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# Successive approximations for the solution of second order advanced differential equations

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ABSTRACT. For the initial value problem associated with second order advanced differential equation on Banach space, it is constructed a numerical method to approximate the solution. The method uses the sequence of Picard successive approximations and the trapezoidal quadrature rule (adapted for Lipschitzian functions with values in Banach spaces).

## 1. INTRODUCTION

Let $X$ be a Banach space and $f : [a,b] \times X \times X \to X$ be a continuous function. We consider the problem:

$$\text{(1.1)} \qquad \begin{cases} x''(t) = f(t, x(t), x(h(t))), & t \in [a,b] \\ x(t) = \psi(t), & t \in [b, b_1] \end{cases}$$

where $\psi \in C^1([b, b_1], X)$

$$\text{(1.2)} \qquad b \le b_1 \ \text{ and } \ h \in C([a,b],[a,b_1]), \quad \text{with } \ t \le h(t) \le b_1, \ \forall\, t \in [a,b].$$

In this paper we construct a numerical method which use the sequence of succesive approximations and the trapezoidal quadrature rule for Lipschitz functions with values in Banach space.

The quadrature rule was obtained in [4] (for real valued functions) and in [5] (for vector valued functions). A similar quadrature rule for functions with fuzzy numbers value was obtained in [1].

In $C([a,b_1],X) \cap C^2([a,b],X)$ the problem (1.1) is equivalent to the integral equation:

$$\text{(1.3)} \quad x(t) = \begin{cases} \psi(b) + (t-b)\psi'(b) - \displaystyle\int_t^b (t-s) f(s, x(s), x(h(s)))\, ds, & t \in [a,b] \\ \psi(t), & t \in [b, b_1]. \end{cases}$$

Indeed, let

$$x \in C([a,b_1],X) \cap C^2([a,b],X)$$

be a solution of (1.3), with $\psi \in C^1([b,b_1],X)$. If we derive in (1.3) by $t$, for $t \in [a,b]$ we obtain:

$$x'(t) = \psi'(b) - \int_t^b f(s, x(s), x(h(s)))\, ds, \ \forall\, t \in [a,b].$$

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Differentiating it follows

$$x''(t) = f(t, x(t), x(h(t))), \ \forall\, t \in [a,b].$$

Then, $x$ is solution of (1.1).

Let

$$x \in C([a,b_1], X) \cap C^2([a,b], X)$$

be a solution of (1.1), with

$$\psi \in C^1([b,b_1], X).$$

For any $t \in [a,b]$, integrating in (1.1) on $[t,b]$, we obtain

$$\psi'(b) - x'(t) = \int_t^b f(s, x(s), x(h(s)))\, ds, \ \forall\, t \in [a,b].$$

Integrating again in the equality

$$x'(t) = \psi'(b) - \int_t^b f(s, x(s), x(h(s)))\, ds,$$

it follows

$$x(t) = \psi(b) + \int_t^b \psi'(b)\, ds + \int_t^b \left( \int_s^b f(v, x(v), x(h(v)))\, dv \right) ds.$$

Since integrating by parts,

$$\int_t^b \left( \int_s^b f(v, x(v), x(h(v)))\, dv \right) ds$$
$$= \int_t^b \left[ (s-t) \left( \int_s^b f(v, x(v), x(h(v))) dv \right) \Big|_t^b + (s-t) f(s, x(s), x(h(s))) \right] ds$$
$$= -\int_t^b (t-s)\, f(s, x(s), x(h(s)))\, ds,$$

we infer that

$$x(t) = \psi(b) + (t-b)\,\psi'(b) - \int_t^b (t-s)\, f(s, x(s), x(h(s)))\, ds,$$

so, $x$ is solution of (1.3).

## 2. Existence, uniqueness and approximation

Denote $Y := C([a,b_1], X)$ and consider the Bielecki's norm,

$$\|x\|_B = \max \left\{ \|x(t)\|\, e^{-\theta(b_1 - t)}, \ t \in [a,b_1] \right\}.$$

With this norm $Y$ became Banach space.

We define the operator

$$A : Y \to Y,$$

$$A(x)(t) = \begin{cases} \psi(b) + (t-b)\psi'(b) - \int_t^b (t-s) f(s, x(s), x(h(s))) ds, & t \in [a,b] \\ \psi(t), & t \in [b,b_1] \end{cases} \tag{2.4}$$

Since $f \in C([a,b] \times X \times X, X)$, we infer that $A(Y) \subset Y$.

