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A note on a paper of Chatzidakis-Pappas

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ABSTRACT. In this brief note we give a new algebraic interpretation, which is also a non-elementary extension, of an assertion due to Chatzidakis-Pappas appeared in J. Symbolic Logic (2001).

1. INTRODUCTION

Suppose G is an infinite abelian p-group and suppose K is a field of the first kind with respect to the prime number p and is of characteristic different from p. Let $K[G]$ be the K-group algebra of G and let $SK[G]$ be the group of all normalized p-units in $K[G]$. As usual, in what follows, $|G|$ denotes the cardinality of G, B_G its basic subgroup, and G^1 its first Ulm subgroup. All other unexplained notions, notations as well as the terminology from the abelian group theory are standard and follow mainly the reference [6]; for instance $\mathbf{Z}(p^\infty)$ and $\mathbf{Z}(p^n)$ are the quasi-cyclic group and the cyclic group of order p^n, respectively.

A problem of some interest in the theory of commutative semi-simple group algebras is the so-termed *Classification Problem* for $SK[G]$ (see, e.g., [8] and, more detailed, [4] plus [5]). In more precise words, one must classify $SK[G]$ up to isomorphism only in terms of K and G. A successful attempt for this was made by us in [4] by calculation of the Ulm-Kaplansky invariants of the quotient group $SK[G]/G$ and by giving a realistic strategy for the complete resolution of the problem. There we have also point out and eliminate some errors in the result of Mollov from [9].

Now, for completeness of the exposition, we provide the reader with a little information about certain important achievements in the theme explored.

If G is of countable power, that is $|G| = \aleph_0$, and the first Ulm factor G/G^1 is infinite, then consulting with [8] we detect that $SK[G] \cong SK[H]$ for any abelian group H if and only if $|G| = |H|$ and $|G/G^1| = |H/H^1|$ or, equivalently, H and H/H^1 are both countable groups. In fact, by virtue of [8], $|SK[G]| = |G|$ and $|SK[H]| = |H|$, hence obviously $|G| = |H|$. Moreover, owing again to [7], $SK[G] \cong SK[H]$ ensures $SK[G]/S^1K[G] \cong SK[G/G^1] \cong SK[H/H^1] \cong SK[H]/S^1K[H]$ whence H/H^1 must be infinite. Thus the necessity follows. To treat the sufficiency, according to [8] and the second Prufer theorem proved in [6], which guarantees that G/G^1 is a direct sum of cyclic groups, we can write $SK[G] \cong \sum_{|G|} \mathbf{Z}(p^\infty) \times \sum_{|G/G^1|} \times_{n<\omega} \mathbf{Z}(p^n)$ and also by a reason of symmetry $SK[H] \cong \sum_{|H|} \mathbf{Z}(p^\infty) \times \sum_{|H/H^1|} \times_{n<\omega} \mathbf{Z}(p^n)$. Evidently $SK[G]$ and $SK[H]$ are isomorphic, thus proving the desired claim.

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When G is infinite reduced, we extract that G/G^1 is infinite, so the cardinality $|G|$ of the countable reduced group G is a full system of invariants for $SK[G]$. In other words, under the above circumstances, the countable groups $SK[G]$ are isomorphic.

In [1], Chatzidakis and Pappas showed that a similar fact is, in general, not possible for the uncountable case by constructing of separable p-groups with special properties. We shall confirm below their affirmation by making use of another approach.

When G is divisible, complying with [8], $SK[G] \cong \sum_{|G|} \mathbf{Z}(p^\infty)$ and $SK[H] \cong \sum_{|H|} \mathbf{Z}(p^\infty)$, so it is clear that the cardinality $|G|$ is a complete set of invariants for the divisible group $SK[G]$.

When G/G^1 is finite no identity (i.e. G is indivisible), we shall show further on that $|G|$ is not enough to determine the isomorphism structure of $SK[G]$ even where G is countable. Thereby, we will establish in the sequel that there are $\aleph_0$ non-isomorphic countable groups $SK[G]$ with G/G^1 finite nontrivial.

Specifically, the goal of this short article is to strengthen the statement obtained by Chatzidakis-Pappas (see [1]) to arbitrary infinite cardinals ($\geq \aleph_0$), which are not necessarily regular, by the construction of special direct sums of co-cyclic (= cyclic and quasi-cyclic) p-groups. This is done in the next section.

2. Main result

The following is essential for the present research exploration.

