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Positive solutions of functional differential equations

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ABSTRACT. We study the existence of positive solutions of the equation

$$u''(t) + a(t)\ f(u(g(t))) = 0,\ 0 < t < 1$$

with linear boundary conditions.We show the existence of at least one positive solution if f is either superlinear or sublinear by an application of a fixed point theorem in cones.

1. INTRODUCTION

In this paper we shall consider the second-order boundary value problem with modified argument

$$\text{(1.1)}\qquad \begin{cases} u''(t) + a(t)\ f(u(g(t))) = 0, & 0 < t < 1 \\ \alpha\, u(0) - \beta\, u'(0) = 0, \\ \gamma\, u(1) + \delta u'(1) = 0, \\ u(t) = k,\ -\theta \le t < 0. \end{cases}$$

Here $g : [0,1] \to [-\theta, 1]\,,\ \ \theta > 0$, and $g(t) < t$ for all $t \in [0,1]\,.$

The following conditions will be assumed throughout:

(A.1) $f \in C([0,\infty), [0,\infty))$ and $g \in C([0,1], [-\theta,1])\,;$

(A.2) $a \in C([0,1], [0,\infty))$ and $a(t)$ is not identically zero on any proper subinterval of $[0,1]$;

(A.3) $\alpha, \beta, \gamma, \delta \ge 0$ and $\rho := \gamma\,\beta + \alpha\,\gamma + \alpha\,\delta > 0.$

The purpose here is to give an existence result for positive solutions to (1.1), assuming that f is either superlinear or sublinear. We seek solutions of (1.1) which are positive in the sense that $u(t) > 0$ for $0 < t < 1$. We introduce the notations

$$f_0 := \lim_{u\to 0} \frac{f(u)}{u}\,, \qquad f_\infty := \lim_{u\to\infty} \frac{f(u)}{u}.$$

The situation $f_0 = 0$ and $f_\infty = \infty$ corresponds to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ to the sublinear one.

The proof of our main result, Theorem 2.1, is based on the following fixed point theorem, due to Krasnoselskii [3].

Theorem 1.1. (Krasnoselskii [3]) *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1,\ \overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_2 - \Omega_1\right) \to K$$

be a completely continuous operator such that either:

(i) $\|Au\| \le \|u\|,\ u \in K \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|,\ u \in K \cap \partial\Omega_2;$ or

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(ii) $\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2$.
Then A *has a fixed point in* $K \cap \left(\overline{\Omega}_2 - \Omega_1\right)$.

2. THE MAIN RESULT

We start with a lemma which improves a result from [2].

Lemma 2.1. *Let* K *be the Green's function of the problem:*

$$\begin{cases} u'' = 0 \\ \alpha\, u(0) - \beta\, u'(0) = 0 \\ \gamma\, u(1) + \delta\, u'(1) = 0 \end{cases}$$

given by

$$K(t,s) = \begin{cases} \dfrac{1}{\rho}\varphi(t)\ \psi(s)\,,\ 0 \leq s \leq t \leq 1 \\ \dfrac{1}{\rho}\varphi(s)\ \psi(t)\,,\ 0 \leq t \leq s \leq 1 \end{cases}$$

where

$$\varphi(t) := \gamma + \delta - \gamma\, t, \qquad \psi(t) := \beta + \alpha\, t,\ \ 0 \leq t \leq 1.$$

Then, for every $n > \dfrac{4}{3}$, *there is* $M(n) > 0$ *such that*

$$\frac{K(t,s)}{K(s,s)} \geq M(n) \quad \text{for } \frac{1}{n} \leq t \leq \frac{3}{4}.$$

Proof. We have

$$\frac{K(t,s)}{K(s,s)} = \begin{cases} \dfrac{\varphi(t)}{\varphi(s)},\ s \leq t \\ \dfrac{\psi(t)}{\psi(s)},\ t \leq s. \end{cases}$$

For $s \leq t,$ we will prove that

$$\frac{\varphi(t)}{\varphi(s)} \geq \frac{\gamma + 4\,\delta}{4\ (\gamma+\delta)}, \qquad t \leq \frac{3}{4}.$$