We will impose the following conditions:

(i) $h \in C([a,b],[a,b_1])$, and $t \le h(t) \le b_1$, $\forall\, t \in [a,b]$;
(ii) $f \in C([a,b] \times X \times X, X)$;
(iii) $\psi \in C^1([b,b_1],X)$;
(iv) there exists $L > 0$ such that

$$\|f(t,u_1,u_2) - f(t,v_1,v_2)\|_X \le L \sum_{i=1}^{2} \|u_i - v_i\|_X ,$$

$\forall\, t \in [a,b]$, $\forall\, u_i, v_i \in X$, $i = \overline{1,2}$.

**Theorem 2.1.** *Under the conditions $(i)-(iv)$, the integral equation $(1.3)$ has in $Y$ an unique solution $x^*$. Moreover, $x^* \in C([a,b_1],X) \cap C^2([a,b],X)$ and it can be obtained by the method of successive approximations,*

(2.5) $\quad x_{m+1} = A(x_m),\ m \in \mathbb{N}$

*starting from any $x_0 \in Y$. In this approximation, the apriori error estimation is:*

(2.6) $$\|x^* - x_m\|_B \le \left(\frac{1}{2}\right)^{m-1} \|x_1 - x_0\|_B ,\ \forall\, m \in \mathbb{N}^*.$$

*Proof.* We have $A(Y) \subset Y$, and elementary computation leads to:

$$\begin{aligned}
&\|A(u_1) - A(u_2)\|_X \\
&\le \int_t^b (s-t)\,\|f(s,u_1(s),u_1(h(s))) - f(s,u_2(s),u_2(h(s)))\|_X\, ds \\
&\le L \int_t^b (s-t) \Big( \|u_1(s) - u_2(s)\|_X\, e^{-\theta(b_1-s)} e^{\theta(b_1-s)} \\
&+ \|u_1(h(s)) - u_2(h(s))\|_X\, e^{-\theta(b_1-h(s))} e^{\theta(b_1-h(s))} \Big)\, ds \\
&\le \frac{2L(b-a)}{\theta} \|u_1 - u_2\|_B \int_t^b \theta e^{\theta(b_1-s)} ds \\
&< \frac{2L(b-a)}{\theta} \|u_1 - u_2\|_B\, e^{\theta(b_1-t)},\ \forall\, t \in [a,b],
\end{aligned}$$

and so,

$$\|A(u_1) - A(u_2)\|_B \le \frac{2L(b-a)}{\theta} \|u_1 - u_2\|_B ,\ \forall u_1, u_2 \in Y.$$

Choosing $\theta = 4L(b-a) + 1$, we infer that $A$ is contraction, with a contraction constant less than $\frac{1}{2}$. Applying the Banach's Fixed Point Principle [6], the equation (1.3) has an unique solution $x^*$ which can be obtained by the method of successive approximations (2.5) starting from any $x_0 \in Y$ and the estimation (2.6) holds. By condition (ii) we infer that

$$x^* \in C^2([a,b],X)$$

and after elementary calculus $x^*$ is the solution of (1.1). □

Similar existence and uniqueness result for second order differential equations of mixed type can be found in [7].

**Remark 2.1.** To approximate the solution of (1.1) we can choose the first term in the sequence of successive approximations as

$$x_0(t) = \begin{cases} \psi(b) + (t-b)\psi'(b), & t \in [a,b] \\ \psi(t), & t \in [b,b_1]. \end{cases}$$

Obviously, $x_0 \in C^1([a,b_1], X)$.

By (2.5), the sequence of successive approximations is:

$$x_0(t) = \begin{cases} \psi(b) + (t-b)\psi'(b), & t \in [a,b] \\ \psi(t), & t \in [b,b_1] \end{cases}$$

$$x_{m+1}(t) = \begin{cases} \psi(b) + (t-b)\psi'(b) - \int_t^b (t-s) f(s, x_m(s), x_m(h(s))) ds, \ t \in [a,b] \\ \psi(t), t \in [b,b_1], \ \forall\, m \in \mathbb{N}. \end{cases} \tag{2.7}$$

## 3. Main result

We will approximate the terms from (2.7) in the particular case $b_1 = b + \tau$, with $\tau > 0$, the advance, and $h(t) = t + \tau$, $\forall\, t \in [a,b]$.

Suppose that exist $l \in \mathbb{N}^*$ such that $b - a = l\tau$. In this case, (1.1) became:

$$\begin{cases} x''(t) = f(t, x(t), x(t+\tau)), \ t \in [a,b] \\ x(t) = \psi(t), \ \ x'(t) = \psi'(t), \ t \in [b, b+\tau] \end{cases} \tag{3.8}$$

with $\psi \in C^1([b, b+\tau], X)$ and in (2.7) we have

$$x_{m+1}(t) = \psi(b) + (t-b)\psi'(b) - \int_t^b (t-s) f(s, x_m(s), x_m(s+\tau)) ds, \tag{3.9}$$

$\forall\, t \in [a,b]$, $\forall\, m \in \mathbb{N}$.

