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Some pairs of multivalued operators

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ABSTRACT. We study the following problem.

Let (X,d) be a metric space and $T_1,T_2 : X \to P(X)$ two multivalued operators. Determine metric conditions on the pair of multivalued operators T_1 and T_2, which imply that, for each $x \in X$, there exists a sequence of successive approximations for the pair (T_1,T_2) or for the pair (T_2,T_1), starting from x, which converges to a common fixed point or to a common strict fixed point of T_1 and T_2 and, for each $x \in X$, there exists a sequence of successive approximations of T_i, starting from x, which converges to a fixed point or to a strict fixed point of T_i, for each $i \in \{1,2\}$.

1. INTRODUCTION

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X, i. e. $P(X) := \{ Y \mid \emptyset \neq Y \subseteq X \}$. Let $T_1,T_2 : X \to P(X)$ be two multivalued operators. We denote by F_{T_1} the fixed points set of T_1, i. e. $F_{T_1} := \{ x \in X \mid x \in T_1(x) \}$, by $(SF)_{T_1}$ the strict fixed points set of T_1, i. e. $(SF)_{T_1} := \{ x \in X \mid T_1(x) = \{x\} \}$ and by $(CF)_{T_1,T_2}$ the common fixed points set of T_1 and T_2, i. e. $(CF)_{T_1,T_2} := \{ x \in X \mid x \in T_1(x) \cap T_2(x) \}$.
A sequence $(x_n)_{n\in\mathbb{N}}$ is called *sequence of successive approximations of* T_1 if $x_0 \in X$ and $x_{n+1} \in T_1(x_n)$, for each $n \in \mathbb{N}$.
A sequence $(x_n)_{n\in\mathbb{N}}$ is called *sequence of successive approximations for the pair* (T_1,T_2) if $x_0 \in X$, $x_{2n+1} \in T_1(x_{2n})$ and $x_{2n+2} \in T_2(x_{2n+1})$, for each $n \in \mathbb{N}$.

Definition 1.1. Let (X,d) be a metric space and $T : X \to P(X)$ a multivalued operator. We say that T is a *weakly Picard multivalued operator* iff for each $x \in X$ and for every $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}^*$;
(iii) sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

For examples of weakly Picard multivalued operators see for instance [15], [16].

Definition 1.2. Let (X,d) be a metric space and $T_1,T_2 : X \to P(X)$ two multivalued operators. We say that $\{T_1,T_2\}$ is a *weakly Picard pair of multivalued operators* iff for each $x \in X$ and for every $y \in T_1(x) \cup T_2(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;

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(ii) $x_{2n+1} \in T_i(x_{2n})$ and $x_{2n+2} \in T_j(x_{2n+1})$, for each $n \in \mathbb{N}$, where $i, j \in \{1, 2\}$, $i \neq j$;

(iii) sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a common fixed point of T_1 and T_2.

For examples of weakly Picard pairs of multivalued operators see [18], [19], [20].

Let (X, d) be a metric space.

We denote by $P_{cl}(X)$ the set of all nonempty and closed subsets of X, i. e. $P_{cl}(X) := \{Y \mid Y \in P(X), Y \text{is a closed set}\}$ and by $P_b(X)$ the set of all nonempty and bounded subsets of X, i. e. $P_b(X) := \{Y \mid Y \in P(X), Y \text{is a bounded set}\}$.

We also recall the functional $D : P(X) \times P(X) \to \mathbb{R}_+$, defined by $D(A, B) = \inf \{\, d(a, b) \mid a \in A,\ b \in B \,\}$, for each $A, B \in P(X)$, and the generalized functionals $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by $\delta(A, B) = \sup\{\, d(a, b) \mid\ a \in A, b \in B\}$, for each $A, B \in P(X)$, and $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by $H(A, B) = \max \{\sup_{a\in A} D(a, B),\ \sup_{b\in B} D(b, A)\}$, for each $A, B \in P(X)$.

The following property of the generalized functional H is well-known.

Lemma 1.1. *Let (X, d) be a metric space, $A, B \in P(X)$ and $q \in \mathbb{R}$, $q > 1$.*

Then for every $a \in A$, there exists $b \in B$ such that $d(a, b) \leq q\, H(A, B)$.

The purpose of this paper is to study the following problem.

