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The SOR method for infinite systems of linear equations (III)

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ABSTRACT. In [3], [4] we presented the extension of the Jacobi and Gauss-Seidel iterative numerical method from the case of finite linear systems to the case of infinite systems.

The purpose of this paper is to extend the classical SOR (successiv over relaxation) method, known for finite linear systems, to infinite systems.

1. VECTOR NORMS

Let $x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{pmatrix}$ be a sequence of real numbers represented in the form of an infinite column vector, and we denote by s the real linear space of these sequences. Let us define $l^1 = \{x \in s \mid \sum_{i=0}^{\infty} |x_i| \text{ is convergent}\}$. It is well known that l^1 is a real linear subspace of s and for every $x \in l^1$ the formula $\|x\|_1 = \sum_{i=0}^{\infty} |x_i|$ defines a norm on l^1. In this way $(l^1, \|\cdot\|_1)$ is not only a normed linear space, but a Banach space, too. We will call it vector space, the elements vectors and the above mentioned norm, vector norm [6], [8]. For this paragraph see also [2].

2. MATRIX NORMS

Let $A = (a_{ij})_{i,j\in\mathbb{N}}$ be an infinite matrix of real numbers and we denote by M the real linear space of these infinite matrices. Let $M^1 = \left\{A \in M \mid \sup_{j\in\mathbb{N}} \sum_{i=0}^{\infty} |a_{ij}| \text{ is finite}\right\}$. Then M^1 is a real linear subspace of M and for every $A \in M^1$ the formula $\|A\|_1 = \sup_{j\in\mathbb{N}} \sum_{i=0}^{\infty} |a_{ij}|$ defines a norm on M^1 called column norm. In this way $(M^1, \|\cdot\|_1)$ becomes not only a real linear normed space, but a Banach space, too.

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Corollary 2.1. *If for the matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ we have $a_{ij} = 0$ for $i > n$ and $j > n$, $n \in \mathbb{N}$, then from the above we obtain the results in the finite dimensional space $\mathbb{R}^n$* [1], [5].

This space will be called matrix space and the above mentioned norm, matrix norm. For this paragraph see also [2], [6], [8].

3. The compatibility of the vector and matrix norms

Let $x \in s$ be a sequence of real numbers, and $A = (a_{ij})_{i,j\in\mathbb{N}} \in M$ an infinite matrix of real numbers.

Definition 3.1. We will define the product $A \cdot x$ if for every $i \in \mathbb{N}$ the series $\sum_{j=0}^{\infty} a_{ij}x_j$ is convergent. In this case the resulting vector $y = A \cdot x$ is a column vector with components $y = \begin{pmatrix} \sum_{j=0}^{\infty} a_{0j}x_j \\ \sum_{j=0}^{\infty} a_{1j}x_j \\ \vdots \\ \sum_{j=0}^{\infty} a_{ij}x_j \\ \vdots \end{pmatrix}$.

Theorem 3.1. *The vector norm $\|\cdot\|_1$ defined on l^1 is compatible with the matrix norm $\|\cdot\|_1$ defined on M^1, i.e. $\|Ax\|_1 \le \|A\|_1 \cdot \|x\|_1$ for every $x \in l^1$ and every $A \in M^1$.*

Proof. We have:

$$\begin{aligned} \|A\cdot x\|_1 &= \sum_{i=0}^{\infty}\left|\sum_{j=0}^{\infty} a_{ij}x_j\right| \le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|a_{ij}|\cdot|x_j| = \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}|a_{ij}|\cdot|x_j| = \sum_{j=0}^{\infty}|x_j|\cdot\sum_{i=0}^{\infty}|a_{ij}| \le \\ &\le \sum_{j=0}^{\infty}|x_j|\cdot\sup_{j\in\mathbb{N}}\sum_{i=0}^{\infty}|a_{ij}| = \sup_{j\in\mathbb{N}}\sum_{i=0}^{\infty}|a_{ij}|\cdot\sum_{j=0}^{\infty}|x_j| = \|A\|_1\cdot\|x\|_1. \end{aligned}$$

□

Corollary 3.2. *If for the matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ we have $a_{ij} = 0$ for $i > n$ and $j > n$, $n \in \mathbb{N}$, then from Theorem 3.1 we reobtain the results in the finite dimensional space $\mathbb{R}^n$* [1], [5].

For this paragraph see also [2], [6], [8].

