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On finite groups of whose all proper subgroups are w-cyclic

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ABSTRACT. A finite group G is called w-cyclic, if G has at most d subgroup, for all divisors d of $|G|$. In this paper, we study the structure of a finite group all of whose proper subgroups are w-cyclic. In the case that G has prime power order, we prove that such a group is elementary abelian of order p^2, p is prime, the quaternion group Q_8 or the generalized quaternion group Q_{16}. We prove that if such a G is not a p-group, then G is solvable and in some cases, we obtain the structure of G. Finally, we characterize the finite groups with w-cyclic proper quotient groups.

1. INTRODUCTION

Let G be a finite group. It is a well known fact that if the equation $x^n = 1$ has at most n solutions, for all divisors n of $|G|$, then G must be cyclic. Indeed, G is cyclic if the number of elements of order n does not exceed n, for any natural number n.

In this connection we might ask about the structure of G, if the number of subgroups of order n of G is at most n, for all positive integers n.

Definition 1.1. A finite group G is called weak cyclic, for abbreviation w-cyclic, if the number of subgroups of order n of G is at most n, for any positive integer n. Furthermore, if any proper subgroup of G is w-cyclic, then G is called $w^{\star}$-cyclic.

It is clear that if G is w-cyclic, then all of proper subgroups of G is also w-cyclic. Therefore, if G is w-cyclic, then G is $w^{\star}$-cyclic, but its converse is not generally true. To see this, it is enough to consider the generalized quaternion group Q_{16} of order 16. This group has exactly five subgroups of order 4 and so it is not w-cyclic.

In this paper, we obtain the structure of finite $w^{\star}$-cyclic groups. Furthermore, we characterize the finite groups in which any proper quotient is w-cyclic. We prove that such a group is solvable and investigate the structure of G.

Throughout this paper, only finite groups are treated. Our notation are standard and taken mainly from [2] and [3].

2. NILPOTENT w-CYCLIC GROUP

In this section we will classify the $w^{\star}$-cyclic p-groups, which will also be of use latter. Moreover, we characterize the $w^{\star}$-cyclic group with an abelian Sylow 2-subgroup.

The proof of the next lemma is elementary.

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Lemma 2.1. *Let G be an abelian $w^\star$-cyclic group. Then G is cyclic or $G \cong Z_p \times Z_p$, for some prime p.*

Lemma 2.2. *Let G be a $w^\star$-cyclic group with an abelian Sylow 2-subgroup. Then G is metacyclic.*

Proof. Suppose p is a prime divisor of $|G|$. Then G has exactly one subgroup of order p. Therefore, every Sylow subgroup of G is cyclic, as desired. □

The following theorem is crucial throughout this paper.

Theorem 2.1. *Let G be a $w^\star$-cyclic p-group. Then G is isomorphic to a cyclic p-group, the elementary abelian group of order p^2, the quaternion group Q_8 or the generalized quaternion group Q_{16}.*

Proof. Suppose G is abelian. Then by Lemma 2.1, G is a cyclic p-group or the elementary abelian group of order p^2. Thus, it is enough to investigate the non-abelian case. On the other hand, it is well known that a p-group which contains only one subgroup of order p is cyclic or a generalized quaternion group (see Theorem 12.5.2 of [1]). Therefore, maximal subgroups of G are cyclic or generalized quaternion. We proceed in two steps.

Step 1. If every maximal subgroup of G is abelian, then $G \cong Q_8$. Since G is not cyclic, it has at least two maximal subgroups, say M and K. By Lemma 2.1, M and K are cyclic and $G = MK$. Set $X = M \cap K$. We can see that $M \leq C_G(X)$, $K \leq C_G(X)$ and so $G = MK \leq C_G(X)$. Thus $X \leq Z(G)$ and $|G : Z(G)|||G : X| = p^2$. But, G is not abelian, so $X = Z(G) = \Phi(G)$. Now we assume that T is a subgroup of order p in G. Then there exists a maximal subgroup Y of G which contains T. Since Y is cyclic, $T \leq \Phi(G)$. Therefore, G has exactly one subgroup of order p and so G is isomorphic to the generalized quaternion group Q_{2^n}. But $|Z(Q_{2^n})| = 2$ and $|Q_{2^n} : Z(Q_{2^n})| = 4$, so $G \cong Q_8$, as desired.