Problem 1.1. *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. Determine metric conditions on the pair of multivalued operators T_1 and T_2, which imply that, for each $x \in X$, there exists a sequence of successive approximations for the pair (T_1, T_2) or for the pair (T_2, T_1), starting from x, which converges to a common fixed point or to a common strict fixed point of T_1 and T_2 and, for each $x \in X$, there exists a sequence of successive approximations of T_i, starting from x, which converges to a fixed point or to a strict fixed point of T_i, for each $i \in \{1, 2\}$.*

For singlevalued operators results of this type are given by Rus [13] and Dien [4] and for multivalued operators results which answer to Problem 1.1 are presented by Dien [4] and Sîntămărian [18], [19], [20].

2. Fixed points and common fixed points

Theorem 2.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued operators for which there exist $a, b \in \mathbb{R}_+^*$, with $a + 2b < 1$, such that*

$$H(T_1(x), T_2(y)) \leq a\, d(x, y) + b\, [D(x, T_1(x)) + D(y, T_2(y))],$$

for each $x, y \in X$.

Then $F_{T_1} = F_{T_2} \in P_{cl}(X)$ and $\{T_1, T_2\}$ is a weakly Picard pair of multivalued operators.

If in addition we have $a + 3b + ab < 1$, then T_1 and T_2 are weakly Picard multivalued operators.

Proof. From a result given by Popa in [10] it follows that $F_{T_1} = F_{T_2} \in P(X)$ and that $\{T_1, T_2\}$ is a weakly Picard pair of multivalued operators. It is easy to verify that F_{T_1} and F_{T_2} are closed sets.

Further on we suppose that $a+3b+ab<1$ and we shall prove that T_1 and T_2 are weakly Picard multivalued operators.

Let $i,j\in\{1,2\}$, $i\neq j$ and $q\in\mathbb{R}$ such that $1<q<[\sqrt{(a+9b)(a+b)}-(a+3b)]/(2ab)$. Let $x_0\in X$ and $x_1\in T_i(x_0)$.

There exits $y_1\in T_j(x_1)$ such that

$$\begin{aligned}d(x_1,y_1)&\le q\,H(T_i(x_0),T_j(x_1))\le\\&\le q\,[a\,d(x_0,x_1)+b\,D(x_0,T_i(x_0))+b\,D(x_1,T_j(x_1))]\le\\&\le q\,[a\,d(x_0,x_1)+b\,d(x_0,x_1)+b\,d(x_1,y_1)]\end{aligned}$$

and so

$$d(x_1,y_1)\le q(a+b)/(1-qb)\,d(x_0,x_1).$$

Also there exits $x_2\in T_i(x_1)$ such that

$$\begin{aligned}d(y_1,x_2)&\le q\,H(T_j(x_1),T_i(x_1))\le\\&\le q\,[a\,d(x_1,x_1)+b\,D(x_1,T_j(x_1))+b\,D(x_1,T_i(x_1))]\le\\&\le qb\,[d(x_1,y_1)+d(x_1,x_2)].\end{aligned}$$

Using the triangle inequality and taking into account the above inequalities we obtain

$$\begin{aligned}d(x_1,x_2)&\le d(x_1,y_1)+d(y_1,x_2)\le\\&\le d(x_1,y_1)+qb\,[d(x_1,y_1)+d(x_1,x_2)]=\\&=(1+qb)\,d(x_1,y_1)+qb\,d(x_1,x_2)\le\\&\le q(a+b)(1+qb)/(1-qb)\,d(x_0,x_1)+qb\,d(x_1,x_2)\end{aligned}$$

and so

$$d(x_1,x_2)\le q(a+b)(1+qb)/(1-qb)^2\,d(x_0,x_1).$$

Now, there exists $y_2\in T_j(x_2)$ such that

$$\begin{aligned}d(x_2,y_2)&\le q\,H(T_i(x_1),T_j(x_2))\le\\&\le q\,[a\,d(x_1,x_2)+b\,D(x_1,T_i(x_1))+b\,D(x_2,T_j(x_2))]\le\\&\le q\,[a\,d(x_1,x_2)+b\,d(x_1,x_2)+b\,d(x_2,y_2)]\end{aligned}$$

and so

$$d(x_2,y_2)\le q(a+b)/(1-qb)\,d(x_1,x_2).$$

Also there exits $x_3\in T_i(x_2)$ such that

$$\begin{aligned}d(y_2,x_3)&\le q\,H(T_j(x_2),T_i(x_2))\le\\&\le q\,[a\,d(x_2,x_2)+b\,D(x_2,T_j(x_2))+b\,D(x_2,T_i(x_2))]\le\\&\le qb\,[d(x_2,y_2)+d(x_2,x_3)].\end{aligned}$$