Theorem 2.1. *Let K be the first kind field with respect to p of $char(K) \neq p$. Then, for each cardinal $\kappa \geq \aleph_0$, there exist $\geq \aleph_0$ and $\leq \kappa$ non-isomorphic groups $SK[G]$ where G is an abelian p-group of power κ. In particular, if $\kappa = \aleph_\delta$ for any ordinal δ, then the number of non-isomorphic groups of normalized p-elements is $\aleph_0$ when $0 \leq \delta \leq \omega_0$ and $|\delta|$ when $\delta \geq \omega_0$.*

Proof. Assume β is a limit ordinal with $|\beta| \leq |G_\alpha|$ (thus $|\beta| = |G_\alpha| = \aleph_0$, or $|\beta| < |G_\alpha|$ whence $|G_\alpha| > \aleph_0$) for the abelian p-groups $G_\alpha = G_d^{(\alpha)} \times C^{(\alpha)}$, where $G_d^{(\alpha)}$ are divisible groups with $|G_d^{(\alpha)}| = |G_d^{(\alpha+1)}| \geq \aleph_0$ whereas $C^{(\alpha)}$ are direct sums of cycles with $|C^{(\alpha)}| < |C^{(\alpha+1)}|$ and $|G_d^{(\alpha)}| > |C^{(\alpha)}|$, over all ordinals $0 \leq \alpha < \beta$. Moreover, when $\beta = \omega_0$, i.e. when all groups $C^{(\alpha)}$ together with the ordinals α are finite, we shall additionally presume that $C^{(\alpha)}$ are cyclic groups of exponent $p^{c+\alpha}$ whenever c is the constant of K about p defined as in [8] and $0 \leq \alpha < \omega_0$.

Therefore, $|G_\alpha| = |G_{\alpha+1}| = |G_d^{(\alpha)}|$. Utilizing [6], the basic subgroups B_{G_α} are isomorphic to $C^{(\alpha)}$ hence $|B_{G_\alpha}| < |B_{G_{\alpha+1}}|$. Certainly, G_α are not isomorphic.

Now, invoking to [3], $B_{SK[G_\alpha]} \cong SK[B_{G_\alpha}]$ and consequently, since from [8] we have that $|SK[B_{G_\alpha}]| = |B_{G_\alpha}| \geq \aleph_0$ or otherwise concerning the finite case $SK[B_{G_\alpha}] \cong SK[B_{G_{\alpha+1}}]$ implies $|B_{G_\alpha}| = |B_{G_{\alpha+1}}|$ (see, for example, [2, formula (4)] plus [8]) only when B_{G_α} and $B_{G_{\alpha+1}}$ are cyclic groups with the same exponent p^t so that t lies in the spectrum of K about p (cf. [8]), we deduce in general that $SK[G_\alpha]$ are non-isomorphic, as promised.

Let now the substitution $\kappa = \aleph_\delta$ holds valid for an arbitrary ordinal number δ. Because of the simple fact, which relies on the construction, that the interval of indices, that is precisely the wanted number of so-defined groups G_α, is of length either $\aleph_0$ or $|\delta|$, we are done.

This concludes the proof. $\square$

Example 2.1. Following the algorithm presented above, plain direct computations lead us to this that, for instance, the putting $\kappa = \aleph_{\omega_1}$ yields that the number of such non-isomorphic groups is equal to $\aleph_1$, $\kappa = \aleph_{\omega_2}$ that the number is $\aleph_2$, and etc.

We close the work with another treatment of the investigated problem.

Remark 2.1. For some (infinite) ordinal α we may also construct a family of non-isomorphic reduced groups $\{SK[G_\rho]\}_{\rho<\alpha}$ in the following manner: Appealing to [7] there are cardinals $\{\chi_\rho\}_{\rho<\alpha}$ with the properties $\aleph_0 \leq \chi_0 < \chi_1 < \chi_2 < \cdots < \chi_\rho < \cdots < \kappa$ and $\kappa \leq \chi_\rho^{\aleph_0}$ for sufficiently many $\rho < \alpha$ whose number is equal to $|\alpha|$. Thus we can take (the choice is correct by the usage of [6]) reduced abelian p-groups $\{G_\rho\}_{\rho<\alpha}$ such that $|G_0| = |G_1| = |G_2| = \cdots = |G_\rho| = \cdots = \kappa$ while $|B_{G_\rho}| = \chi_\rho$, for ρ as already described.

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