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On S_3-actions on spin 4-manifolds

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ABSTRACT. Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$ and signature $\sigma(X)$. In this paper we use Seiberg-Witten theory to prove that if X admits an even type symmetric group S_3 action preserving the spin structure, then $b_2^+(X) \geq |\sigma(X)|/8 + 2$ under some non-degeneracy conditions.

1. INTRODUCTION

Let X be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of X. The following conjecture, credited to Kas and Kirby [7], is well known and has been called the $11/8$ - conjecture:

(1.1) $$b_2(X) \geq \frac{11}{8}|\sigma(X)|.$$

All complex surfaces and their connected sums satisfy the conjecture (see [10]).

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold X is

$$-2kE_8 \oplus mH, \qquad k \geq 0,$$

where E_8 is the 8×8 intersection form matrix and H is the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $11/8$ is optimal, if the $11/8$ - conjecture is true.

Donaldson has proved that if $k > 0$ then $m \geq 3$ [4]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [12], Furuta [8] proved that

(1.2) $$b_2(X) \geq \frac{5}{4}|\sigma(X)| + 2.$$

This estimate has been dubbed the $10/8$ - theorem. In fact, if the intersection form of X is definite, i.e., $m = 0$, then Donaldson proved that $b_2(X)$ and $\sigma(X)$ are zero [4, 5]. Thus, Furuta assumed that m is not zero. Inequality (1.2) follows by a surgery argument from the non-positive signature, $b_1(X) = 0$ case:

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Theorem 1.1. (Furuta [8]). *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Then, if $m \neq 0$,*

$$2k + 1 \leq m.$$

Throughout this paper we will assume that m is not zero and $b_1(X) = 0$, unless stated otherwise.

A $Z/2^p$-action is called a spin action if the generator of the action $\tau : X \to X$ lifts to an action $\hat{\tau} : P_{Spin} \to P_{Spin}$ of the Spin bundle P_{Spin}. Such an action is of even type if $\hat{\tau}$ has order 2^p and is of odd type if $\hat{\tau}$ has order 2^{p+1}.

In [2], Bryan (see also [6]) improved the above bound by p under the assumption that X has a spin odd type $Z/2^p$-action satisfying some non-degeneracy conditions analogous to the condition $m \neq 0$. More precisely, he proved

Theorem 1.2. (Bryan [2]). *Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$. Assume that $\tau : X \to X$ generates a spin smooth $Z/2^p$-action of odd type. Let X_i denote the quotient of X by $Z/2^i \subset Z/2^p$. Then*

$$2k + 1 + p \leq m$$

if $m \neq 2k + b_2^+(X_1)$ and $b_2^+(X_i) \neq b_2^+(X_j) > 0$ for $i \neq j$.

In the paper [9], Kim gave the same bound for smooth, spin, even type $Z/2^p$-action on X satisfying some non-degeneracy conditions analogous to Bryan's.

The purpose of this paper is to study the spin symmetric group S_3 actions of even type on spin 4-manifolds, we prove that $b_2^+(X) \geq |\sigma(X)|/8 + 2$ under some non-degeneracy conditions.

We organize this paper as follows. In section 2, we give some preliminaries to prove the main theorems. We refer the readers to the Bryan's excellent exposition [2] for more details. In section 3, we we use equivariant K-theory and representation theory to study the G-equivariant properties of the moduli space. In the last section we give our main result.

2. Notations and preliminaries

We assume that we have completely every Banach spaces with suitable Sobolev norms. Let $S = S^+ \oplus S^-$ denote the decomposition of the spinor bundle into the positive and negative spinor bundles. Let $D : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator, and $\rho : \Lambda_C^* \to End_C(S)$ be the Clifford multiplication. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}R) \times \Gamma(S^+)$ and they are

$$D\phi + \rho(a)\phi = 0, \qquad \rho(d^+a) - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2 id = 0, \qquad d^*a = 0.$$

Let

$$V = \Gamma(\sqrt{-1}\Lambda^1 \oplus S^+),$$
$$W' = \Gamma(S^- \oplus \sqrt{-1}su(S^+) \oplus \sqrt{-1}\Lambda^0).$$

We can think of the equation as the zero set of a map

$$\mathcal{D} + \mathcal{Q} : V \to W,$$

where $\mathcal{D}(a, \phi) = (D\phi, \rho(d^+a), d^*a))$, $\mathcal{Q}(a, \phi) = (\rho(a)\phi, \phi \otimes \phi^* - \frac{1}{2}|\phi|^2 id, 0)$, and W is defined to be the orthogonal complement to the constant functions in W'.