Using again the triangle inequality and taking into account the above two inequalities we get

$$\begin{aligned}d(x_2,x_3)&\le d(x_2,y_2)+d(y_2,x_3)\le d(x_2,y_2)+qb\,[d(x_2,y_2)+d(x_2,x_3)]=\\&=(1+qb)\,d(x_2,y_2)+qb\,d(x_2,x_3)\le\\&\le q(a+b)(1+qb)/(1-qb)\,d(x_1,x_2)+qb\,d(x_2,x_3)\end{aligned}$$

and so

$$d(x_2, x_3) \leq q(a+b)(1+qb)/(1-qb)^2 \; d(x_1, x_2).$$

By induction we obtain that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T_i, starting from (x_0, x_1), with the property that

$$d(x_n, x_{n+1}) \leq q(a+b)(1+qb)/(1-qb)^2 \; d(x_{n-1}, x_n),$$

for each $n \in \mathbb{N}^*$.

It follows that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X, d) is a complete metric space and $q(a+b)(1+qb)/(1-qb)^2 < 1$. Let $x^* = \lim_{n\to\infty} x_n$.

We have

$$\begin{aligned}
&D(x^*, T_j(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T_j(x^*)) \leq \\
&\leq d(x^*, x_{n+1}) + H(T_i(x_n), T_j(x^*)) \leq \\
&\leq d(x^*, x_{n+1}) + a\, d(x_n, x^*) + b\, [D(x_n, T_i(x_n)) + D(x^*, T_j(x^*))] \leq \\
&\leq d(x^*, x_{n+1}) + a\, d(x_n, x^*) + b\, d(x_n, x_{n+1}) + b\, D(x^*, T_j(x^*)),
\end{aligned}$$

for each $n \in \mathbb{N}$.

From this we get

$$D(x^*, T_j(x^*)) \leq (1-b)^{-1} \, [d(x^*, x_{n+1}) + a\, d(x_n, x^*) + b\, d(x_n, x_{n+1})],$$

for each $n \in \mathbb{N}$, which implies, by letting n to tend to infinity, that $D(x^*, T_j(x^*)) = 0$. Taking into account the fact that $T_j(x^*)$ is a closed set, we are able to write that $x^* \in T_j(x^*)$, which means that $x^* \in F_{T_j}$. But $F_{T_1} = F_{T_2}$ and therefore $x^* \in F_{T_i}$. □

Remark 2.1. *If we take $a = 0$ in Theorem 2.1, then we obtain a result presented in Theorem 2.2 from [18].*

Theorem 2.2. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued operators. We suppose that:*

(i) *there exist $a_1, b_1 \in \mathbb{R}_+$, with $a_1 + 2b_1 < 1$, such that for each $x \in X$, any $u_x \in T_1(x)$ and for all $y \in X$, there exists $u_y \in T_2(y)$ so that*

$$d(u_x, u_y) \leq a_1 \, d(x, y) + b_1 \, [d(x, u_x) + d(y, u_y)];$$

(ii) *there exist $a_2, b_2 \in \mathbb{R}_+$, with $a_2 + 2b_2 < 1$, such that for each $x \in X$, any $u_x \in T_2(x)$ and for all $y \in X$, there exists $u_y \in T_1(y)$ so that*

$$d(u_x, u_y) \leq a_2 \, d(x, y) + b_2 \, [d(x, u_x) + d(y, u_y)].$$

Then $F_{T_1} = F_{T_2} \in P_{cl}(X)$ and $\{T_1, T_2\}$ is a weakly Picard pair of multivalued operators.

If in addition we have that $b_1 + b_2 + \max\{a_1 + b_1 + a_1 b_2, a_2 + b_2 + a_2 b_1\} < 1$, then T_1 and T_2 are weakly Picard multivalued operators.

Proof. From Theorem 2.2 in [17] it follows that $F_{T_1} = F_{T_2} \in P_{cl}(X)$ and that $\{T_1, T_2\}$ is a weakly Picard pair of multivalued operators.

Further on we suppose that $b_1 + b_2 + \max\{a_1 + b_1 + a_1 b_2, a_2 + b_2 + a_2 b_1\} < 1$. We shall prove that T_1 and T_2 are weakly Picard multivalued operators.