Step 2. If G has a non-abelian maximal subgroup, then $G \cong Q_{16}$. Suppose M is a non-abelian maxiaml subgroup of G. If every maximal subgroup of M is abelian, then by Step 1, $M \cong Q_8$. Since $M \trianglelefteq G$, $Z(M) \trianglelefteq G$. Now, if S is another subgroup of G of order 2, then $H = SZ(M)$ is a subgroup of G with three subgroup of order 2, a contradiction. Therefore, G has exactly one subgroup of order 2 and so $G \cong Q_{16}$. Otherwise, with an inductive argument, we can assume that M has a subgroup N isomorphic to Q_8. Suppose T is a subgroup of M with $|T : N| = 2$. Using a similar arqument as in above, we can show that $T \cong Q_{16}$. Therefore, G has a proper subgroup isomorphic to Q_{16}. But, Q_{16} has exactly five subgroups of order 4, which is a contradiction. This completes the proof. □

Corollary 2.1. *If G is a w-cyclic p-group, then G is cyclic or it is isomorphic to Q_8.*

Corollary 2.2. *There is no finite $w^\star$-cyclic group G such that $|G| > 16$ and a Sylow 2-subgroup of G is isomorphic to Q_{16}.*

Proof. Suppose $|G| > 16$ and a Sylow 2-subgroup of G is isomorphic to Q_{16}. Then G has a subgroup with five subgroups of order four, which is a contradiction. □

3. MAIN RESULTS

The aim of this section is to characterize the structure of $w^\star$-cyclic groups with a Sylow 2-subgroup isomorphic to the quaternion group Q_8. In the following lemma we investigate the structure of G, when G is w-cyclic.

Lemma 3.3. *Let G be a w-cyclic group and a Sylow 2-subgroup S of G is isomorphic to Q_8. Then $S \trianglelefteq G$.*

Proof. Suppose that S is not normal in G and T is another Sylow 2-subgroup of G. It is clear that $|S \cap T| = 1, 2, 4$. If $|S \cap T| = 1, 2$ then G has at least six subgroups of order 4, a contradiction. Otherwise, G has at least five subgroups of order 4, which is a contradiction. Therefore, $S \trianglelefteq G$. $\square$

Corollary 3.3. *Let G be a $w^\star$-cyclic group and a Sylow 2-subgroup S of G is isomorphic to Q_8. If there exists a proper normal subgroup N of G such that $S \subseteq N$ then $S \trianglelefteq G$.*

Proof. By assumption, N is a w-cyclic group and a Sylow 2-subgroup of N is isomorphic to Q_8. Therefore, $S \trianglelefteq N$ and so $S \trianglelefteq G$. $\square$

Theorem 3.2. *Let G be a finite $w^\star$-cyclic group and a Sylow 2-subgroup Q of G is isomorphic to Q_8. Then one of the following holds:*

a) $G \cong Z_3 \propto Q_8$, the semi-direct product of Z_3 by Q_8,

b) $G \cong Q_8 \times H$, in which H is a metacyclic odd order group,

c) $G \cong P \propto (Q_8 \times K)$ such that P is a Sylow 3-subgroup of G, $(|K|, 6) = 1$ and $|P : C_P(Q_8)| = 3$.