Now we describe the group of symmetries of the equations. Define $Pin(2) \subset SU(2)$ to be the normalizer of $S^1 \subset SU(2)$. Regarding $SU(2)$ as the group of unit quaternions and taking S^1 to be elements of the form $e^{\sqrt{-1}\theta}$, then $Pin(2)$ consists of the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta}J$. Define the action of $Pin(2)$ on V and W as follows: Since S^+ and S^- are $SU(2)$ bundles, $Pin(2)$ naturally acts on $\Gamma(S^{\pm})$ by multiplication on the left. $Z/2$ acts on $\Gamma(\Lambda_C^*)$ by multiplication by ± 1 and this pulls back to an action of $Pin(2)$ by the natural map $Pin(2) \to Z/2$. A calculation shows that this pullback also describes the induced action of $Pin(2)$ on $\sqrt{-1}su(S^+)$. Both $\mathcal{D}$ and $\mathcal{Q}$ are seen to be $Pin(2)$ equavariant maps.

If X is a smooth closed spin 4-manifold. Suppose that X admits a spin structure preserving action by a compact Lie group (or finite group) G'. We may assume a Riemannian metric on X so that G' acts by isometries. If the action is of even type, Both $\mathcal{D}$ and $\mathcal{Q}$ are $G = Pin(2) \times G'$ equavariant maps.

Now we define V_λ to be the subspace of V spanned by the eigenspaces $\mathcal{D}^*\mathcal{D}$ with eigenvalues less than or equal to $\lambda \in R$. Similarly, define W_λ using $\mathcal{D}\mathcal{D}^*$. The virtual G-representation $[V_\lambda \otimes C] - [W_\lambda \otimes C] \in R(G)$ is the G-index of $\mathcal{D}$ and can be determined by the G-index and is independent of $\lambda \in R$, where $R(G)$ is the complex representation of G. In particular, since $V_0 = KerD$ and $W_0 = CokerD \oplus Coker\, d^+$, we have

$$[V_\lambda \otimes C] - [W_\lambda \otimes C] = [V_0 \otimes C] - [W_0 \otimes C] \in R(G).$$

Note that $Coker\, d^+ = H_+^2(X, R)$.

The symmetric group S_3 is the group of all permutations of a set $\{a, b, c\}$ having three elments. It has 4 elements, partitioned into 3 conjugacy classes:

the identity element 1;

3 transpositions: $(ab), (ac), (bc)$;

2 elements of order 3: (abc), (acb).

And we have the following character table for S_3 [11]:

	1	$x_1 = (ab)$	$x_2 = (abc)$
χ_0	1	1	1
θ	1	-1	1
η	2	0	-1

3. THE INDEX OF $\mathcal{D}$ AND THE CHARACTER FORMULA FOR THE K-THEORETIC DEGREE

The virtual representation $[V_{\lambda,C}] - [W_{\lambda,C}] \in R(G)$ is the same as $Ind(\mathcal{D}) = [ker\mathcal{D}] - [Coker\mathcal{D}]$. Furuta determines $Ind(\mathcal{D})$ as a $Pin(2)$ representation; denoting the restriction map $r :\ R(G) \to R(Pin(2))$, Furuta shows

$$r(Ind(\mathcal{D})) = 2kh - m\tilde{1}$$

where $k = |\sigma(X)|/16$ and $m = b_2^+(X)$. Thus $Ind(\mathcal{D}) = sh - t\tilde{1}$ where s and t are polynomials such that $s(1) = 2k$ and $t(1) = m$. we suppose $\sigma(X) \leq 0$ in the following. For spin S_3 action of even type, $G = Pin(2) \times S_3$, and we could write

$$s(\theta, \eta) = a_0 + b_0\theta + c_0\eta, \text{ and } (\theta, \eta) = a_1 + b_1\theta + c_1\eta,$$

such that $a_0 + b_0 + 2c_0 = 2k$ and $a_1 + b_1 + 2c_1 = m$.

The Thom isomorphism theory in equivariant K-theory for a general compact Lie group is a deep theory proved using elliptic operator [1]. The subsequent character formula of this section uses only elementary properties of the Bott class.

Let V and W be complex Γ representations for some compact Lie group Γ. Let BV and BW denote balls in V and W and let $f : BV \to BW$ be a Γ-map preserving the boundaries SV and SW. By definition $K_\Gamma(V)$ is $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator the Bott class $\lambda(V)$. Applying the K-theory functor to f we get a map

$$f^* : K_\Gamma(W) \to K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$. The element α_f is called the K-theoretic degree of f.

