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Weak forms of open and closed functions via semi-θ-open sets

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ABSTRACT. In this paper, we introduce and study two new classes of functions by using the notions of semi-θ-open sets and semi-θ-closure operator called weakly semi-θ-open and weakly semi-θ-closed functions. The connections between these functions and other existing well-known related functions are investigated.

1. INTRODUCTION AND PRELIMINARIES

In 1987, Di Maio and Noiri [13] initiated a brief study of the concepts of semi-θ-open and semi-θ-closed sets which provide a formulation of semi-θ-closure of a set in a topological space. Noiri [24] defined and studied the concept of θ-semicontinuous functions by involving these sets. Mukherjee and Basu [21] continued the work of Di Maio and Noiri and defined the concepts of semi-θ-connectedness, semi-θ- components and semi-θ-quasi-components. Also Park and Park [26] have used these sets to define the notion of weaker forms of irresolute functions. Dontchev and Noiri [14] obtained, among others, that a topological space is semi-Hausdorff if and only if each singleton is semi-θ-closed. Recently the authors [7, 8, 9] have also obtained several new and important results and notions related to these sets. The aim of this paper is to present the class of weakly semi θ-openness (resp. weakly semi θ-closedness) as a new generalization of semi θ-openness (resp. semi θ-closedness). We investigate some of the fundamental properties of this class of functions, w.r.t. these notions.

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of a space X, then $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively. Recall that a subset A of X is called regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). The subset A is called δ-open [31] if $A = Int_\delta(A)$, where $Int_\delta(A)$ is the union of all regular open sets of X which are contained in A.

Definition 1.1. A subset A of a space (X,τ) is said to be semi-open [17], if there exists an open set U such that $U \subset A \subset Cl(U)$, or equivalently if $A \subset Cl(Int(A))$.

The complement of a semi-open set is said to be semi-closed [12]. The intersection of all semi-closed sets containing A is called the semi-closure [12] of A and is denoted by $sCl(A)$. The semi-interior of A denoted by $sInt(A)$, is defined by

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the union of all semi-open sets contained in A.The family of all semiopen (resp. semiclosed) sets of a space X is denoted by $SO(X,\tau)$ (resp. $SC(X,\tau)$).

Definition 1.2. A point $x \in X$ is called a semi θ-cluster [13] (resp. θ-cluster [31]) point of A if $A \cap sCl(U) \neq \emptyset$ (resp. $A \cap Cl(U) \neq \emptyset$) for each semiopen (resp. open) set U containing x.

The set of all semi θ-cluster (resp. θ-cluster) points of A is called the semi θ-closure (resp. θ-closure) of A and is denoted by $sCl_\theta(A)$ (resp. $Cl_\theta(A)$. Hence, a subset A is called semi-θ-closed [13] (resp. θ-closed [31]) if $sCl_\theta(A) = A$ (resp. $Cl_\theta(A) = A$). The complement of a semi-θ-closed (resp. θ-closed) set is called semi-θ-open (resp. θ-open) set. The semi θ-interior (resp. θ-interior) of A is defined by the union of all semi-θ-open (resp. θ-open) sets contained in A and is denoted by $sInt_\theta(A)$ (resp. $Int_\theta(A)$. The family of all semi-θ-open (resp. semi-θ-closed) sets of a space X is denoted by $S\theta O(X,\tau)$ (resp. $S\theta C(X,\tau)$). Recall that a subset A of a space (X,τ) is said to be δ-semi-open [27] if $A \subset Cl(Int_\delta(A))$.

Definition 1.3. A subset $A \subset X$ is called preopen [20] (resp. α-open [22] and β-open [1] (or semi-preopen [2]), if $A \subset Int(Cl(A))(resp.A \subset Int(Cl(Int(A))$ and $A \subset Cl(Int(Cl(A)))$. The complement of a preopen (resp. α-open) set is called preclosed (resp. α-closed).

Remark 1.1. We have the following diagram [25] in which the converses of the implications need not be true.