Let $i, j \in \{1, 2\}$, $i \neq j$. Let $x_0 \in X$ and $x_1 \in T_i(x_0)$.

It follows that there exits $y_1 \in T_j(x_1)$ such that

$$d(x_1, y_1) \leq a_i \, d(x_0, x_1) + b_i \, [d(x_0, x_1) + d(x_1, y_1)]$$

and so

$$d(x_1, y_1) \leq (a_i + b_i)/(1 - b_i)\ d(x_0, x_1).$$

Also there exits $x_2 \in T_i(x_1)$ such that

$$d(y_1, x_2) \leq a_j\ d(x_1, x_1) + b_j\ [d(x_1, y_1) + d(x_1, x_2)] = b_j\ [d(x_1, y_1) + d(x_1, x_2)].$$

Using the triangle inequality we obtain

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, y_1) + d(y_1, x_2) \leq d(x_1, y_1) + b_j\ [d(x_1, y_1) + d(x_1, x_2)] = \\ &= (1 + b_j)\ d(x_1, y_1) + b_j\ d(x_1, x_2) \leq \\ &\leq (1 + b_j)(a_i + b_i)/(1 - b_i)\ d(x_0, x_1) + b_j\ d(x_1, x_2) \end{aligned}$$

and therefore

$$d(x_1, x_2) \leq (1 + b_j)/(1 - b_j)(a_i + b_i)/(1 - b_i)\ d(x_0, x_1).$$

Now, there exists $y_2 \in T_j(x_2)$ such that

$$d(x_2, y_2) \leq a_i\ d(x_1, x_2) + b_i\ [d(x_1, x_2) + d(x_2, y_2)]$$

and so

$$d(x_2, y_2) \leq (a_i + b_i)/(1 - b_i)\ d(x_1, x_2).$$

Also there exits $x_3 \in T_i(x_2)$ such that

$$\begin{aligned} d(y_2, x_3) &\leq a_j\ d(x_2, x_2) + b_j\ [d(x_2, y_2) + d(x_2, x_3)] = \\ &= b_j\ [d(x_2, y_2) + d(x_2, x_3)]. \end{aligned}$$

Using again the triangle inequality and taking into account the above two inequalities we get

$$\begin{aligned} d(x_2, x_3) &\leq d(x_2, y_2) + d(y_2, x_3) \leq d(x_2, y_2) + b_j\ [d(x_2, y_2) + d(x_2, x_3)] = \\ &= (1 + b_j)\ d(x_2, y_2) + b_j\ d(x_2, x_3) \leq \\ &\leq (1 + b_j)(a_i + b_i)/(1 - b_i)\ d(x_1, x_2) + b_j\ d(x_2, x_3) \end{aligned}$$

and therefore

$$d(x_2, x_3) \leq (1 + b_j)/(1 - b_j)(a_i + b_i)/(1 - b_i)\ d(x_1, x_2).$$

By induction we obtain that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T_i, starting from (x_0, x_1), with the property that

$$d(x_n, x_{n+1}) \leq (1 + b_j)/(1 - b_j)(a_i + b_i)/(1 - b_i)\ d(x_{n-1}, x_n),$$

for each $n \in \mathbb{N}^*$.

It follows that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because (X, d) is a complete metric space and $(1 + b_j)/(1 - b_j)(a_i + b_i)/(1 - b_i) < 1$. Let $x^* = \lim_{n\to\infty} x_n$.

From $x_n \in T_i(x_{n-1})$ we have that there exists $u_n \in T_j(x^*)$ such that

$$d(x_n, u_n) \leq a_i\ d(x_{n-1}, x^*) + b_i\ [d(x_{n-1}, x_n) + d(x^*, u_n)],$$

for all $n \in \mathbb{N}^*$.