For (3.9) we use the trapezoidal quadrature rule variant for integrals of Lipschitzian functions with values on Banach spaces from [5]

$$\int_a^b F(t) dt = \frac{b-a}{2n} \sum_{i=0}^{n-1} \left[ F\left(a + \frac{i(b-a)}{n}\right) + F\left(a + \frac{(i+1)(b-a)}{n}\right)\right] + R_n(f) \tag{3.10}$$

$$\|R_n(f)\| \le \frac{(b-a)^2 L_F}{4n} \tag{3.11}$$

where $L_F > 0$ is the Lipschitz constant of $F : [a,b] \to X$.

Formula (3.10)+(3.11) was used in [2] to approximate the solution of nonlinear Fuzzy-Fredholm integral equations. Similar method as below, is built in [3] for first order delay ODE's.

Consider $n \in \mathbb{N}^*$ and $\Delta'_n \in Div[b, b+\tau]$ an uniform division,

$$\Delta'_n : b = t_q < t_{q+1} < ... < t_{q+n-1} < t_{q+n} = b + \tau$$

$$t_{q+i} = b + \frac{i\tau}{n}, \ \forall i = \overline{0,n},$$

and realize a similar uniform division of $[a,b]$, by

$$\Delta''_n : a = t_0 < t_1 < ... < t_{q-1} < t_q = b,$$

with $q = nl$, and $t_j - t_{j-1} = \dfrac{\tau}{n}$, $\forall\, j = \overline{1,q}$.

Let $\Delta_n = \Delta_n^{''} \cup \Delta_n^{'}$ be the obtained uniform division of $[a, b+\tau]$. We see that $x_m(t_i) = \psi(t_i), \forall\, i = \overline{q, q+n}, \forall m \in \mathbb{N}$ and

$$x_0(t_i) = \psi(b) + (t_i - b)\,\psi'(b)\,,\ \forall i = \overline{0, q}.$$

Moreover,

$$x_m(t_i) = \psi(b) + (t_i - b)\,\psi'(b) - \int_{t_i}^{b} (t_i - s)\, f(s, x_{m-1}(s), x_{m-1}(s+\tau))\, ds, \tag{3.12}$$

$\forall\, m \in \mathbb{N}^*, \forall\, i = \overline{0, q}$.

Applying in (3.12) the quadrature rule (3.10) we obtain the numerical method given by:

$$\begin{aligned} x_m(t_i) &= \psi(b) + (t_i - b)\,\psi'(b) - \frac{b-a}{2nl}[(t_i - b)\, f(b, x_{m-1}(b), x_{m-1}(b+\tau)) \\ &+ 2\sum_{j=i+1}^{q-1} (t_i - t_j)\, f(t_j, x_{m-1}(t_j), x_{m-1}(t_j+\tau))] + R_{m,i},\ \forall\, i = \overline{0, q}. \end{aligned} \tag{3.13}$$

Consider the functions

$$\begin{aligned} &F_{m,i} : [a, b] \to X, \\ &F_{m,i}(s) = (t_i - s)\, f(s, x_m(s), x_m(s+\tau))\,,\ \forall\, s \in [a, b]\,,\ \forall i = \overline{0, q},\ \forall\, m \in \mathbb{N}. \end{aligned}$$

Since $f \in C([a, b] \times X \times X, X)$, $\psi \in C^1([b, b_1], X)$ and $x_0 \in C([a, b], X)$ we infer that

$$x_m \in C([a, b_1], X)\,,\ \forall\, m \in \mathbb{N}.$$

Impose the Lipschitz condition: $\exists \alpha > 0$ such that

$$\|f(s_1, u, v) - f(s_2, u, v)\|_X \le \alpha\, |s_1 - s_2|\,, \forall\, s_1, s_2 \in [a, b]\,, \forall\, u, v \in X, \tag{3.14}$$

and in addition we suppose that

$$\exists\, M > 0 \text{ such that } \|f(t, u, v)\|_X \le M,\ \forall; t \in [a, b]\,,\ \forall\, u, v \in X.$$

Then $F_{m,i}$ are bounded, $\forall\, i = \overline{0, q}, \forall\, m \in \mathbb{N}$.