DIAGRAM

$$
\begin{array}{ccccc}
\theta\text{-}open & \rightarrow & \delta\text{-}open & \rightarrow & open \\
\downarrow & & \downarrow & & \downarrow \\
semi-\theta-open & \rightarrow & \delta-semi-open & \rightarrow & semi-open
\end{array}
$$

Lemma 1.1. (Di Maio and Noiri [13]) *Let A be a subset of a topological space (X,τ).*
(1) *If $A \in SO(X,\tau)$, then $sCl(A) = sCl_\theta(A)$.*
(2) *If A is open in X, then $sCl(A) = Int(Cl(A))$.*

Lemma 1.2. (Mukherjee and Basu [21]) *For any subset A of a topological space (X,τ), $sCl_\theta(A)$ is semi-θ-closed, for every $A \subset X$.*

A space X is called extremally disconnected (E.D) [32] if the closure of each open set in X is open. A space X is called semi θ-connected[21] if X can not be expressed as the union of two nonempty disjoint semi-θ-open sets.

Definition 1.4. A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is called:
(i) semi θ-open (resp. semi θ-closed) if for each open set U (resp. closed set F) of X, $f(U) \in S\theta O(Y,\sigma)$ (resp. $f(F) \in S\theta C(Y,\sigma)$;
(ii) θ-semicontinuous [24] if for each $x \in X$ and each open set V containing $f(x)$, there exists $U \in SO(x)$ such that $f(sCl(U)) \subset Cl(V)$;
(iii) θ-irresolute [26] if for each $x \in X$ and each $V \in SO(Y)$ containing $f(x)$, there exists $U \in SO(x)$ such that $f(sCl(U) \subset sCl(V)$;
(iv) almost open in the sense of Singal and Singal , written as (a.o.S) [30] if the image of each regular open set U of X is open set in Y;

(v) preopen [20] (resp. preclosed [15], β-open [1], α-open [22])if for each open set U (resp. closed set F, open set U, open set U) of X, $f(U)$ is preopen (resp. $f(F)$ is preclosed, $f(U)$ is β-open, $f(U)$ is α-open) set in Y;

(vi) contra-open [4] (resp. contra-closed [4]) if $f(U)$ is closed (resp. open) in Y for each open (resp. closed) set U of X;

(vii) weak semi-θ-continuous [10] if $f^{-1}(V) \subset sInt_\theta(f^{-1}(Cl(V)))$ for each open set V of Y.

2. WEAKLY SEMI-θ-OPEN FUNCTIONS

We define in this section the concept of weak semi-θ-openness as natural dual to the weak semi-θ-continuity.

Definition 2.5. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly semi-θ-open if $f(U) \subset sInt_\theta(f(Cl(U)))$ for each open set U of X.

Clearly, every semi-θ-open function is also weakly semi-θ-open, but the converse is not generally true.

Example 2.1. A weakly semi-θ-open function need not be semi-θ-open.
Let $X = \{a, b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is clearly weakly semi-θ-open, but is not semi-θ-open since $f(b)$ is not a semi-θ-open set in Y.

Theorem 2.1. *Let X be a regular space. Then $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-open if and only if f is semi-θ-open.*

Proof. The sufficiency is clear. Necessity. Let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subset Cl(U_x) \subset W$. Hence we obtain that $W = \cup\{U_x : x \in W\} = \cup\{Cl(U_x) : x \in W\}$ and , $f(W) = \cup\{f(U_x) : x \in W\} \subset \cup\{sInt_\theta(f(Cl(U_x))) : x \in W\} \subset sInt_\theta(f(\cup\{Cl(U_x) : x \in W\}) = sInt_\theta(f(W))$. Thus f is semi-θ-open. □

Theorem 2.2. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent :*

(i) *f is weakly semi-θ-open.*

(ii) *$f(Int_\theta(A)) \subset sInt_\theta(f(A))$ for every subset A of X.*

(iii) *$Int_\theta(f^{-1}(B)) \subset f^{-1}(sInt_\theta(B))$ for every subset B of Y.*

(iv) *$f^{-1}(sCl_\theta(B)) \subset Cl_\theta(f^{-1}(B))$ for every subset B of Y.*

(v) *For each $x \in X$ and each open set U of X containing x, there exists a semi-θ-open set V containing $f(x)$ such that $V \subset f(Cl(U))$.*

