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Fixed point theorems for multivalued generalized contractions on complete gauge spaces

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ABSTRACT. We present fixed point theorems for generalized multivalued contractions on complete gauge spaces.

1. INTRODUCTION

This paper presents fixed point theorems for a class of generalized contractive multivalued maps defined on a complete gauge space. The results are in connection with similar theorems established by Frigon [8], [9], Granas and Frigon [10], Chiş and Precup [4], Agarwal and O'Regan [2], and Chiş [5]. The results is based on similar arguments to the single valued case [1], [3], [11]. More exactely, we deal in this paper with geneneralized contractive multivalued maps of the Riech-Rus type

$$D_\alpha(Fx,Fy) \le a_\alpha dist_\alpha(x,Fx) + b_\alpha dist_\alpha(y,Fy) + c_\alpha d_\alpha(x,y),$$

where $a_\alpha, b_\alpha, c_\alpha$ are non-negative numbers with $a_\alpha + b_\alpha + c_\alpha < 1$ (see Precup [12], Rus [13]).

Throughout this article $(X, \{d_\alpha\}_{\alpha\in A})$ (A is a directed set) will be a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in A\}$, see [7]. If $x_0 \in X$ and $r = \{r_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ denote by $B(x_0, r) = \{x \in X : d_\alpha(x_0, x) \le r$ for all $\alpha \in A\}$. For $Y \subset X$ and $x \in X$ fixed, by $dist_\alpha(x, Y)$ we mean $\inf\limits_{y\in Y} d_\alpha(x, y)$. We denote by D_α the generalized Pompeiu-Hausdorff pseudo-metric induced by d_α, that is

$$D_\alpha(Z,Y) = \inf\{\varepsilon > 0 : \forall\, x \in Z, \forall\, y \in Y,\ \exists x^* \in Z,\ \exists y^* \in Y$$

such that $d_\alpha(x, y^*) < \varepsilon, d_\alpha(x^*, y) < \varepsilon\}$ for $Y, Z \subset X$, with the convention that $\inf(\emptyset) = \infty$.

2. FIXED POINT RESULTS

We begin with a fixed point result for generalized contractive multimaps with closed values defined on a complete gauge space.

Theorem 2.1. *Let $(X, \{d_\alpha\}_{\alpha\in A})$ be a complete gauge space, $r \in (0,\infty)^A$, $x_0 \in X$ and $F : B(x_0, r) \longrightarrow C(X)$; here $C(X) = \{Y \subset X : Y$ is closed and nonempty$\}$. Suppose for every $x, y \in B(x_0, r)$, and every $\alpha \in A$ we have*

$$D_\alpha(Fx,Fy) \le a_\alpha dist_\alpha(x,Fx) + b_\alpha dist_\alpha(y,Fy) + c_\alpha d(x,y),$$

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where $a_\alpha, b_\alpha, c_\alpha$ *are non-negative numbers with* $a_\alpha + b_\alpha + c_\alpha < 1$. *In addition assume the following two properties hold:*

(i) *for every* $\alpha \in A$,

$$dist_\alpha(x_0, Fx_0) < \left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha \tag{2.1}$$

(ii) *for every* $x \in B(x_0, r)$ *and every* $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ *there exists* $y \in Fx$ *with*

$$d_\alpha(x, y) \leq dist_\alpha(x, Fx) + \varepsilon_\alpha \text{ for all } \alpha \in A. \tag{2.2}$$

Then there exists $x \in B(x_0, r)$ *with* $x \in Fx$.