Let V_g and W_g denote the subspaces of V and W fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \to W_g$ be the restriction of f and let $d(f^g)$ denote the ordinary topological degree of f^g (by definition, $d(f^g) = 0$ if $dim\, V_g \neq dim\, W_g$). For any $\beta \in R(\Gamma)$, let $\lambda_{-1}\beta$ denote the alternating sum $\Sigma(-1)^i\lambda^i\beta$ of exterior powers.

T. tom Dieck proves the following character formula for the degree α_f:

Theorem 3.3. *Let $f : BV \to BW$ be a Γ-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the K-theory degree. Then*

$$tr_g(\alpha_f) = d(f^g)tr_g(\lambda_{-1}(W_g^\perp - V_g^\perp))$$

where tr_g is the trace of the action of an element $g \in \Gamma$.

This formula is very useful in the case where $dim\, V_g \neq dim\, W_g$ so that $d(f^g) = 0$.

Recall that $\lambda_{-1}(\Sigma_i a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i}$ and that for a one dimensional representation r, we have $\lambda_{-1} r = (1 - r)$. A two dimensional representation such as h has $\lambda_{-1} h = (1 - h + \Lambda^2 h)$. In this case, since h comes from an $SU(2)$ representation, $\Lambda^2 h = det\, h = 1$ so $\lambda_{-1} h = (2 - h)$.

In the following by using the character formula to examine the K-theory degree α_{f_λ} of the map $f_\lambda : BV_{\lambda,C} \to BW_{\lambda,C}$ coming from the Seiberg-Witten equations. We will abbreviate α_{f_λ} as α and $V_{\lambda,C}$ and $W_{\lambda,C}$ as just V and W. Let $\phi \in S^1 \subset Pin(2) \subset G$ be the element generating a dense subgroup of S^1, and recall that there is the element $J \in Pin(2)$ coming from the quaternion. Note that the action of J on h has two invariant subspaces on which J acts by multiplication with $\sqrt{-1}$ and $-\sqrt{-1}$.

4. The main result

Consider $\alpha = \alpha_{f_\lambda} \in R(Pin(2) \times S_3)$, it has the following form

$$\alpha = \alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i.$$

where $\alpha_i = m_i + n_i\theta + l_i\eta,\ i \geq 0$ and $\tilde{\alpha_0} = \tilde{m_0} + \tilde{n_0}\theta + \tilde{l_0}\eta$. Denote by $< x_i >$ the subgroup of S_3 generated by x_i, $i = 1, 2$.

Since ϕ acts non-trivially on h and trivially on $\tilde{1}$, so

$$dim(V(\theta,\eta)h)_\phi - dim(W(\theta,\eta)\tilde{1})_\phi = -(a_1+b_1+2c_1) = -m = -b_2^+(X).$$

So if $b_2^+(X)>0$, $tr_\phi \alpha = 0$.

Since ϕx_1 acts non-trivially on $V(\theta,\eta)h$, ϕ acts trivially on $\tilde{1}$ and ϕx_1 acts trivially on a_1, non-trivially on $b_1\theta$, and acts on $c_1\eta$ as $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, so we have

$$dim(V(\theta,\eta)h)_{\phi x_1} - dim(W(\theta,\eta)\tilde{1})_{\phi x_1} = -(a_1+c_1) = -b_2^+(X/<x_1>).$$

So if $b_2^+(X/<x_1>)>0$, $tr_{\phi x_1}\alpha = 0$.

Finally since ϕx_2 acts non-trivially on $V(\theta,\eta)h$, ϕ acts trivially on $\tilde{1}$ and ϕx_2 acts trivially on a_1 and $b_1\theta$, and acts on $c_1\eta$ non-trivially. So we have

$$dim(V(\theta,\eta)h)_{\phi x_2} - dim(W(\theta,\eta)\tilde{1})_{\phi x_2} = -(a_1+b_1) = -b_2^+(X/<x_2>).$$

So if $b_2^+(X/<x_2>)>0$, $tr_{\phi x_2}\alpha = 0$.