Using the triangle inequality we obtain

$$\begin{aligned} d(x^*, u_n) &\leq d(x^*, x_n) + d(x_n, u_n) \leq \\ &\leq d(x^*, x_n) + a_i\ d(x_{n-1}, x^*) + b_i\ [d(x_{n-1}, x_n) + d(x^*, u_n)] \end{aligned}$$

and so

$$d(x^*, u_n) \leq (1 - b_i)^{-1} \, [d(x^*, x_n) + a_i \, d(x_{n-1}, x^*) + b_i \, d(x_{n-1}, x_n)],$$

for all $n \in \mathbb{N}^*$.
This implies that $d(x^*, u_n) \to 0$, as $n \to \infty$. Since $u_n \in T_j(x^*)$, for all $n \in \mathbb{N}^*$ and $T_j(x^*)$ is a closed set, it follows that $x^* \in T_j(x^*)$. Therefore $x^* \in F_{T_j} = F_{T_i}$. □

Remark 2.2. *It is not difficult to verify that the sequence $(x_n)_{n\in\mathbb{N}}$ from the proof of Theorem* 2.2 *has the property that*

$$d(x_n, x^*) \leq \left(\frac{1 + b_j}{1 - b_j} \cdot \frac{a_i + b_i}{1 - b_i} \right)^n \frac{(1 - b_i)(1 - b_j)}{1 - a_i - 2b_i - b_j - a_i b_j} \, d(x_0, x_1),$$

for each $n \in \mathbb{N}$.

Remark 2.3. *If we take $a = 0$ in Theorem* 2.2*, then we obtain a result presented in Theorem* 2 *from* [19].

3. Strict fixed point and common strict fixed point

There are many strict fixed point and common strict fixed point theorems for multivalued operators which satisfy metric conditions in which functional δ appears (see, for example, Reich [11], Ćirić [2], [3], Rus [12], Avram [1], Fisher [5], Khan-Khan-Kubiaczyk [6], Dien [4], Kubiaczyk [8], Khan-Kubiaczyk [7]).

The following result gives an answer to Problem 1.1 for two multivalued operators which satisfy a metric condition in which functional δ appears.

Theorem 3.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_b(X)$ two multivalued operators for which there exist $a, b \in \mathbb{R}_+$, with $a + 2b < 1$, such that*

$$\delta(T_1(x), T_2(y)) \leq a \, d(x, y) + b \, [\delta(x, T_1(x)) + \delta(y, T_2(y))],$$

for each $x, y \in X$.

Then $F_{T_1} = F_{T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}$ and, for each $i, j \in \{1, 2\}$, with $i \neq j$, any sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for the pair (T_i, T_j) converges to x^* and

$$d(x_n, x^*) \leq \left(\frac{a + b}{1 - b} \right)^n \frac{1 - b}{1 - (a + 2b)} \, \delta(x_0, T_i(x_0)),$$

for every $n \in \mathbb{N}$.

Also, for each $i \in \{1, 2\}$, any sequence $(y_n)_{n\in\mathbb{N}}$ of successive approximations of T_i converges to x^* and

$$d(y_n, x^*) \leq \left(\frac{a + b}{1 - b} \right)^{n-1} [a \, d(y_0, x^*) + b \, \delta(y_0, T_i(y_0))],$$

for every $n \in \mathbb{N}^*$.

Proof. The fact that T_1 and T_2 have a unique common fixed point, which is a strict fixed point both of T_1 and of T_2, it is a known result. In order to prove some other parts of the conclusion we shall take again the proof.

Let $i, j \in \{1, 2\}$, $i \neq j$. Let $x_0 \in X$, $x_{2n-1} \in T_i(x_{2n-2})$ and $x_{2n} \in T_j(x_{2n-1})$, for each $n \in \mathbb{N}^*$.

We have

$$\delta(T_i(x_0),T_j(x_1)) \leq a\,d(x_0,x_1) + b\,[\delta(x_0,T_i(x_0)) + \delta(x_1,T_j(x_1))] \leq$$
$$\leq a\,\delta(x_0,T_i(x_0)) + b\,[\delta(x_0,T_i(x_0)) + \delta(T_i(x_0),T_j(x_1))] =$$
$$= (a+b)\,\delta(x_0,T_i(x_0)) + b\,\delta(T_i(x_0),T_j(x_1))$$

and so

$$d(x_1,x_2) \leq \delta(T_i(x_0),T_j(x_1)) \leq (a+b)/(1-b)\,\delta(x_0,T_i(x_0)).$$