Proof. From conditions (2.1) and (2.2) we may choose a $x_1 \in Fx_0$ with

$$d_\alpha(x_1, x_0) < \left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha \text{ for every } \alpha \in A. \tag{2.3}$$

Notice from (2.3) we have that $x_1 \in B(x_0, r)$. In what follows, for each $\alpha \in A$ we choose an $\varepsilon_\alpha > 0$ such that

$$\frac{a_\alpha + c_\alpha}{1 - b_\alpha} d_\alpha(x_0, x_1) + \frac{\varepsilon_\alpha}{1 - b_\alpha} < \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha. \tag{2.4}$$

According to (ii), there exists $x_2 \in Fx_1$ with $d_\alpha(x_1, x_2) \leq dist_\alpha(x_1, Fx_1) + \varepsilon_\alpha$ for all $\alpha \in A$. Then:

$$\begin{aligned} d_\alpha(x_1, x_2) &\leq dist_\alpha(x_1, Fx_1) + \varepsilon_\alpha \\ &\leq D_\alpha(Fx_0, Fx_1) + \varepsilon_\alpha \\ &\leq a_\alpha dist_\alpha(x_0, Fx_0) + b_\alpha dist_\alpha(x_1, Fx_1) + c_\alpha d_\alpha(x_0, x_1) + \varepsilon_\alpha \\ &\leq a_\alpha d(x_0, x_1) + b_\alpha d_\alpha(x_1, x_2) + c_\alpha d_\alpha(x_0, x_1) + \varepsilon_\alpha. \end{aligned}$$

Hence

$$(1 - b_\alpha) d_\alpha(x_1, x_2) \leq (a_\alpha + c_\alpha) d_\alpha(x_0, x_1) + \varepsilon_\alpha.$$

Then, also using (2.4) we obtain

$$d_\alpha(x_1, x_2) \leq \frac{a_\alpha + c_\alpha}{1 - b_\alpha} d_\alpha(x_0, x_1) + \frac{\varepsilon_\alpha}{1 - b_\alpha} \leq \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha$$

for all $\alpha \in A$. Then

$$\begin{aligned} d_\alpha(x_0, x_2) &\leq d_\alpha(x_0, x_1) + d_\alpha(x_1, x_2) \\ &< \left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha + \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha \\ &= \left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha \left(1 + \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) \\ &\leq \left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha \left[1 + \frac{a_\alpha + c_\alpha}{1 - b_\alpha} + \left(\frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right)^2 + \left(\frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right)^3 + \ldots\right] \\ &= \left(1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha}\right) r_\alpha \frac{1}{1 - \dfrac{a_\alpha + c_\alpha}{1 - b_\alpha}} = r_\alpha. \end{aligned}$$

So we have $d_\alpha(x_0,x_2) < r_\alpha$ for all $\alpha \in A$, that is $x_2 \in B(x_0,r)$.

Next for $\alpha \in A$, choose $\delta_\alpha > 0$ such that

$$\frac{a_\alpha + c_\alpha}{1-b_\alpha} d_\alpha(x_1,x_2) + \frac{\delta_\alpha}{1-b_\alpha} < \left(\frac{a_\alpha + c_\alpha}{1-b_\alpha}\right)^2 \left(1 - \frac{a_\alpha + c_\alpha}{1-b_\alpha}\right) r_\alpha. \tag{2.5}$$

Now we choose $x_3 \in Fx_2$ with $d_\alpha(x_2,x_3) \le dist_\alpha(x_2,Fx_2) + \delta_\alpha$ for all $\alpha \in A$. Then

$$\begin{aligned} d_\alpha(x_2,x_3) &\le dist_\alpha(x_2,Fx_2) + \delta_\alpha \\ &\le D_\alpha(Fx_1,Fx_2) + \delta_\alpha \\ &\le a_\alpha dist_\alpha(x_1,Fx_1) + b_\alpha dist_\alpha(x_2,Fx_2) + c_\alpha d_\alpha(x_1,x_2) + \delta_\alpha \\ &\le a_\alpha d(x_1,x_2) + b_\alpha d_\alpha(x_2,x_3) + c_\alpha d_\alpha(x_1,x_2) + \delta_\alpha. \end{aligned}$$

This together with (2.5) implies that

$$d_\alpha(x_2,x_3) \le \frac{a_\alpha + c_\alpha}{1-b_\alpha} d_\alpha(x_1,x_2) + \frac{\varepsilon_\alpha}{1-b_\alpha} \le \left(\frac{a_\alpha + c_\alpha}{1-b_\alpha}\right)^2 \left(1 - \frac{a_\alpha + c_\alpha}{1-b_\alpha}\right) r_\alpha$$

for all $\alpha \in A$.
As above we obtain that $d_\alpha(x_0,x_3) < r_\alpha$ for all $\alpha \in A$, that is $x_3 \in B(x_0,r)$.