Now if $b_2^+(X)>0$, $b_2^+(X/<x_1>)>0$, $b_2^+(X/<x_2>)>0$, we have $tr_\phi\alpha = tr_{\phi x_1}\alpha = tr_{\phi x_2}\alpha = 0$ which implies that

$$\begin{aligned}
0 &= tr_\phi\alpha = tr_\phi(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i h_i) \\
&= tr_\phi\alpha_0 + tr_\phi\tilde{\alpha_0} + \sum_{i=1}^{\infty} tr_\phi\alpha_i(\phi^i+\phi^{-i}) \\
&= (m_0+n_0+l_0) + (\tilde{m_0}+\tilde{n_0}+\tilde{l_0}) + \sum_{i=1}^{\infty} tr_\phi\alpha_i(\phi^i+\phi^{-i}),
\end{aligned}$$

and so on. From these equations we have $\alpha_0 = -\tilde{\alpha_0}$ and $\alpha_i = 0$, $i\geq 0$, that is $\alpha = \alpha_0(1-\tilde{1})$.

Next we calculate $tr_J\alpha$. Since J acts non-trivially on both h and $\tilde{1}$, $dimV_J = dimW_J = 0$ so $d(f^J)=1$ and the character formula gives $tr_J(\alpha) = tr_J(\lambda_{-1}(m\tilde{1}-2kh)) = tr_J((1-\tilde{1})^m(2-h)^{-2k}) = 2^{m-2k}$ using $tr_J h = 0$ and $tr_J\tilde{1} = -1$.

Now we calculate $tr_{Jx_1}\alpha$. Since Jx_1 acts non-trivially on $V(\theta,\eta)h$. J acts on $\tilde{1}$ by multiplication by -1, and x_1 acts trivially on a_1, on $b_1\theta$ by multiplication by -1 and acts on $c_1\eta$ as $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. So

$$dim(V(\theta,\eta)h)_{Jx_1} - dim(W(\theta,\eta)\tilde{1})_{Jx_1} = -(b_1+c_1) = -(b_2^+(X)-b_2^+(X/<x_1>)).$$

So if $b_2^+(X/<x_1>)\neq b_2^+(X)$, $tr_{Jx_1}\alpha = 0$.

At last, we look at $tr_{Jx_2}\alpha$. Since Jx_2 acts non-trivially on both $V(\theta,\eta)h$ and $W(\theta,\eta)\tilde{1}$, so $d(f^{Jx_2})=1$. By tom Dieck formula, we have

$$\begin{aligned}
tr_{Jx_2}(\alpha) &= tr_{Jx_2}[\lambda_{-1}((a_1+b_1\theta+c_1\eta)\tilde{1} - (a_0+b_0\theta+c_0\eta)h)] \\
&= tr_{Jx_2}[(1-\tilde{1})^{a_1}(1-\theta\tilde{1})^{b_1}(1-\eta\tilde{1})^{c_1}(1-h)^{-a_0}(1-\theta h)^{-b_0}(1-\eta h)^{-c_0}] \\
&= 2^{a_1}2^{b_1}(1+\omega)^{c_1}(1+\omega^2)^{c_1}2^{-a_0}2^{-b_0}(1+\omega)^{-c_0}(1+\omega^2)^{-c_0} \\
&= 2^{(a_1+b_1)-(a_0+b_0)}.
\end{aligned}$$

By direct calculation we have

(4.3) $tr_1\alpha_0 = m_0 + n_0 + 2l_0 = 2^{m-2k-1}$

(4.4) $tr_{x_1}\alpha_0 = m_0 - n_0$

(4.5) $tr_{x_2}\alpha_0 = m_0 + n_0 - l_0 = 2^{(a_1+b_1)-(a_0+b_0)-1}$.

Here we use $0 = tr_{Jx}\alpha = tr_x(2\cdot\alpha_0) = 2\cdot tr_x\alpha_0$.

So if $b_1 + c_1 = b_2^+(X) - b_2^+(X/ < x_1 >) \neq 0$, $tr_{x_2}\alpha_0 = m_0 - n_0 = 0$, with the equation (4.3), we get $2(m_0 + l_0) = 2^{m-2k-1}$, that is $m_0 + l_0 = 2^{m-2k-2}$. So we have the following theorem.

Theorem 4.4. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. If S_3 acts on X such that the action is spin even type. Then $2k + 2 \leq m$ if $b_2^+(X) > 0$, $b_2^+(X/ < x_1 >) > 0$, $b_2^+(X/ < x_2 >) > 0$, and $b_2^+(X) \neq b_2^+(X/ < x_1 >)$.*

Remark 4.1. From the Theorem, we know that the genuine $K3$ surface can not admit a nontrivial spin S_3 action of even type. In fact, the standard action of S_3 on Fermat quartic surface X which is defined by the equation $\sum_{i=0}^{3} z_i^4 = 0$ in CP^3 which is given as permutation of variables is not even type, since the fixed point set is not isolated.

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