + b \delta(y_0, T_i(y_0))], \]
for each \(n \in \mathbb{N}^*\).

It follows that \((y_n)_{n \in \mathbb{N}}\) is a convergent sequence and its limit is \(x^*\). \(\square\)
Corollary 3.1. Let \((X, d)\) be a complete metric space and \(T_1, T_2 : X \to P_b(X)\) two multivalued operators for which there exist \(a, b \in \mathbb{R}_+\), with \(a + 2b < 1\), such that
\[
\delta(T_1(x), T_2(y)) \leq a d(x, y) + b[\delta(x, T_1(x)) + \delta(y, T_2(y))],
\]
for each \(x, y \in X\).
Then \(F_{T_1} = F_{T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}\) and
\[
d(x_0, x^*) \leq (1 + b)/(1 - a) \min \{\delta(x_0, T_1(x_0)), \delta(x_0, T_2(x_0))\},
\]
for each \(x_0 \in X\).

Proof. From Theorem 3.1 we have that \(F_{T_1} = F_{T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}\).
Let \(i \in \{1, 2\}\). Let \(x_0 \in X\) and \(x_1 \in T_i(x_0)\). We have
\[
d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq \delta(x_0, T_i(x_0)) + a d(x_0, x^*) + b \delta(x_0, T_i(x_0)) = \]
\[
a d(x_0, x^*) + (1 + b) \delta(x_0, T_i(x_0))
\]
and so
\[
d(x_0, x^*) \leq (1 + b)/(1 - a) \delta(x_0, T_i(x_0)).
\]

\(\square\)

References

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