Indeed, for $0 \leq s \leq t \leq \dfrac{3}{4}$ one has

$$\begin{aligned} &\frac{\gamma+\delta-\gamma\, t}{\gamma+\delta-\gamma\, s} \geq \frac{\gamma+4\,\delta}{4\ (\gamma+\delta)}, \\ &4\ (\gamma+\delta)^2 - 4\,\gamma\ (\gamma+\delta)\ t \geq (\gamma+4\,\delta)\ (\gamma+\delta) - \gamma\ (\gamma+4\,\delta)\ s, \\ &4\ \left(\gamma^2 + 2\,\gamma\,\delta + \delta^2\right) - \left(\gamma^2 + 5\,\gamma\,\delta + 4\,\delta^2\right) \geq \gamma\ [4\ (\gamma+\delta)\ t - (\gamma+4\,\delta)\ s]\,, \\ &3\,\gamma^2 + 3\,\gamma\,\delta \geq \gamma\ [4\ (\gamma+\delta)\ t - (\gamma+4\,\delta)\ s]\,, \\ &3\ (\gamma+\delta) \geq 4\ (\gamma+\delta)\ t - (\gamma+4\,\delta)\ s, \\ &(\gamma+\delta)\ (3-4\,t) \geq -(\gamma+4\,\delta)\ s, \end{aligned}$$

that is obviously for $\gamma, \delta \geq 0$ and $t \leq \dfrac{3}{4}$.

For $t \leq s$, we will prove that

$$\frac{\psi(t)}{\psi(s)} \geq \frac{\alpha + n\,\beta}{n\,(\alpha+\beta)}$$

for $t \geq \dfrac{1}{n}$, $n > \dfrac{4}{3}$, which may be written equivalently using the following inequalities:

$$n\,\beta\,(\alpha+\beta) + n\,\alpha\,(\alpha+\beta)\,t \geq \beta\,(\alpha+n\,\beta) + \alpha\,(\alpha+n\,\beta)\,s,$$
$$n\,\beta\,\alpha + n\,\beta^2 + n\,\alpha^2\,t + n\,\alpha\,\beta\,t \geq \beta\,\alpha + n\,\beta^2 + \alpha\,(\alpha+n\,\beta)\,s,$$
$$n\,\beta + n\,t\,\alpha + n\,t\,\beta \geq \beta + (\alpha+n\,\beta)\,s,$$

(2.2) $\quad B(t,s) := \beta\,(n-1) + n\,t\,(\alpha+\beta) - (\alpha+n\beta)\,s \geq 0.$

But we have:

$$B(t,s) \geq \min_s B(t,s) = B(t,1) = \beta\,(n-1) + n\,t\,(\alpha+\beta) - \alpha - n\,\beta =$$
$$= n\,t\,(\alpha+\beta) - (\alpha+\beta) = (\alpha+\beta)\,(n\,t-1) \geq 0$$

for $\alpha, \beta > 0$, $t \geq \dfrac{1}{n}$, so (2.1) is true, which proves the conclusion.

Now we choose

$$M(n) = \min\left\{\frac{\gamma+4\,\delta}{4\,(\gamma+\delta)}, \frac{\alpha+n\,\beta}{n\,(\alpha+\beta)}\right\},$$

in order that for $\dfrac{1}{n} \leq t \leq \dfrac{3}{4}$,

$$\frac{K(t,s)}{K(s,s)} \geq M(n).$$

□

Theorem 2.2. *Assume that conditions* $(A.1)-(A.3)$ *hold. If* $g \in C^1[0,1]$, $g' > 0$, $g(1) > 0$ *and* g *is bijective, then problem* (1.1) *has at least one positive solution in each of the cases:*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear); or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. Superlinear case: Suppose that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1.1). Now (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 K(t,s)\,a(s)\,f(u(g(s)))\,ds := Au(t), \qquad u \in C[0,1].$$

Here $K(t,s)$ denotes the Green's function (expressed in Lemma 2.1) for the boundary values problem (BVP)

(2.3) $$\begin{cases} u'' = 0 \\ \alpha\,u(0) - \beta\,u'(0) = 0 \\ \gamma\,u(1) + \delta\,u'(1) = 0. \end{cases}$$

Observe that

$$K(t,s) \le \frac{1}{\rho}\,\varphi(s)\;\psi(s) = K(s,s), \qquad 0 \le t,\ s \le 1 \tag{2.4}$$

We let K be the cone in $C[0,1]$ given by

$$K = \left\{ u \in C[0,1] : u(t) \ge 0,\ \min_{a_0 \le t \le b} u(t) \ge M\,\|u\| \right\}$$

where $\|u\| = \sup_{t\in[0,1]} |u(t)|$, a_0 and b will be specified later and $M = M(n)$ is from Lemma 2.1.