Proceeding inductively we obtain

$$d(x_{n+1},x_n) < \left(\frac{a_\alpha + c_\alpha}{1-b_\alpha}\right)^n \left(1 - \frac{a_\alpha + c_\alpha}{1-b_\alpha}\right) r_\alpha, \text{ for all } \alpha \in A,$$

where $x_n \in Fx_{n-1}$ and $x_n \in B(x_0,r)$.

Since $\frac{a_\alpha + c_\alpha}{1-b_\alpha} \in [0,1)$ it follows that $\left(\frac{a_\alpha + c_\alpha}{1-b_\alpha}\right)^n \to 0$, as $n \to \infty$ and, consequently (x_n) is a Cauchy sequence and since X is complete, (x_n) converges to $x \in B(x_0,r)$. We claim now that $x \in Fx$. Indeed, for all $\alpha \in A$ we have some

$$\begin{aligned} d_\alpha(x,Fx) &\le d_\alpha(x,x_n) + dist_\alpha(x_n,Fx) \le d_\alpha(x,x_n) + D_\alpha(Fx_{n-1},Fx) \\ &\le d_\alpha(x,x_n) + a_\alpha dist_\alpha(x_{n-1},Fx_{n-1}) + b_\alpha dist_\alpha(x,Fx) + c_\alpha d_\alpha(x_{n-1},x). \end{aligned}$$

Hence

$$(1-b_\alpha) dist_\alpha(x,Fx) \le d_\alpha(x,x_n) + c_\alpha d_\alpha(x_{n-1},x) + a_\alpha d_\alpha(x_{n-1},x_n).$$

In the last inequality letting $n \to \infty$ we obtain

$$(1-b_\alpha) dist_\alpha(x,Fx) \le 0.$$

So $dist_\alpha(x,F(x)) = 0$, that is $x \in \overline{F(x)} = F(x)$. Thus the proof of Theorem 2.1 is complete. □

Next we present an homotopy result for this type of multivalued generalized contractions.

Theorem 2.2. *Let $(X, \{d_\alpha\}_{\alpha \in A})$ be a complete gauge space. Let $U \subset X$ be an open subset of X. Assume $H : \overline{U} \times [0,1] \longrightarrow C(X)$ satisfies the following conditions:*

(i) $x \notin H(x,\lambda)$ for $x \in \overline{U} \setminus U$ and $\lambda \in [0,1]$;

(ii) for every $\lambda \in [0,1]$, $\alpha \in A$ and $x,y \in \overline{U}$ we have

$$D_\alpha(H(x,\lambda),H(y,\lambda)) \le a_\alpha dist_\alpha(x,H(x,\lambda) + b_\alpha dist_\alpha(y,H(y,\lambda)) + c_\alpha d_\alpha(x,y).$$

Here $a_\alpha, b_\alpha, c_\alpha$ *are non-negative numbers with* $a_\alpha + b_\alpha + c_\alpha < 1$;

(iii) *for every* $\lambda \in [0,1]$ *and every* $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ *there exists* $y \in H(x,\lambda)$ *with* $d_\alpha(x,y) \leq dist_\alpha(x, H(x,\lambda)) + \varepsilon_\alpha$ *for every* $\alpha \in A$;

(iv) *for every* $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ *there exists* $\delta = \delta(\varepsilon) > 0$ *(which does not depend on* α*) such that for* $\lambda, \mu \in [0,1]$ *with* $|\lambda - \mu| < \delta$*, we have* $D_\alpha(H(x,\lambda), H(x,\mu)) < \varepsilon_\alpha$ *for all* $x \in \overline{U}$ *and all* $\alpha \in A$;

(v) *there exists* $\alpha \in A$*, with* $\inf\{dist_\alpha(x, H(x,\lambda)) : x \in \partial U$ *and* $\lambda \in [0,1]\} > 0$;

In addition assume H_0 *has a fixed point. Then* $H_\lambda(.) = H(.,\lambda)$ *has a fixed point for each* $\lambda \in [0,1]$.