4. THE MATRIX NORM SUBORDINATED TO A GIVEN VECTOR NORM

For every $x \in l^1$ and $A \in M^1$ we have $\|Ax\|_1 \leq \|A\|_1 \cdot \|x\|_1$ according to Theorem 3.1. If $x \neq \theta_{l^1}$ (the null element of the vector space l^1), then $\frac{\|Ax\|_1}{\|x\|_1} \leq \|A\|_1$ and we can define $\sup\left\{\frac{\|Ax\|_1}{\|x\|_1} \mid x \in l^1 \setminus \{\theta_{l^1}\}\right\}$.

It is known that this formula defines a matrix norm on M^1, which we call the matrix norm subordinated to the vector norm $\|\cdot\|_1$ defined on l^1 and we denote it by $\|A\|_1^* = \sup\left\{\frac{\|Ax\|_1}{\|x\|_1} \mid x \in l^1 \setminus \{\theta_{l^1}\}\right\}$. It is immediately that $\|A\|_1^* \leq \|A\|_1$, for every $A \in M^1$. Actually, we have

Theorem 4.2. $\|A\|_1^* = \|A\|_1$.

Proof. We must prove that $\|A\|_1^* \geq \|A\|_1$. From $\|A\|_1 = \sup_{j\in\mathbb{N}} \sum_{i=0}^{\infty} |a_{ij}|$ we obtain: for every $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} |a_{ij_0}| \geq \|A\|_1 - \varepsilon$. Let us choose the vector $x \in l^1$ such that all the components of x are zero except for the component j_0. So

$$\begin{aligned}
\|A \cdot x\|_1 &= \sum_{i=0}^{\infty}\left|\sum_{j=0}^{\infty} a_{ij}x_j\right| = \sum_{i=0}^{\infty} |a_{ij_0}x_{j_0}| = \\
&= \sum_{i=0}^{\infty} |a_{ij_0}| \cdot |x_{j_0}| = \left(\sum_{i=0}^{\infty} |a_{ij_0}|\right) \cdot |x_{j_0}| \geq \\
&\geq (\|A\|_1 - \varepsilon) \cdot |x_{j_0}| = (\|A\|_1 - \varepsilon) \cdot \|x\|_1.
\end{aligned}$$

Consequently

$$\frac{\|Ax\|_1}{\|x\|_1} \geq \|A\|_1 - \varepsilon,$$

i.e.

$$\|A\|_1^* = \sup\left\{\frac{\|Ax\|_1}{\|x\|_1} \mid x \in l^1 \setminus \{\theta_{l^1}\}\right\} \geq \|A\|_1 - \varepsilon,$$

for every $\varepsilon > 0$. This means that $\|A\|_1^* \geq \|A\|_1$. □

Corollary 4.3. *If for the matrix* $A = (a_{ij})_{i,j\in\mathbb{N}}$ *we have* $a_{ij} = 0$ *for* $i > n$ *and* $j > n$, $n \in \mathbb{N}$, *then from Theorem* 4.2 *we reobtain the results in the finite dimensional space* $\mathbb{R}^n$ [1], [5].

For this paragraph see also [2], [6], [8].

The above presented vector and matrix spaces will be used to extend the iterative Jacobi's and Gauss-Seidel's methods, from finite linear systems to the case of infinite systems. In this way we can study the linear stationary processes with infinite but countable number of parameters.

5. The SOR method for infinite systems of linear equations

Let us consider the infinite system of linear equations $Ax = b$, where $A \in M$ and $x, b \in s$.

Definition 5.2. For a given $A \in M$ and $b \in s$ we will say that $x^* \in s$ is a solution of the infinite system of linear equations $Ax = b$ if we have $Ax^* = b$.

This means that the series $\sum_{j=0}^{\infty} a_{ij} x_j^*$ is convergent and we have $\sum_{j=0}^{\infty} a_{ij} x_j^* = b_i$, for every $i \in \mathbb{N}$.