Proof. We first assume that $S \ntrianglelefteq G$ and show that G is 2-nilpotent. By Theorem 2.27 of [3], it is enough to show that for any 2-subgroup U of G, $\frac{N_G(U)}{C_G(U)}$ is a 2-group. If $|U| \leq 4$, then $|Aut(U)| = 1, 2$ and so $|\frac{N_G(Q)}{C_G(Q)}| = 1, 2$, as desired. It remains to show that $\frac{N_G(S)}{C_G(S)}$ is a 2-group. If $3 \not| |\frac{N_G(S)}{C_G(S)}|$ then $|\frac{N_G(S)}{C_G(S)}| = 1, 2, 4$. Also, it is easy to see that $4 || \frac{N_G(S)}{C_G(S)}|$. Therefore, we have $|\frac{N_G(S)}{C_G(S)}| = 12$. Suppose $P \in Syl_3(G)$. If $|P| = 3$ then $P \subseteq C_G(S)$, which is a contradiction. We now assume that $P_0 \in Syl_3(N_G(S)$ and $P_0 \subset P$. If $P_0 \trianglelefteq N_G(S)$ then $SP_0 \cong S \times P_0$ and so $P_0 \leq C_G(S)$, a contradiction. This implies that $P \ntrianglelefteq G$ and so $N_G(P) \neq G$. If $C_G(P) = N_G(P)$ then G has a normal 3-complement and by Corollary 2.7, $S \trianglelefteq G$, which is impossible. Therefore, $C_G(P) < N_G(P)$ and we can see that $|\frac{N_G(P)}{C_G(P)}| = 2$. Suppose S_0 is a Sylow 2-subgroup of $N_G(P)$. If $|S_0| = 8$ then by Lemma 2.6, $S_0 \trianglelefteq N_G(P)$ and so $S_0 \leq C_G(P)$, which is a contradiction. So, $|S_0| = 4$. Suppose $P_1 < P_0$ has order 3. Using similar argument as in above, $P_1 \subseteq C_G(S_0)$. Thus, $N_G(P) = S_0 C_{N_G(P)}(P)$. On the other hand, $P_1 \subseteq C_G(S_0)$, $P_1 \subseteq C_{N_G(P)}(P)$ and so $P_1 \subseteq Z(N_G(P))$. Thereofre, $C_P(N_G(P)) \neq 1$ and so G is 3-nilpotent. Now by Lemma 2.6, $S \trianglelefteq G$, which is a contradiction. This shows that G is 2-nilpotent.

We now show that $G \cong Q_8 \propto Z_3$. Assume p is the least prime factor of $|G|$, Q a subgroup of order p and H is a normal 2-complement for G. It is clear that $P \leq H$. But p is the least prime factor of H, so $P \leq Z(H)$. If $|H| \neq p$ then by Lemma 2.6, $S \trianglelefteq SQ$ and so $Q \leq C_G(S)$. Now since $G = SH$, so $P \leq Z(G)$. This shows that G is p-nilpotent and by Corollay 2.7, $S \trianglelefteq G$, which is impossible. Therefore, $|G| = 8p$, p is odd prime. If $Q \ntrianglelefteq G$ then $p = 3$ or 7. It is an easy fact that a group of order 56 has either a normal Sylow 2-subgroup or a normal Sylow 7-subgroup. Also, there is no $w^\star$-cyclic group of order 24 with a non-normal Sylow 2- and 3-subgroup. Thus, $Q \trianglelefteq G$. We now consider the group $\frac{G}{C_G(H)}$. If $G = C_G(H)$ then G has a normal p-complement and so $S \trianglelefteq G$, a contradiction. Otherwise $|\frac{G}{C_G(H)}| = 2$. Let K be a subgroup of order 4 of S and $K \nsubseteq C_G(H)$. Then since $K \ntrianglelefteq KH$, KH has exactly p Sylow 2-subgroup. But, KH is a w-cyclic group, so $p \leq 4$. This implies that $p = 3$ and $G \cong Q_8 \propto Z_3$.

Next we assume that $S \trianglelefteq G$. If $|G : C_G(S)| = 4$ then G has a normal 2-complement H and we have $G \cong S \times H$. Otherwise, we can assume that $|G : C_G(S)| = 12$. Suppose $|C_G(S)| = 2n$, where n is odd. Then $C_G(S)$ has a normal 2-complement K which is normal in G. Since $|S \cap K| = 1$ and S, K are normal subgroups of G, $SK \cong S \times K$. We now consider a Sylow 3-subgroup P of G. If $|P| = 3$ then $G \cong Z_3 \propto (S \times K)$, $6 \not| \ |K|$ and $|P : C_P(S)| = 3$, as required. Otherwise, $|P| > 3$ and $P \cap K$ contains a subgroup P_0 of order 3. Since N is w-cyclic, $P_0 \subseteq Z(K)$. Set $H = PK$. Then since $P_0 \leq C_G(P)$, $P_0 \subseteq Z(H)$. But, $P_0 \subseteq C_G(S)$ and $G = SH$, so $P_0 \subseteq Z(G)$. This implies that G has a normal 3-complement, which completes the proof. □