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \text{ there exists } x \in U \text{ with } x \in H(x,\lambda)\}.$$

Since H_0 has a fixed point and (i) holds, we have $0 \in \Lambda$, and so the set Λ is nonempty. We will show Λ is open and closed in $[0,1]$ and so by the connectedness of $[0,1]$ we have $\Lambda = [0,1]$ and the proof will be finished.

First we show that Λ is closed in $[0,1]$.

Let (λ_k) be a sequence in Λ with $\lambda_k \to \lambda \in [0,1]$ as $k \to \infty$. By definition of Λ, for each k, there exists $x_k \in U$ such that $x_k \in H(x_k, \lambda_k)$. We claim that

$$\inf_{k \geq 1} dist_\alpha(x_k, X \backslash U) > 0 \quad \text{(here } \alpha \text{ is as in (v)).} \tag{2.6}$$

Suppose our claim is true and then there exists $\varepsilon_\alpha > 0$ with $d_\alpha(x_k, z) > \varepsilon_\alpha$ for all $k \geq 1$ and $z \in X \backslash U$. So there exists $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ with $B(x_k, \varepsilon) \subset U$ for all $k \geq 1$. Now fix $\alpha \in A$. From (iv) we have that there exists an integer n_0, (which does not depend on α) with

$$dist_\alpha(x_{n_0}, H(x_{n_0}, \lambda_{n_0})) \leq D_\alpha(H(x_{n_0}, \lambda_{n_0}), H(x_{n_0}, \lambda)) < (1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha})\varepsilon_\alpha$$

if $|\lambda - \lambda_{n_0}| < \delta$.

Now Theorem 2.1 implies that H_λ has a fixed point $x_{\lambda, n_0} \in B(x_{n_0}, \varepsilon) \subset U$. Hence $\lambda \in \Lambda$ and so Λ is closed in $[0,1]$.

We prove now Λ is open in $[0,1]$.

Let $\lambda_0 \in \Lambda$ and $x_0 \in U$ such that $x_0 \in H(x_0, \lambda_0)$. From U open we know that there exists $\delta_{1,} \delta_2, ..., \delta_m > 0$ with $U(x_{0,} \delta_1) \cap ... \cap U(x_0, \delta_m) \subset U$. Here $U(x_0, \delta_i) = \{x \in X : d_{\alpha_i}(x, x_0) \leq \delta_i\}$, for $i = 1, 2, ..., m$ (here $\alpha_i \in A$ for $i \in \{1, 2, ..., m\}$).Then there exists $\delta = \{\delta_\alpha\}_{\alpha \in A} \in (0,\infty)^A$ with $B(x_0, \delta) \subset U$. Fix $\alpha \in A$. Now using the condition (iv) we have: there exists $\eta = \eta(\delta) > 0$ such that for each $\lambda \in [0,1]$ $|\lambda - \lambda_0| \leq \eta$ with $D_\alpha(H(x,\lambda), H(x,\lambda_0)) < \varepsilon$ for any $x \in B(x_0, \delta)$. So this property holds for x_0 too, and then we have

$$dist_\alpha(x_0, H(x_0, \lambda)) \leq D_\alpha(H(x_0, \lambda_0), H(x_0, \lambda)) < (1 - \frac{a_\alpha + c_\alpha}{1 - b_\alpha})\delta_\alpha$$

$$\text{for } \lambda \in [0,1] \text{ with } |\lambda - \lambda_0| \leq \eta.$$

Using now (ii), (iv) and (v) together with the Theorem 2.1 (in this case $r = \delta$ and $F = H_\lambda$) we obtain that there exists $x_\lambda \in B(x_0, \delta) \subset U$ with $x_\lambda \in H_\lambda(x_\lambda)$ for $\lambda \in [0,1]$, $|\lambda - \lambda_0| \leq \eta$. Consequently Λ is open in $[0,1]$.