(vi) *For each closed subset F of X, $f(Int(F)) \subset sInt_\theta(f(F))$.*

(vii) *For each open subset U of X, $f(Int(Cl(U))) \subset sInt_\theta(f(Cl(U)))$.*

$(viii)$ *For each preopen subset U of X, $f(Int(Cl(U))) \subset sInt_\theta(f(Cl(U)))$.*

(ix) *For every α-open subset U of X, $f(U) \subset sInt_\theta(f(Cl(U)))$.*

Proof. $(i) \to (ii)$ Let A be any subset of X and $x \in Int_\theta(A)$. Then, there exists an open set U such that $x \in U \subset Cl(U) \subset A$. Then, $f(x) \in f(U) \subset f(Cl(U)) \subset f(A)$. Since f is weakly semi-θ-open, $f(U) \subset sInt_\theta(f(Cl(U))) \subset sInt_\theta(f(A))$. It implies that $f(x) \in sInt_\theta(f(A))$. This shows that $x \in f^{-1}(sInt_\theta(f(A)))$. Thus $Int_\theta(A) \subset f^{-1}(sInt_\theta(f(A)))$, and so, $f(Int_\theta(A)) \subset sInt_\theta(f(A))$.

$(ii) \to (i)$ Let U be an open set in X. As $U \subset Int_\theta(Cl(U))$ implies, $f(U) \subset f(Int_\theta(Cl(U))) \subset sInt_\theta(f(Cl(U)))$. Hence f is weakly semi-θ-open.

$(ii) \to (iii)$ Let B be any subset of Y. Then by (ii), $f(Int_\theta(f^{-1}(B)) \subset sInt_\theta(B)$. Therefore $Int_\theta(f^{-1}(B)) \subset f^{-1}(sInt_\theta(B))$.

$(iii) \to (ii)$ This is obvious.

$(iii) \to (iv)$ Let B be any subset of Y. Using (iii), we have $X - Cl_\theta(f^{-1}(B)) = Int_\theta(X - f^{-1}(B)) = Int_\theta(f^{-1}(Y - B)) \subset f^{-1}(sInt_\theta(Y - B)) = f^{-1}(Y - sCl_\theta(B)) = X - (f^{-1}(sCl_\theta(B))$. Therefore, we obtain $f^{-1}(sCl_\theta(B)) \subset Cl_\theta(f^{-1}(B))$.

$(iv) \to (iii)$ Similarly we obtain, $X - f^{-1}(sInt_\theta(B)) \subset X - Int_\theta(f^{-1}(B))$, for every subset B of Y, i.e., $Int_\theta(f^{-1}(B)) \subset f^{-1}(sInt_\theta(B))$.

$(i) \to (v)$ Let $x \in X$ and U be an open set in X with $x \in U$. Since f is weakly semi-θ-open. $f(x) \in f(U) \subset sInt_\theta(f(Cl(U)))$. Let $V = sInt_\theta(f(Cl(U)))$. Hence $V \subset f(Cl(U))$, with V containing $f(x)$.

$(v) \to (i)$ Let U be an open set in X and $y \in f(U)$. It follows from (ii) that $V \subset f(Cl(U))$ for some V semi-θ-open in Y containing y. Hence we have, $y \in V \subset sInt_\theta(f(Cl(U)))$. This shows that $f(U) \subset sInt_\theta(f(Cl(U)))$, i.e., f is a weakly semi-θ-open function.

$(i) \to (vi) \to (vii) \to (viii) \to (iv) \to (i)$ This is obvious. □

Theorem 2.3. *Let $f : (X,\tau) \to (Y,\sigma)$ be a bijective function. Then the following statements are equivalent.*

(i) *f is weakly semi-θ-open,*

(ii) *$sCl_\theta(f(U)) \subset f(Cl(U))$ for each U open of X,*

(iii) *$sCl_\theta(f(Int(F)) \subset f(F)$ for each F closed of X.*

Proof. $(i) \to (iii)$ Let F be a closed set in X. Then we have $f(X-F) = Y - f(F) \subset sInt_\theta(f(Cl(X-F)))$ and so $Y - f(F) \subset Y - sCl_\theta(f(Int(F)))$.
Hence $sCl_\theta(f(Int(F))) \subset f(F)$.