So, for $u \in K$, we have, using (2.4):

$$Au(t) = \int_0^1 K(t,s)\;a(s)\;f(u(g(s)))\,ds \le \int_0^1 K(s,s)\;a(s)\;f(u(g(s)))\,ds$$

and hence

$$\|Au\| \le \int_0^1 K(s,s)\;a(s)\;f(u(g(s)))\,ds \tag{2.5}$$

We choose $a_0 = \dfrac{1}{n}$ (n from Lemma 2.1.), and $b = \min\left\{ g(1), \dfrac{3}{4} \right\}$.

For $u \in K$, using (2.5) and Lemma 2.1, we obtain

$$\begin{aligned} \min_{a_0 \le t \le b} Au(t) &= \min_{a_0 \le t \le b} \int_0^1 K(t,s)\;a(s)\;f(u(g(s)))\,ds \ge \\ &\ge M \int_0^1 K(s,s)\;a(s)\;f(u(g(s)))\,ds \ge M\,\|Au\|. \end{aligned}$$

Therefore, $A(K) \subset K$. Moreover, it is easy to see that $A : K \to K$ is completely continuous. Now, since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \le \eta\, u$, for $0 < u \le H_1$, where $\eta > 0$ can be chosen conveniently. For $0 < u \le H_1$, we have:

$$\begin{aligned} \|Au\| &\le \int_0^1 K(s,s)\;a(s)\;f(u(g(s)))\,ds \le \\ &\le \eta \int_0^1 K(s,s)\;a(s)\;u(g(s))\,ds = \\ &= \eta \int_{g(0)}^{g(1)} K\left(g^{-1}(y), g^{-1}(y)\right)\;\frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)}\;u(y)\,dy \le \|u\| \end{aligned} \tag{2.6}$$

making

$$g(s) = y,\ s = g^{-1}(y),\ ds = \frac{1}{g'\left(g^{-1}(y)\right)}\,dy,$$

if we choose η so that

$$\eta \int_{g(0)}^{g(1)} K\left(g^{-1}(y),\ g^{-1}(y)\right)\;\frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy \le 1.$$

Now if we let $\Omega_1 = \{u \in E : \|u\| < H_1\}$ then (2.6) shows that $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$.

Making the same change of variables, we have

$$Au(t) = \int_{g(0)}^{g(1)} K\left(t, g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} f(u(y))\, dy.$$

Furthermore, since $f_\infty = \infty$, there exists $\overline{H}_2 > 0$ such that $f(u) \geq \mu\, u$, $u \geq \overline{H}_2$, where $\mu > 0$ can be chosen conveniently. Let

$$\Omega_2 = \{u \in E : \|u\| < H_2\},$$

where

$$H_2 = \max\left\{2\, H_1, \frac{\overline{H}_2}{M}\right\}.$$

Using the hypothesis $g(1) > 0$, $g(0) < 0$, $g' > 0$, we obtain for $a_0 \leq t_0 \leq b$, $u \in K$, $\|u\| = H_2$, that

$$\min_{a_0 \leq t \leq b} u(t) \geq M\, \|u\| \geq \overline{H}_2$$

and so

$$\begin{aligned} Au(t_0) &= \int_{g(0)}^{g(1)} K\left(t_0, g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} f(u(y))\, dy \geq \\ &\geq \mu \int_{g(0)}^{g(1)} K\left(t_0, g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} u(y)\, dy \geq \\ &\geq \mu\, M \left[\int_{a_0}^{b} K\left(t_0, g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)}\, dy\right] \|u\| \geq \|u\| \end{aligned}$$

if we choose $\mu > 0$ so that

$$\mu\, M \left[\int_{a_0}^{b} K\left(t_0, g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)}\, dy\right] \geq 1.$$

Hence $\|Au\| \geq \|u\|$, for $u \in K \cap \partial\Omega_2$. We observe that for $g(1) > 0$ we may choose $n > \dfrac{1}{g(1)}$ in Lemma 2.1, so $g(1) > a_0$ (obviously, for $g(1) < \dfrac{3}{4}$, we may choose $n > \dfrac{1}{g(1)} > \dfrac{4}{3}$, how we need in Lemma 2.1). Therefore,by the first part of the Fixed Point Theorem,it follows that A has a fixed point in $K \cap \left(\overline{\Omega_2} - \Omega_1\right)$, such that $H_1 \leq \|u\| \leq H_2$. Furthermore, since $K(t,s) > 0$, it follows that $u(t) > 0$ for $0 < t < 1$. This completes the superlinear part of the Theorem 2.1.