Let us suppose that $a_{ii} \neq 0$ for every $i \in \mathbb{N}$ and let us consider the constants $\omega_i \in \mathbb{R} \setminus \{0\}$ for every $i \in \mathbb{N}$. The initial system of linear equations $Ax = b$ is equivalent to the following iterative system of linear equations:

$$\begin{cases} x_0 = (1-\omega_0)x_0 - \omega_0 \sum_{j=1}^{\infty} \frac{a_{0j}}{a_{00}} x_j + \omega_0 \frac{b_0}{a_{00}} \\ x_1 = -\omega_1 \frac{a_{10}}{a_{11}} x_0 + (1-\omega_1)x_1 - \omega_1 \sum_{j=2}^{\infty} \frac{a_{1j}}{a_{11}} x_j + \omega_1 \frac{b_1}{a_{11}} \\ x_2 = -\omega_2 \frac{a_{20}}{a_{22}} x_0 - \omega_2 \frac{a_{21}}{a_{22}} x_1 + (1-\omega_2)x_2 - \omega_2 \sum_{j=3}^{\infty} \frac{a_{2j}}{a_{22}} x_j + \omega_2 \frac{b_2}{a_{22}} \\ \vdots \\ x_i = -\omega_i \sum_{j=0}^{i-1} \frac{a_{ij}}{a_{ii}} x_j + (1-\omega_i)x_i - \omega_i \sum_{j=i+1}^{\infty} \frac{a_{ij}}{a_{ii}} x_j + \omega_i \frac{b_i}{a_{ii}} \\ \vdots \end{cases}$$

Using this system of linear equations, let us choose $x^0 \in s$ and we generate the sequence $(x^k)_{k\in\mathbb{N}} \subset s$ by the following iterative formula:

$$\begin{cases} x_0^{k+1} = (1-\omega_0)x_0^k - \omega_0 \sum_{j=1}^{\infty} \frac{a_{0j}}{a_{00}} x_j^k + \omega_0 \frac{b_0}{a_{00}} \\ x_1^{k+1} = -\omega_1 \frac{a_{10}}{a_{11}} x_0^{k+1} + (1-\omega_1)x_1^k - \omega_1 \sum_{j=2}^{\infty} \frac{a_{1j}}{a_{11}} x_j^k + \omega_1 \frac{b_1}{a_{11}} \\ x_2^{k+1} = -\omega_2 \frac{a_{20}}{a_{22}} x_0^{k+1} - \omega_2 \frac{a_{21}}{a_{22}} t x_1^{k+1} + (1-\omega_2)x_2^k - \omega_2 \sum_{j=3}^{\infty} \frac{a_{2j}}{a_{22}} x_j^k + \omega_2 \frac{b_2}{a_{22}} \\ \vdots \\ x_i^{k+1} = -\omega_i \sum_{j=0}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + (1-\omega_i)x_i^k - \omega_i \sum_{j=i+1}^{\infty} \frac{a_{ij}}{a_{ii}} x_j^k + \omega_i \frac{b_i}{a_{ii}} \\ \vdots \end{cases} \tag{5.1}$$

Consequently, starting from the vector x^k, we generate the vector x^{k+1} by the recursion formula $x^{k+1} = B_\omega \cdot x^k + c$.

Definition 5.3. The matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ is l^1 diagonal dominated if there exists the positive real number $\lambda > 0$ such that for every $j \in \mathbb{N}$ we have $\lambda|a_{jj}| > \sum_{\substack{i=0\\ i\neq j}}^{\infty} |a_{ij}|$.

It is immediately that A is l^1 diagonal dominated if and only if $\sup_{j\in\mathbb{N}} \sum_{\substack{i=0\\ i\neq j}}^{\infty} \left|\frac{a_{ij}}{a_{jj}}\right|$ is a finite real number.

Let us denote by $\lambda := \sup_{j\in\mathbb{N}} \sum_{\substack{i=0\\ i\neq j}}^{\infty} \left|\frac{a_{ij}}{a_{jj}}\right|$, $\omega^* = \sup\{|\omega_i|/\, i \in \mathbb{N}\} \in \mathbb{R}$ and $\omega^{**} = \sup\{|1-\omega_i|/\, i \in \mathbb{N}\} \in \mathbb{R}$. Let us suppose $\omega^* \cdot \lambda < 1$.