$(iii) \to (ii)$ Let U be a open set in X. Since $Cl(U)$ is a closed set and $U \subset Int(Cl(U))$ by (iii) we have $sCl_\theta(f(U)) \subset sCl_\theta(f(Int(Cl(U))) \subset f(Cl(U))$.

$(ii) \to (iii)$ Similar to $(iii) \to (ii)$.

$(iii) \to (i)$ Clear. □

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is said to be strongly continuous [18], if for every subset A of X, $f(Cl(A)) \subset f(A)$.

Theorem 2.4. *If $f : (X,\tau) \to (Y,\sigma)$ is weakly semi-θ-open and strongly continuous, then f is semi-θ-open.*

Proof. Let U be an open subset of X. Since f is weakly semi-θ-open $f(U) \subset sInt_\theta(f(Cl(U)))$. However, because f is strongly continuous, $f(U) \subset sInt_\theta(f(U))$. Therefore $f(U)$ is semi-θ-open. □

A function $f : (X,\tau) \to (Y,\sigma)$ is said to be contra semi-θ-closed if $f(U)$ is a semi-θ-open set of Y, for each closed set U in X.

Theorem 2.5. *If $f : (X,\tau) \to (Y,\sigma)$ is contra semi-θ-closed, then f is a weakly semi-θ-open function.*

Proof. Let U be an open subset of X. Then, we have $f(U) \subset f(Cl(U))) = sInt_\theta(f(Cl(U)))$. □

Example 2.2. The converse of Theorem 2.5 does not hold. Example 2.1 shows that a weakly semi-θ-open function need not be contra semi-θ-closed.

Next, we define a dual form, called complementary weakly semi-θ-open function as follows:

Definition 2.6. A function $f : (X, \tau) \to (Y, \sigma)$ is called complementary weakly semi-θ-open (written as c.w.sθ.o) if for each open set U of X, $f(Fr(U))$ is semi-θ-closed in Y, where $Fr(U)$ denotes the frontier of U.

Example 2.3. A weakly semi-θ-open function need not be c.w.sθ.o.
Let $X = \{a, b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is clearly weakly semi-θ-open, but it is not c.w.sθ.o., since $F_r(\{b\}) = Cl(\{b\}) - \{b\} = \{a\}$ and $f(F_r(\{b\})) = \{x\}$ is not a semi-θ-closed set in Y.

Theorem 2.6. *Let $S\theta O(X, \tau)$ be closed under finite intersections. If $f : (X, \tau) \to (Y, \sigma)$ is bijective weakly semi-θ-open and c.w.sθ.o, then f is semi-θ-open.*

Proof. Let U be an open subset in X with $x \in U$. Since f is weakly semi-θ-open, by Theorem 2.4 there exists a semi-θ-open set V containing $f(x) = y$ such that $V \subset f(Cl(U))$. Now $Fr(U) = Cl(U) - U$ and thus $x \notin Fr(U)$. Hence $y \notin f(Fr(U))$ and therefore $y \in V - f(Fr(U))$. Put $V_y = V - f(Fr(U))$ a semi-θ-open set since f is c.w.sθ.o. Since $y \in V_y$, $y \in f(Cl(U))$. But $y \notin f(Fr(U))$ and thus $y \notin f(Fr(U)) = f(Cl(U)) - f(U)$. It follows that $y \in f(U)$. Therefore $f(U) = \cup\{V_y : V_y \in S\theta O(Y, \sigma), y \in f(U)\}$. Hence f is semi-θ-open. □

The following theorem is a variation of a result of Baker [4] in which contra-closedness is replaced by weakly semi-θ-open and closed by contra-pre-semi-θ-closed, where, $f : (X, \tau) \to (Y, \sigma)$ is said to be contra-pre-semi-θ-closed provided that $f(F)$ is semi-θ-open for each semi-θ-closed subset F of X.