For each $n \in \mathbb{N}^*$ we have

$$\delta(T_j(x_{2n-1}),T_i(x_{2n})) \leq a\,d(x_{2n-1},x_{2n}) + b\,[\delta(x_{2n-1},T_j(x_{2n-1})) + \delta(x_{2n},T_i(x_{2n}))] \leq$$
$$\leq a\,\delta(T_i(x_{2n-2}),T_j(x_{2n-1})) + b\,[\delta(T_i(x_{2n-2}),T_j(x_{2n-1})) + \delta(T_j(x_{2n-1}),T_i(x_{2n}))] =$$
$$= (a+b)\,\delta(T_i(x_{2n-2}),T_j(x_{2n-1})) + b\,\delta(T_j(x_{2n-1}),T_i(x_{2n}))$$

and from here we get that

$$d(x_{2n},x_{2n+1}) \leq \delta(T_j(x_{2n-1}),T_i(x_{2n})) \leq (a+b)/(1-b)\,\delta(T_i(x_{2n-2}),T_j(x_{2n-1})).$$

Also, for each $n \in \mathbb{N}^*$ we have

$$\delta(T_i(x_{2n}),T_j(x_{2n+1})) \leq a\,d(x_{2n},x_{2n+1}) + b\,[\delta(x_{2n},T_i(x_{2n})) + \delta(x_{2n+1},T_j(x_{2n+1}))] \leq$$
$$\leq a\,\delta(T_j(x_{2n-1}),T_i(x_{2n})) + b\,[\delta(T_j(x_{2n-1}),T_i(x_{2n})) + \delta(T_i(x_{2n}),T_j(x_{2n+1}))] =$$
$$= (a+b)\,\delta(T_j(x_{2n-1}),T_i(x_{2n})) + b\,\delta(T_i(x_{2n}),T_j(x_{2n+1}))$$

and so

$$d(x_{2n+1},x_{2n+2}) \leq \delta(T_i(x_{2n}),T_j(x_{2n+1})) \leq (a+b)/(1-b)\,\delta(T_j(x_{2n-1}),T_i(x_{2n})).$$

Now, we are able to write that

$$d(x_n,x_{n+1}) \leq [(a+b)/(1-b)]^n\,\delta(x_0,T_i(x_0)),$$

for each $n \in \mathbb{N}$.

Let $p \in \mathbb{N}^*$. Using the triangle inequality we obtain

$$d(x_n,x_{n+p}) \leq [(a+b)/(1-b)]^n(1-b)/[1-(a+2b)]\,\delta(x_0,T_i(x_0)),$$

for each $n \in \mathbb{N}$. It follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and so a convergent sequence, because (X,d) is a complete metric space and $(a+b)/(1-b) < 1$. Let $x^* = \lim_{n\to\infty} x_n$.

Letting p to tend to infinity in the above inequality we get that

$$d(x_n,x^*) \leq [(a+b)/(1-b)]^n(1-b)/[1-(a+2b)]\,\delta(x_0,T_i(x_0)),$$

for every $n \in \mathbb{N}$.

We have

$$\delta(x^*,T_i(x^*)) \leq d(x^*,x_{2n+2}) + \delta(x_{2n+2},T_i(x^*)) \leq$$
$$\leq d(x^*,x_{2n+2}) + \delta(T_j(x_{2n+1}),T_i(x^*)) \leq$$
$$\leq d(x^*,x_{2n+2}) + ad(x_{2n+1},x^*) + b\,[\delta(x_{2n+1},T_j(x_{2n+1})) + \delta(x^*,T_i(x^*))] \leq$$
$$\leq d(x^*,x_{2n+2}) + ad(x_{2n+1},x^*) + b\,[\delta(T_i(x_{2n}),T_j(x_{2n+1})) + \delta(x^*,T_i(x^*))] \leq$$
$$\leq d(x^*,x_{2n+2}) + a\,d(x_{2n+1},x^*) +$$
$$+ b\,\{[(a+b)/(1-b)]^{2n+1}\,\delta(x_0,T_i(x_0)) + \delta(x^*,T_i(x^*))\},$$

for all $n \in \mathbb{N}$.

From this we get that

$$\delta(x^*, T_i(x^*)) \leq (1-b)^{-1} \{d(x^*, x_{2n+2}) + a\, d(x_{2n+1}, x^*) + \\ + b[(a+b)/(1-b)]^{2n+1}\, \delta(x_0, T_i(x_0))\},$$

for each $n \in \mathbb{N}$.

Letting n to tend to infinity it follows that $\delta(x^*, T_i(x^*)) = 0$, so $T_i(x^*) = \{x^*\}$. It is easy to verify that $(CF)_{T_1,T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}$.