Theorem 5.3. *If* $\dfrac{\omega^*\lambda + \omega^{**}}{1-\omega^*\lambda} < 1$, *then the iterative sequence* $(x^k)_{k\in\mathbb{N}}$ *generated by* (5.1) *is convergent in* l^1*, for every* $x^0 \in l^1$*. The limit point* $x^* \in l^1$ *is the unique solution of the linear system* $Ax = b$*.*

Proof. We prove that condition $\dfrac{\omega^*\lambda + \omega^{**}}{1-\omega^*\lambda} < 1$ implies $\|B_\omega\|_1 < 1$. Indeed, if $y = B_\omega x$, then

$$
\begin{aligned}
\|y\|_1 &= \sum_{i=0}^{\infty} |y_i| = \sum_{i=0}^{\infty} \left| -\omega_i \sum_{j=0}^{i-1} \frac{a_{ij}}{a_{ii}} y_j + (1-\omega_i)x_i - \omega_i \sum_{j=i+1}^{\infty} \frac{a_{ij}}{a_{ii}} x_j \right| \leq \\
&\leq \sum_{i=0}^{\infty} \left(|\omega_i| \sum_{j=0}^{i-1} \left|\frac{a_{ij}}{a_{ii}}\right| |y_j| + |1-\omega_i||x_i| + |\omega_i| \sum_{j=i+1}^{\infty} \left|\frac{a_{ij}}{a_{ii}}\right| |x_j| \right) = \\
&= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i-1} |\omega_i| \left|\frac{a_{ij}}{a_{ii}}\right| |y_j| + \sum_{j=i+1}^{\infty} |\omega_i| \left|\frac{a_{ij}}{a_{ii}}\right| |x_j| \right) + \sum_{i=0}^{\infty} |1-\omega_i||x_i| = \\
&= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j-1} |\omega_i| \left|\frac{a_{ij}}{a_{jj}}\right| |x_j| + \sum_{i=j+1}^{\infty} |\omega_i| \left|\frac{a_{ij}}{a_{jj}}\right| |y_j| \right) + \sum_{i=0}^{\infty} |1-\omega_i||x_i| = \\
&= \sum_{j=0}^{\infty} \left(|x_j| \sum_{i=0}^{j-1} |\omega_i| \left|\frac{a_{ij}}{a_{jj}}\right| + |y_j| \sum_{i=j+1}^{\infty} |\omega_i| \left|\frac{a_{ij}}{a_{jj}}\right| \right) + \sum_{i=0}^{\infty} |1-\omega_i||x_i| \leq \\
&\leq \sum_{j=0}^{\infty} \left(|x_j| \sum_{i=0}^{j-1} \omega^* \left|\frac{a_{ij}}{a_{jj}}\right| + |y_j| \sum_{i=j+1}^{\infty} \omega^* \left|\frac{a_{ij}}{a_{jj}}\right| \right) + \sum_{i=0}^{\infty} \omega^{**}|x_i| \leq \\
&\leq \sum_{j=0}^{\infty} (|x_j|\omega^*\lambda + |y_j|\omega^*\lambda) + \sum_{i=0}^{\infty} \omega^{**}|x_i| = \\
&= \omega^*\lambda\|x\|_1 + \omega^*\lambda\|y\|_1 + \omega^{**}\|x\|_1.
\end{aligned}
$$

Consequently:

$$\|y\|_1 \le \omega^*\lambda\|x\|_1 + \omega^*\lambda\|y\|_1 + \omega^{**}\|x\|_1,$$

which is equivalent to

$$\frac{\|y\|_1}{\|x\|_1} \le \frac{\omega^*\lambda + \omega^{**}}{1-\omega^*\lambda}.$$

This means that

$$\|B_\omega\|_1 = \sup_{x\neq\theta_{l^1}} \frac{\|B_\omega x\|_1}{\|x\|_1} = \sup_{x\neq\theta_{l^1}} \frac{\|y\|_1}{\|x\|_1} \le \frac{\omega^*\lambda + \omega^{**}}{1-\omega^*\lambda} < 1.$$

Now we can apply the Banach fixed point theorem for the iteration map $\Phi : l^1 \to l^1$, $\Phi(x) = B_\omega x + c$. Indeed, Φ is a contraction, because

$$\|\Phi(x) - \Phi(y)\|_1 = \|(B_\omega x + c) - (B_\omega y + c)\|_1 = \|B_\omega(x-y)\|_1 \le \|B_\omega\|_1\|x-y\|_1.$$

This means that the sequence $(x^k)_{k\in\mathbb{N}}$ is convergent in l^1 for every $x^0 \in l^1$ and its limit point $x^* \in l^1$ is the unique fixed point of Φ in l^1, i.e. $\Phi(x^*) = x^*$. So $B_\omega x^* + c = x^*$, which is equivalent to $Ax^* = b$. □

Corollary 5.4. *If for the matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ we have $a_{ij} = 0$ when $i > n$, $j > n$ and $b_i = 0$ for $i > n$, $n \in \mathbb{N}$, then we reobtain the linear system with finite number of equations and finite number of unknowns. In this way from Theorem 5.3 we obtain the classical SOR iterative numerical method to solve finite systems of linear equations* [7].