In what follows we show that the claim (2.6) is true. Suppose (2.6) is false, that is,

$$\inf_{k\geq 1} dist_\alpha(x_k, X\backslash U) = 0 \quad \text{(here } \alpha \text{ is as in (v))}.$$

Fix $i \in \{1, 2, ...\}$. Then there exists $n_i \in \{1, 2, ...\}$ and $y_{n_i} \in X\backslash U$ with $d_\alpha(x_{n_i}, y_{n_i}) < \dfrac{1}{i}$. Hence there exists a subsequence S_α of $\{1, 2, ...\}$ and a sequence (y_i) in $X\backslash U$, for $i \in S_\alpha$, with

$$d_\alpha(x_i, y_i) < \frac{1}{i} \quad \text{for } i \in S_\alpha. \tag{2.7}$$

The last inequality together with (v) implies

$$\begin{aligned} &0 < \inf\{dist_\alpha(x, H(x,\lambda)) : x \in \partial U, \lambda \in [0,1]\} \leq \\ &\leq \lim_{i\to\infty,\, i\in S_\alpha} \inf\, dist_\alpha(y_i, H(y_i, \lambda_i)). \end{aligned} \tag{2.8}$$

We show now that

$$\lim_{i\to\infty,\, i\in S_\alpha} \inf\, dist_\alpha(y_i, H(y_i, \lambda_i)) = 0. \tag{2.9}$$

We have

$$\begin{aligned} &\lim_{i\to\infty} \inf_{in\, S_\alpha} dist_\alpha(y_i, H(y_i,\lambda_i)) \leq \lim_{i\to\infty} \inf_{in\, S_\alpha} [d_\alpha(y_i, x_i) + dist_\alpha(x_i, H(y_i,\lambda_i))] \\ &\leq \lim_{i\to\infty} \inf_{in\, S_\alpha} \left[\frac{1}{i} + D_\alpha(H(x_i,\lambda_i), H(y_i,\lambda_i))\right] \\ &= \lim_{i\to\infty} \inf_{in\, S_\alpha} D_\alpha(H(x_i,\lambda_i), H(y_i,\lambda_i)) \\ &\leq \lim_{i\to\infty} \inf_{in\, S_\alpha} [a_\alpha dist_\alpha(x_i, H(x_i,\lambda_i) + b_\alpha dist_\alpha(y_i, H(y_i,\lambda_i)) + c_\alpha d_\alpha(x_i, y_i)] \\ &\leq \lim_{i\to\infty} \inf_{in\, S_\alpha} [0 + b_\alpha dist_\alpha(y_i, H(y_i,\lambda_i)) + c_\alpha d_\alpha(x_i, y_i)] \\ &\leq b_\alpha \lim_{i\to\infty} \inf_{in\, S_\alpha} dist_\alpha(y_i, H(y_i,\lambda_i)) + c_\alpha \lim_{i\to\infty} d_\alpha(x_i, y_i) \\ &\leq b_\alpha \lim_{i\to\infty} \inf_{in\, S_\alpha} dist_\alpha(y_i, H(y_i,\lambda_i)) + c_\alpha \lim_{i\to\infty} \frac{1}{i}. \end{aligned}$$

Hence

$$(1 - b_\alpha) \lim_{i\to\infty,\, i\in S_\alpha} \inf\, dist_\alpha(y_i, H(y_i,\lambda_i)) \leq 0.$$

As a result we have that assumption (2.9) is true. But the assumption (2.9) is in contradiction with (2.8) so the claim (2.6) is true. Now the proof of Theorem 2.2 is complete. □

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