In order to prove that $F_{T_i} = \{x^*\}$, let $x \in F_{T_i}$. Then we have

$$d(x, x^*) \leq \delta(T_i(x), T_j(x^*)) \leq a\, d(x, x^*) + b\, [\delta(x, T_i(x)) + \delta(x^*, T_j(x^*))] = \\ = a\, d(x, x^*) + b\, \delta(x, T_i(x))$$

and therefore

$$d(x, x^*) \leq b/(1-a)\, \delta(x, T_i(x)).$$

We also have

$$\delta(x, T_i(x)) \leq \delta(T_i(x), T_i(x)) \leq \delta(T_i(x), T_j(x^*)) + \delta(T_j(x^*), T_i(x)) = \\ = 2\, \delta(T_i(x), T_j(x^*)) \leq 2\, [a\, d(x, x^*) + b\, \delta(x, T_i(x))] \leq \\ \leq 2[ab/(1-a)\delta(x, T_i(x)) + b\delta(x, T_i(x))] = 2b/(1-a)\delta(x, T_i(x)).$$

From this we get that $\delta(x, T_i(x)) = 0$, so $T_i(x) = \{x\}$, i. e. $x \in (SF)_{T_i}$.

Let $y_0 \in X$ and $y_{n+1} \in T_i(y_n)$, for each $n \in \mathbb{N}$. We have

$$d(y_1, x^*) \leq \delta(T_i(y_0), T_j(x^*)) \leq \\ \leq a\, d(y_0, x^*) + b\, [\delta(y_0, T_i(y_0)) + \delta(x^*, T_j(x^*))] = \\ = a\, d(y_0, x^*) + b\, \delta(y_0, T_i(y_0)).$$

Taking into account the above inequality we are able to write

$$\delta(T_i(y_0), T_i(y_1)) \leq \delta(T_i(y_0), T_j(x^*)) + \delta(T_j(x^*), T_i(y_1)) \leq \\ \leq a\, d(y_0, x^*) + b\, [\delta(y_0, T_i(y_0)) + \delta(x^*, T_j(x^*))] + \\ + a\, d(x^*, y_1) + b\, [\delta(x^*, T_j(x^*)) + \delta(y_1, T_i(y_1))] = \\ = a\, d(y_0, x^*) + b\, \delta(y_0, T_i(y_0)) + a\, d(x^*, y_1) + b\, \delta(y_1, T_i(y_1)) \leq \\ \leq a\, d(y_0, x^*) + b\, \delta(y_0, T_i(y_0)) + a\, d(x^*, y_1) + b\, \delta(T_i(y_0), T_i(y_1)) \leq \\ \leq (1+a)\, [a\, d(y_0, x^*) + b\, \delta(y_0, T_i(y_0))] + b\, \delta(T_i(y_0), T_i(y_1))$$

and from here we get

$$\delta(T_i(y_0), T_i(y_1)) \leq \frac{1+a}{1-b}\, [a\, d(y_0, x^*) + b\, \delta(y_0, T_i(y_0))].$$

Now we have

$$d(y_2, x^*) \leq \delta(T_i(y_1), T_j(x^*)) \leq a\, d(y_1, x^*) + b\, [\delta(y_1, T_i(y_1)) + \delta(x^*, T_j(x^*))] = \\ = a\, d(y_1, x^*) + b\, \delta(y_1, T_i(y_1)) \leq a\, d(y_1, x^*) + b\, \delta(T_i(y_0), T_i(y_1)) \leq \\ \leq \frac{a+b}{1-b}\, [a\, d(y_0, x^*) + b\, \delta(y_0, T_i(y_0))].$$

Using this result we obtain

$$\begin{aligned}
&\delta(T_i(y_1), T_i(y_2)) \le \delta(T_i(y_1), T_j(x^*)) + \delta(T_j(x^*), T_i(y_2)) \le \\
&\le a\, d(y_1, x^*) + b\,[\delta(y_1, T_i(y_1)) + \delta(x^*, T_j(x^*))] + \\
&+ a\, d(x^*, y_2) + b\,[\delta(x^*, T_j(x^*)) + \delta(y_2, T_i(y_2))] = \\
&= a\, d(y_1, x^*) + b\,\delta(y_1, T_i(y_1)) + a\, d(x^*, y_2) + b\,\delta(y_2, T_i(y_2)) \le \\
&\le a\, d(y_1, x^*) + b\,\delta(T_i(y_0), T_i(y_1)) + a\, d(x^*, y_2) + b\,\delta(T_i(y_1), T_i(y_2)) \le \\
&\le \frac{a+b}{1-b}\,(1+a)\,[a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))] + b\,\delta(T_i(y_1), T_i(y_2))
\end{aligned}$$

and so

$$\delta(T_i(y_1), T_i(y_2)) \le \frac{a+b}{1-b} \cdot \frac{1+a}{1-b}\,[a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))].$$