In the following we consider the particular case when $\omega_i = \omega$, for every $i \in \mathbb{N}$. So, from (5.1) we can deduce: let us choose $x^0 \in s$ and we generate the sequence $(x^k)_{k\in\mathbb{N}} \subset s$ by the following iterative formula:

$$(5.2)\quad \begin{cases} x_0^{k+1} = (1-\omega)x_0^k - \omega\displaystyle\sum_{j=1}^{\infty}\frac{a_{0j}}{a_{00}}x_j^k + \omega\frac{b_0}{a_{00}} \\ x_1^{k+1} = -\omega\dfrac{a_{10}}{a_{11}}x_0^{k+1} + (1-\omega)x_1^k - \omega\displaystyle\sum_{j=2}^{\infty}\frac{a_{1j}}{a_{11}}x_j^k + \omega\frac{b_1}{a_{11}} \\ x_2^{k+1} = -\omega\dfrac{a_{20}}{a_{22}}x_0^{k+1} - \omega\dfrac{a_{21}}{a_{22}}x_1^{k+1} + (1-\omega)x_2^k - \omega\displaystyle\sum_{j=3}^{\infty}\frac{a_{2j}}{a_{22}}x_j^k + \omega\frac{b_2}{a_{22}} \\ \vdots \\ x_i^{k+1} = -\omega\displaystyle\sum_{j=0}^{i-1}\frac{a_{ij}}{a_{ii}}x_j^{k+1} + (1-\omega)x_i^k - \omega\sum_{j=i+1}^{\infty}\frac{a_{ij}}{a_{ii}}x_j^k + \omega\frac{b_i}{a_{ii}} \\ \vdots \end{cases}$$

In this case from Theorem 5.3 we obtain:

Corollary 5.5. *If* $\dfrac{|\omega|\lambda + |1-\omega|}{1-|\omega|\lambda} < 1$, $(|\omega|\lambda < 1)$ *then the corresponding iterative sequence $(x^k)_{k\in\mathbb{N}}$ given by (5.2) is convergent in l^1 for every $x^0 \in l^1$. The limit point $x^* \in l^1$ is the unique solution of the linear system $Ax = b$.*

In the following we consider the particular case, when $\omega_i = \omega = 1$ for every $i \in \mathbb{N}$. So from (5.2) we can deduce: let us choose $x^0 \in s$ and we generate the sequence $(x^k)_{k\in\mathbb{N}} \subset s$ by the following iterative formula:

$$(5.3)\quad \begin{cases} x_0^{k+1} = -\sum_{j=1}^{\infty} \frac{a_{0j}}{a_{00}} x_j^k + \frac{b_0}{a_{00}} \\ x_1^{k+1} = -\frac{a_{10}}{a_{11}} x_0^{k+1} - \sum_{j=2}^{\infty} \frac{a_{1j}}{a_{11}} x_j^k + \frac{b_1}{a_{11}} \\ x_2^{k+1} = -\frac{a_{20}}{a_{22}} x_0^{k+1} - \frac{a_{21}}{a_{22}} x_1^{k+1} - \sum_{j=3}^{\infty} \frac{a_{2j}}{a_{22}} x_j^k + \frac{b_2}{a_{22}} \\ \quad\vdots \\ x_i^{k+1} = -\sum_{j=0}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} - \sum_{j=i+1}^{\infty} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}} \\ \quad\vdots \end{cases}$$

From Theorem 5.3 and Corollary 5.5 we obtain the author's result, the Gauss-Seidel's iterative method for infinite systems of linear equations [4]:

Corollary 5.6. *If $\lambda < \frac{1}{2}$, then the corresponding iterative sequence $(x^k)_{k\in\mathbb{N}}$ given by (5.3) is convergent in l^1 for every $x^0 \in l^1$. The limit point $x^* \in l^1$ is the unique solution of the linear system $Ax = b$.*

We can obtain similar results if we replace the space l^1 by the space l^∞ or l^p, for $p \in (1, +\infty)$.

We mention that all these results are valid in the complex case, too.

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