Theorem 2.7. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-open, $S\theta O(Y, \sigma)$ closed under finite unions and if for each semi-θ-closed subset F of X and each fiber $f^{-1}(y) \subset X - F$ there exists an open subset U of X for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \phi$, then f is contra-pre-semi-θ-closed.*

Proof. Assume that F is a semi-θ-closed subset of X and let $y \in Y - f(F)$. Thus $f^{-1}(y) \subset X - F$. Hence there exists an open subset U of X for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \phi$. Therefore $y \in Y - f(Cl(U)) \subset Y - f(F)$. Since f is weakly semi-θ-open $f(U) \subset sInt_\theta(f(Cl(U)))$. We obtain $y \in sCl_\theta(Y - f(Cl(U))) \subset Y - f(F)$. Let $B_y = sCl_\theta(Y - f(Cl(U)))$. Then B_y is a semi-θ-closed subset of Y containing y. Hence $Y - f(F) = \cup\{B_y : y \in Y - f(F)\}$ is semi-θ-closed. Therefore $f(F)$ is semi-θ-open. □

Theorem 2.8. *If $f : (X, \tau) \to (Y, \sigma)$ is an a.o.S and closed function, then it is a weakly semi-θ-open function.*

Proof. Let U be an open set in X. Since f is a.o.S and $Int(Cl(U))$ is regular open, $f(Int(Cl(U)))$ is open in Y. Since f is closed,

$$f(U) \subset f(Int(Cl(U)) \subset Int(f(Cl(U))) \subset sInt_\theta(f(Cl(U))).$$

This shows that f is weakly semi-θ-open. □

Lemma 2.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous function, then for any subset U of X, $f(Cl(U)) \subset Cl(f(U))$* [32].

Theorem 2.9. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly semi-θ-open and continuous function, then f is a β-open function.*

Proof. Let U be an open set in X. Then by weak semi-θ-openness of f, $f(U) \subset sInt_\theta(f(Cl(U)))$. Since f is continuous $f(Cl(U)) \subset Cl(f(U))$. Hence we obtain that, $f(U) \subset sInt_\theta(f(Cl(U))) \subset sInt_\theta(Cl(f(U))) \subset sInt(Cl(f(U))) \subset Cl(Int(Cl(f(U))))$. Therefore, $f(U) \subset Cl(Int(Cl(f(U)))$ which shows that $f(U)$ is a β-open set in Y and we are done. □

Since every strongly continuous function is continuous we have the following,

Corollary 2.1. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly semi-θ-open and strongly continuous function. Then f is a β-open function.*

Theorem 2.10. *If $f : (X, \tau) \to (Y, \sigma)$ is a bijective weakly semi-θ-open function from a space X onto a semi-θ-connected space Y, then X is connected.*

Proof. Assume that $X$is not connected. Then there exist non-empty open sets U_1 and U_2 such that $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$. Hence we have $f(U_1) \cap f(U_2) = \phi$ and $f(U_1) \cup f(U_2) = Y$. Since f is bijective weakly semi-θ-open, we have $f(U_i) \subset sInt_\theta(f(Cl(U_i)))$ for i=1, 2 and since U_i is open and also closed, we have $f(Cl(U_i) = f(U_i)$ for i=1, 2. Hence $f(U_i)$ is semi-θ-open in Y for i=1, 2. Thus, Y has been decomposed into two non-empty disjoint semi-θ-open sets. This is contrary to the hypothesis that Y is a semi-θ-connected space. Thus X is connected. □

Definition 2.7. A space X is said to be hyperconnected [23] if every nonempty open subset of X is dense in X.

Theorem 2.11. *If X is a hyperconnected space, then a function $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-open if and only if $f(X)$ is semi-θ-open in Y.*

Proof. The sufficiency is clear. For the necessity observe that for any open subset U of $X, f(U) \subset f(X) = sInt_\theta(f(X) = sInt_\theta(f(Cl(U)))$. □

3. Weakly semi-θ-closed functions

Now, we define the generalized form of semi-θ-closed functions.

Definition 3.8. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly semi-θ-closed if $sCl_\theta(f(Int(F))) \subset f(F)$ for each closed set F of X.

Clearly, every semi-θ-closed function is weakly semi-θ-closed function, but the converse is not generally true.