Sublinear case. Suppose next that $f_0 = \infty$ and $f_\infty = 0$. We first choose $H_1 > 0$ such that $f(u) \geq \overline{\eta}\, u$ for $0 < u \leq H_1$, where $\overline{\eta} > 0$ may be chosen conveniently. Let's consider M from the first part of the proof. Then, for $a_0 \leq t_0 \leq b, u \in K$ and

$\|u\| = H_1$, we have:

$$
\begin{aligned}
Au(t_0) &= \int_{g(0)}^{g(1)} K\left(t_0, g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} \ f(u(y))\, dy \geq \\
&\geq \int_{a_0}^{b} K\left(t_0, g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} \ f(u(y))\, dy \geq \\
&\geq \overline{\eta} \ \int_{a_0}^{b} K\left(t_0, g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} \ u(y)\, dy \geq \\
&\geq \left[\overline{\eta}\, M \ \int_{a_0}^{b} K\left(t_0, g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy\right] \ \|u\| \geq \|u\|
\end{aligned}
$$

if we choose $\overline{\eta} > 0$ such that

$$
\overline{\eta}\, M \ \int_{a_0}^{b} K\left(t_0, g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy \geq 1.
$$

Thus, we may let

$$
\Omega_1 := \{u \in E : \|u\| < H_1\},
$$

so that $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$.

Now, since $f_\infty = 0$, there exists $\overline{H}_2 > 0$ so that $f(u) \leq \lambda\, u$ for $u \geq \overline{H}_2$ where $\lambda > 0$ may be chosen conveniently.

We consider two cases:

Case (i). Suppose f is bounded, say $f(u) \leq N$ for all $u \in (0, \infty)$. In this case choose

$$
H_2 := \max\left\{2\, H_1, N \ \int_{g(0)}^{g(1)} K\left(g^{-1}(y)\, ,\ g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy\right\}
$$

so that for $u \in K$ with $\|u\| = H_2$ we have:

$$
\begin{aligned}
Au(t) &= \int_{g(0)}^{g(1)} K\left(t, g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} \ f(u(y))\, dy \leq \\
&\leq N \ \int_{g(0)}^{g(1)} K\left(g^{-1}(y), g^{-1}(y)\right) \ \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy \leq H_2
\end{aligned}
$$

and therefore $\|Au\| \leq \|u\|$.

Case (ii). If f is unbounded, then let $H_2 > max\left\{2\, H_1, \overline{H}_2\right\}$ and such that $f(u) \leq f(H_2)$ for $0 < u \leq H_2$(we are able to do this since f is unbounded). Then for $u \in K$ and $\|u\| = H_2$ we have:

$$\begin{aligned} Au(t) &= \int_{g(0)}^{g(1)} K\left(t, g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} f(u(y))\, dy \leq \\ &\leq \int_{g(0)}^{g(1)} K\left(g^{-1}(y), g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} f(u(y))\, dy \leq \\ &\leq \int_{g(0)}^{g(1)} K\left(g^{-1}(y), g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} f(H_2)\, dy \leq \\ &\leq \lambda\, H_2 \int_{g(0)}^{g(1)} K\left(g^{-1}(y), g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy \leq H_2 = \|u\| \end{aligned}$$

if we choose $\lambda > 0$ such that

$$\lambda \int_{g(0)}^{g(1)} K\left(g^{-1}(y), g^{-1}(y)\right) \frac{a\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)} dy \leq 1.$$

Therefore, in either case we may put

$$\Omega_2 := \{u \in E : \|u\| < H_2\}$$

and for $u \in K \cap \partial\Omega_2$ we have $\|Au\| \leq \|u\|$.

By the second part of Fixed Point Theorem it follows that (1.1) has a positive solution, and this completes the proof of the Theorem 2.1. □

Remark 2.1. If we choose $g(t) = t - h$, $h \in [0, 1)$, we note that $g(1) > 0$, $g' \equiv 1$, so g satisfies the hypothesis of Theorem 2.1. In this case problem (1.1) concerns a delay equation.

Remark 2.2. If we choose $g(t) = \dfrac{t}{\xi}$, $\xi > 1$, we note that $g(1) > 0$, $g' \equiv \dfrac{1}{\xi}$ so satisfies the hypothesis of Theorem 2.1. In this case we do not need the condition $u(t) = k, -\theta \leq t < 0$.

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