We have

$$\begin{aligned}
d(y_3, x^*) &\le \delta(T_i(y_2), T_j(x^*)) \le a\, d(y_2, x^*) + b\,[\delta(y_2, T_i(y_2)) + \delta(x^*, T_j(x^*))] = \\
&= a\, d(y_2, x^*) + b\,\delta(y_2, T_i(y_2)) \le a\, d(y_2, x^*) + b\,\delta(T_i(y_1), T_i(y_2)) \le \\
&\le \left(\frac{a+b}{1-b}\right)^2 [a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))].
\end{aligned}$$

Using this result we obtain

$$\begin{aligned}
&\delta(T_i(y_2), T_i(y_3)) \le \delta(T_i(y_2), T_j(x^*)) + \delta(T_j(x^*), T_i(y_3)) \le \\
&\le a\, d(y_2, x^*) + b\,[\delta(y_2, T_i(y_2)) + \delta(x^*, T_j(x^*))] + \\
&+ a\, d(x^*, y_3) + b\,[\delta(x^*, T_j(x^*)) + \delta(y_3, T_i(y_3))] = \\
&= a\, d(y_2, x^*) + b\,\delta(y_2, T_i(y_2)) + a\, d(x^*, y_3) + b\,\delta(y_3, T_i(y_3)) \le \\
&\le a\, d(y_2, x^*) + b\,\delta(T_i(y_1), T_i(y_2)) + a\, d(x^*, y_3) + b\,\delta(T_i(y_2), T_i(y_3)) \le \\
&\le \left(\frac{a+b}{1-b}\right)^2 (1+a)\,[a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))] + b\,\delta(T_i(y_2), T_i(y_3)),
\end{aligned}$$

which implies

$$\delta(T_i(y_2), T_i(y_3)) \le \left(\frac{a+b}{1-b}\right)^2 \frac{1+a}{1-b}\,[a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))].$$

By induction can be proved that the sequence $(y_n)_{n\in\mathbb{N}}$ has the following properties:

$$d(y_n, x^*) \le \left(\frac{a+b}{1-b}\right)^{n-1} [a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))]$$

and

$$\delta(T_i(y_{n-1}), T_i(y_n)) \le \left(\frac{a+b}{1-b}\right)^{n-1} \frac{1+a}{1-b}\,[a\, d(y_0, x^*) + b\,\delta(y_0, T_i(y_0))],$$

for each $n \in \mathbb{N}^*$.

It follows that $(y_n)_{n\in\mathbb{N}}$ is a convergent sequence and its limit is x^*. □

Corollary 3.1. *Let* (X, d) *be a complete metric space and* $T_1, T_2 : X \to P_b(X)$ *two multivalued operators for which there exist* $a, b \in \mathbb{R}_+$, *with* $a + 2b < 1$, *such that*

$$\delta(T_1(x), T_2(y)) \leq a\ d(x, y) + b\ [\delta(x, T_1(x)) + \delta(y, T_2(y))],$$

for each $x, y \in X$.

Then $F_{T_1} = F_{T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}$ *and*

$$d(x_0, x^*) \leq (1 + b)/(1 - a)\ \min\{\delta(x_0, T_1(x_0)), \delta(x_0, T_2(x_0))\},$$

for each $x_0 \in X$.

Proof. From Theorem 3.1 we have that $F_{T_1} = F_{T_2} = (SF)_{T_1} = (SF)_{T_2} = \{x^*\}$.

Let $i \in \{1, 2\}$. Let $x_0 \in X$ and $x_1 \in T_i(x_0)$. We have

$$d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq \delta(x_0, T_i(x_0)) + a\ d(x_0, x^*) + b\ \delta(x_0, T_i(x_0)) =$$

$$= a\ d(x_0, x^*) + (1 + b)\ \delta(x_0, T_i(x_0))$$

and so

$$d(x_0, x^*) \leq (1 + b)/(1 - a)\ \delta(x_0, T_i(x_0)).$$

□

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