Example 3.4. Let $f : (X, \tau) \to (Y, \sigma)$ be the function from Example 2.1. Then it is shown that f is weakly semi-θ-closed which is not semi-θ-closed.

Theorem 3.12. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent.*

(i) *f is weakly semi-θ-closed.*
(ii) *$sCl_\theta(f(U)) \subset f(Cl(U))$ for every open set U of X.*

Proof. $(i) \to (ii)$. Let U be any open subset of X. Then
$sCl_\theta(f(U)) = sCl_\theta(f(Int(U))) \subset sCl_\theta(f(Int(Cl(U))) \subset f(Cl(U))$.
$(ii) \to (i)$. Let F be any closed subset of X. Then,
$sCl_\theta(f(Int(F))) \subset f(Cl(Int(F))) \subset f(Cl(F)) = f(F)$. □

The proof of the following result is mostly straightforward and is therefore omitted.

Theorem 3.13. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:*
(i) *f is weakly semi-θ-closed,*
(ii) *$sCl_\theta(f(U)) \subset f(Cl(U))$ for each open set U in X,*
(iii) *$sCl_\theta(f(Int(F))) \subset f(F)$ for each preclosed subset F in X,*
(iv) *$sCl_\theta(f(Int(F))) \subset f(F)$ for every α-closed subset F in X.*

Theorem 3.14. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:*
(i) *f is weakly semi-θ-closed,*
(ii) *$sCl_\theta(f(U)) \subset f(Cl(U))$ for each regular open subset U of X,*
(iii) *For each subset F in Y and each open set U in X with $f^{-1}(F) \subset U$, there exists a semi-θ-open set A in Y with $F \subset A$ and $f^{-1}(F) \subset Cl(U)$,*
(iv) *For each point y in Y and each open set U in X with $f^{-1}(y) \subset U$, there exists a semi-θ-open set A in Y containing y and $f^{-1}(A) \subset Cl(U)$,*
(v) *$sCl_\theta(f(Int(Cl(U)))) \subset f(Cl(U))$ for each set U in X,*
(vi) *$sCl_\theta(f(Int(Cl_\theta(U)))) \subset f(Cl_\theta(U))$ for each set U in X,*
(vii) *$sCl_\theta(f(U)) \subset f(Cl(U))$ for each preopen set U in X.*

Proof. It is clear that: $(i) \to (vi), (iii) \to (iv)$, and $(i) \to (v) \to (vii) \to (ii) \to (i)$. To show that $(ii) \to (iii)$, let F be a subset in Y and U be open in X with $f^{-1}(F) \subset U$. Then $f^{-1}(F) \cap Cl(X - Cl(U)) = \phi$ and consequently, $F \cap f(Cl(X - Cl(U))) = \phi$. Since $X - Cl(U)$ is regular open, $F \cap sCl_\theta(f(X - Cl(U))) = \phi$ by (ii). Let $A = Y - sCl_\theta(f(X - Cl(U)))$. Then A is semi-θ-open with $F \subset A$ and $f^{-1}(A) \subset X - f^{-1}(sCl_\theta(f(X - Cl(U)))) \subset X - f^{-1}f(X - Cl(U)) \subset Cl(U)$.

$(vi) \to (i)$ It suffices to see that $Cl_\theta(U) = Cl(U)$ for every open sets U in X.

$(iv) \to (i)$ Let F be closed in X and $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F$, there exists a semi-θ-open A in Y with $y \in A$ and $f^{-1}(A) \subset Cl(X - F) = X - Int(F)$ by (iv). Therefore $A \cap f(Int(F)) = \phi$, so that $y \in Y - sCl_\theta(f(Int(F)))$. Thus $(iv) \to (i)$.

$(vi) \to (vii)$ Note that $Cl_\theta(U) = Cl(U)$ for each preopen subset U in X. □

Remark 3.2. By Theorem 2.3, if $f : (X, \tau) \to (Y, \sigma)$ is a bijective function, then f is weakly semi-θ-open if and only if f is weakly semi-θ-closed.

Next we investigate conditions under which weakly semi-θ-closed functions are semi-θ-closed.