Corollary 3.4. *Let G be a $w^\star$-cyclic group, then G is a non-abelian group of order pq, p and q are prime numbers, or $|Z(G)| \neq 1$.*

Proof. We first assume that a Sylow 2-subgroup of G is cyclic. Then by Lemma 2.2, G is metacyclic. Suppose G is a non-abelian group of order $p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r}$, where $p_1 < \cdots p_r$ are prmes and $P_i \in Syl_{p_i}, 1 \leq i \leq r$. If $r = 1$ or $r > 2$, then $Z(G)$ contains an element of order p_1. So, we can assume that $r = 2$. Assume that A is a subgroup of order p_1 and $B = AP_2$. If $B \neq G$ then B is w-cyclic and so B is cyclic. This implies that $A \leq Z(G)$. Suppose $G = AP_2$. We now assume that D is a subgroup of G of order p_2. Since $D \trianglelefteq G$, AD is a subgroup of order p_1p_2 of G. Again, if $G \neq AD$, then AD is w-cyclic and so $AD \cong A \times D$. This shows that $C_G(D) = G$, i.e., $D \leq Z(G)$, as desired.

Next we assume that a Sylow 2-subgroup of G is isomorphic to Q_8. If $S \trianglelefteq G$ then $Z(S) \leq Z(G)$, so we can assume that $S \ntrianglelefteq G$. In this case, by Theorem 3.2, $G \cong Q_8 \propto Z_3$ and the proof is complete. □

In the end of this paper we obtain the structure of finite groups in which every proper quotient is $w^\star$-cyclic. We have:

Theorem 3.3. *Let G be a finite group. If for every proper normal subgroup N of G, $\frac{G}{N}$ is $w^\star$-cyclic then one of the following holds:*

$a)$ *G is cyclic,*

$b)$ *$G \cong Z_p \times Z_p$, p is prime,*

$c)$ *G is the quaternion group Q_8,*

$d)$ *G is a two generators group presented by,*

$$G = \langle x, y | x^n = y^q = 1, x^{-1}yx = x^\alpha \rangle$$

in which $(n, q) = 1$ and there exists a prime factor p of n such that $p < q$, $q \equiv 1$ (mod p) and $\alpha^p \equiv 1$ (mod p).

Proof. By Lemma 2.1, we can assume that G is non-abelian. Suppose a Sylow 2-subgroup of G is cyclic. Then by a result of Taunt [4], G is centerless and its hypercenter is $Z(G)$. Therefore, by Corollary 3.4, $\frac{G}{Z(G)}$ is a non-abelian group of order pq, $p < q$ are primes, and $q \equiv 1 \pmod p$. If R is a Sylow r-subgroup of G such that $r \neq p$ and $r \neq q$ then $R \subseteq Z(G)$. Set $T = R_1 R_2 \cdots R_n$, in which R_i's are distinct Sylow r_i-subgroups of G and for $1 \leq i \leq n$, $r_i \notin \{p, q\}$. Therefore, there is a subgroup U of G such that $G \cong U \times T$. It is easy to see that $|U| = p^a q^b$, p, q are distinct primes and a, b are positive integers. Suppose $Q \in Syl_q(G)$ and $P \in Syl_p(G)$. Then $Q \trianglelefteq G$ and $\frac{G}{Q} \cong P \times T$, so $G' \leq Q$. If $b > 1$ then $G' \cap Z(G) \neq 1$, a contradiction. Thus, $b = 1$ and $G' = Q$. This shows that G is a two generators group presented by,

$$G = \langle x, y | x^n = y^q = 1, x^{-1} y x = x^{\alpha} \rangle$$

in which $(n, q) = 1$ and there exists a prime factor p of n such that $p < q$, $q \equiv 1 \pmod p$ and $\alpha^p \equiv 1 \pmod p$.

Secondly, if $G \cong Q_{16}$ then $\frac{G}{Z(G)} \cong D_8$, a contradiction. So, we can assume that a Sylow 2-subgroup of G is isomorphic to Q_8. By Theorem 3.2, $S \trianglelefteq G$ or $G \cong Q_8 \propto Z_3$. In any case, $Z(S) \trianglelefteq G$ and a Sylow 2-subgroup of $\frac{G}{Z(S)} \cong Z_2 \times Z_2$. Therefore, $G \cong Q_8$ which completes the proof. $\square$

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