Theorem 3.15. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-closed and if for each closed subset F of X and each fiber $f^{-1}(y) \subset X - F$, there exists an open U of X such that $f^{-1}(y) \subset U \subset Cl(U) \subset X - F$. Then f is semi-θ-closed.*

Proof. Let F be any closed subset of X and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and hence $f^{-1}(y) \subset X - F$. By hypothesis, there exists an open U of X such that $f^{-1}(y) \subset U \subset Cl(U) \subset X - F$. Since f is weakly semi-θ-closed by Theorem 3.14, there exists a semi-θ-open V in Y with $y \in V$ and $f^{-1}(V) \subset Cl(U)$. Therefore, we obtain $f^{-1}(V) \cap F = \phi$ and thus $V \cap f(F) = \phi$, this shows that $y \notin sCl_\theta(f(F))$. Therefore, $f(F)$ is semi-θ-closed in Y and f is semi-θ-closed. □

Theorem 3.16. *(i) If $f : (X, \tau) \to (Y, \sigma)$ is preclosed and contra-closed, then f is weakly semi-θ-closed.*

(ii) If $f : (X, \tau) \to (Y, \sigma)$ is contra-semi-θ-open, then f is weakly semi-θ-closed.

Proof. (i) Let F be a closed subset of X. Since f is preclosed $Cl(Int(f(F))) \subset f(F)$ and f is contra-closed, $f(F)$ is open. Therefore $sCl_\theta(f(Int(F))) \subset sCl_\theta(f(F)) \subset Cl(Int(f(F))) \subset f(F)$.

(ii) Let F be a closed subset of X. Then, $sCl_\theta(f(Int(F))) \subset f(Int(F)) \subset f(F)$. □

Theorem 3.17. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective weakly semi-θ-closed function, then for every subset F in Y and every open set U in X with $f^{-1}(F) \subset U$, there exists a semi-θ-closed set B in Y such that $F \subset B$ and $f^{-1}(B) \subset Cl(U)$.*

Proof. Let F be a subset of Y and U be an open subset of X with $f^{-1}(F) \subset U$. Put $B = sCl_\theta(f(Int(Cl(U))))$, then B is a semi-θ-closed set of Y such that $F \subset B$ since $F \subset f(U) \subset f(Int(Cl(U)) \subset sCl_\theta(f(Int(Cl(U)))) = B$. By weakly semi-$\theta$-closedness of f, it follows that $f^{-1}(B) \subset Cl(U)$. □

Taking the set F in Theorem 3.17 to be y for $y \in Y$ we obtain the following result,

Corollary 3.2. *If $f : (X, \tau) \to (Y, \sigma)$ is one-one weakly semi-θ-closed, then for every point y in Y and every open set U in X with $f^{-1}(y) \subset U$, there exists a semi-θ-closed set B in Y containing y such that $f^{-1}(B) \subset Cl(U)$.*

Recall that, a set F in a space X is θ-compact [29] if for each cover Ω of F by open U in X, there is a finite family $U_1, ..., U_n$ in Ω such that $F \subset Int(\cup\{Cl(U_i) : i = 1, 2, ..., n\})$.

Theorem 3.18. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-closed with all fibers θ-closed, then $f(F)$ is semi-θ-closed for each θ-compact set F in X.*

Proof. Let F be θ-compact set and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and for each $x \in F$ there is an open U_x containing x in X and $Cl(U_x) \cap f^{-1}(y) = \phi$. Clearly $\Omega = \{U_x : x \in F\}$ is an open cover of F. Since F is θ-compact, there is a finite family $\{U_{x_1}, ..., U_{x_n}\}$ in Ω such that $F \subset Int(A)$, where $A = \cup\{Cl(U_{x_i}) : i = 1, ..., n\}$. Since f is weakly semi-θ-closed by Theorem 2.2 there exists a semi-θ-open B in Y with $f^{-1}(y) \subset f^{-1}(B) \subset Cl(X - A) = X - Int(A) \subset X - F$. Therefore $y \in B$ and $B \cap f(F) = \phi$. Thus $y \in Y - sCl_\theta(f(F))$. This shows that $f(F)$ is semi-θ-closed. □

Two non empty subsets A and B in X are strongly separated [29], if there exist open sets U and V in X with $A \subset U$ and $B \subset V$ and $Cl(U) \cap Cl(V) = \phi$. If A and

B are singleton sets we may speak of points being strongly separated. We will use the fact that in a normal space, disjoint closed sets are strongly separated.

Recall that a space X is said to be semi-θ-Hausdorff (briefly semi-θ-T_2 [8]) if for every distinct pair of points x and y, there exist two semi-θ-open sets U and V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Theorem 3.19. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-closed surjection and all pairs of disjoint fibers are strongly separated, then Y is semi-θ-T_2.*

Proof. Let y and z be two points in Y. Let U and V be open sets in X such that $f^{-1}(y) \subset U$ and $f^{-1}(z) \subset V$, respectively, with $Cl(U) \cap Cl(V) = \phi$. By weak semi-θ-closedness (Theorem 3.14 (iv)) there are semi-θ-open sets F and B in Y such that $y \in F$ and $z \in B$, $f^{-1}(F) \subset Cl(U)$ and $f^{-1}(B) \subset Cl(V)$. Therefore $F \cap B = \phi$, since $Cl(U) \cap Cl(V) = \phi$ and f is a surjection. Therefore Y is semi-θ-T_2. □

Corollary 3.3. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly semi-θ-closed surjection with all closed fibers and X is normal, then Y is semi-θ-T_2.*

Corollary 3.4. *If $f : (X, \tau) \to (Y, \sigma)$ is continuous weakly semi-θ-closed surjection with X compact T_2 space and Y a T_1 space, then Y is compact semi-θ-T_2 space.*

Proof. Since f is a continuous surjection and Y is a T_1 space, Y is compact and all fibers are closed. Since X is normal Y is also semi-θ-T_2. □

Definition 3.9. A topological space X is said to be quasi H-closed [11] (resp. N sθ-closed), if every open (resp. semi-θ-closed) cover of X has a finite subfamily whose closures cover X. A subset A of a topological space X is quasi H-closed relative to X (resp. N sθ-closed relative to X) if every cover of A by open (resp. N sθ-closed) sets of X has a finite subfamily whose closures cover A.

Lemma 3.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if for each $B \subset Y$, $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B))$* [19].

Theorem 3.20. *Let X be an extremally disconnected space and $S\theta O(X, \tau)$ closed under finite intersections. Let $f : (X, \tau) \to (Y, \sigma)$ be an open weakly semi-θ-closed function one-one such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is N sθ-closed relative to Y then $f^{-1}(G)$ is quasi H-closed.*

Proof. Let $\{V_\beta : \beta \in I\}$, I being the index set be a open cover of $f^{-1}(G)$. Then for each $y \in G \cap f(X)$, $f^{-1}(y) \subset \cup\{Cl(V_\beta) : \beta \in I(y)\} = H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $Cl(V_\beta)$ is open, hence H_y is open in X. So by Corollary 3.2, there exists a semi-θ-closed set U_y containing y such that $f^{-1}(U_y) \subset Cl(H_y)$. Then, $\{U_y : y \in G \cap f(X)\} \cup \{Y - f(X)\}$ is a semi-θ-closed cover of G, $\quad G \subset \cup\{Cl(U_y) : y \in K\} \cup \{Cl(Y - f(X))\}$ for some finite subset K of $g \cap f(X)$. Hence and by Lemma 3.4, $f^{-1}(G) \subset \cup\{f^{-1}(Cl(U_y) : y \in K\} \cup \{f^{-1}(Cl(Y - f(X))\} \subset \cup\{Cl(f^{-1}(U_y) : y \in K\} \cup \{Cl(f^{-1}(Y - f(X)))\} \subset \{Cl(f^{-1}(U_y)) : y \in K\}$, so $f^{-1}(G) \subset \cup\{Cl(V_\beta) : \beta \in I(y), y \in K\}$. Therefore $f^{-1}(G)$ is quasi H-closed. □

Corollary 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be as in Theorem 3.20. If Y is N sθ-closed, then X is quasi-H-closed.*

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