Now we investigate the Lipschitz property of the functions $F_{m,i}$, $i = \overline{0, q}$, $m \in \mathbb{N}$.

$$\begin{aligned} &\|F_{m,i}(s_1) - F_{m,i}(s_2)\|_X \\ &= \|(t_i - s_1) f(s_1, x_m(s_1), x_m(s_1+\tau)) - (t_i - s_2) f(s_2, x_m(s_2), x_m(s_2+\tau))\|_X \\ &\le \|(t_i - s_1) f(s_1, x_m(s_1), x_m(s_1+\tau)) - (t_i - s_2) f(s_1, x_m(s_1), x_m(s_1+\tau)) \\ &+ (t_i - s_2) f(s_1, x_m(s_1), x_m(s_1+\tau)) - (t_i - s_2) f(s_2, x_m(s_2), x_m(s_2+\tau))\|_X \\ &\le |(t_i - s_1) - (t_i - s_2)|\, \|f(s_1, x_m(s_1), x_m(s_1+\tau))\|_X \\ &+ |t_i - s_2|\, \|f(s_1, x_m(s_1), x_m(s_1+\tau)) - f(s_2, x_m(s_2), x_m(s_2+\tau))\|_X \\ &\le |s_1 - s_2| M + (b-a)\big[\alpha |s_1 - s_2| \\ &+ L(\|x_m(s_1) - x_m(s_2)\|_X + \|x_m(s_1+\tau) - x_m(s_2+\tau)\|_X)\big] \end{aligned} \tag{3.15}$$

and

$$\|x_m(s_1) - x_m(s_2)\|_X \le \|(s_1 - b)\psi'(b) - (s_2 - b)\psi'(b)\|_X \tag{3.16}$$
$$+ \left\| \int_{s_1}^{b} (s_1 - \eta) f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau))\, d\eta \right.$$
$$\left. - \int_{s_2}^{b} (s_2 - \eta) f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau))\, d\eta \right\|_X \le |s_1 - s_2| \, \|\psi'(b)\|_X$$
$$+ \int_{s_1}^{b} \| (s_1 - \eta) f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau))$$
$$- (s_2 - \eta) f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau)) \|_X d\eta$$
$$+ \int_{s_1}^{s_2} \|(s_2 - \eta) f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau))\|_X \, d\eta$$
$$\le |s_1 - s_2| \, \|\psi'(b)\|_X + \int_{s_1}^{b} \|f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau))\|_X \, |s_1 - s_2| \, d\eta$$
$$+ \int_{s_1}^{s_2} |s_2 - \eta| \, \|f(\eta, x_{m-1}(\eta), x_{m-1}(\eta + \tau))\|_X \, d\eta$$
$$\le |s_1 - s_2| \, \|\psi'(b)\|_X + (b - a) M |s_1 - s_2| + \int_{s_1}^{s_2} (b - a) M d\eta$$
$$= [\|\psi'(b)\|_X + 2(b - a) M] \, |s_1 - s_2|, \ \forall m \in \mathbb{N}^*, \ \ \forall s_1, s_2 \in [a, b]$$

$$\|x_0(s_1) - x_0(s_2)\|_X \le \|\psi'\|_C \, |s_1 - s_2|, \ \ \forall s_1, s_2 \in [a, b] \tag{3.17}$$

where $\|\psi'\|_C = \max\{\|\psi'(t)\|, \ t \in [b, b_1]\}$.

From (3.16) and (3.17) we infer that

$$\|x_m(s_1) - x_m(s_2)\|_X \le [\|\psi'\|_C + 2(b - a) M] \, |s_1 - s_2|, \tag{3.18}$$

$\forall\, s_1, s_2 \in [a, b]$, $\forall\, m \in \mathbb{N}$.

From (3.15) and (3.18) follows:

$$\|F_{m,i}(s_1) - F_{m,i}(s_2)\|_X$$
$$\le [M + (b - a)(\alpha + 2L(\|\psi'\|_C + 2(b - a) M))] \, |s_1 - s_2|,$$

$\forall\, s_1, s_2 \in [a, b]$, $\forall\, i = \overline{0, q}$, $\forall\, m \in \mathbb{N}$.

Let

$$\gamma := M + (b - a)[\alpha + 2L(\|\psi'\|_C + 2(b - a) M)],$$

be the Lipschitz constant of all functions $F_{m,i}$, $i = \overline{0, q}$, $\forall\, m \in \mathbb{N}$.

Then in (3.13) the remainder estimation is:

$$\|R_{m,i}\|_X \le \frac{(b - a)^2 \gamma}{4nl}, \ \forall\, i = \overline{0, q}, \ \forall m \in \mathbb{N}. \tag{3.19}$$

The relations (3.13) and (3.19) lead to the following algorithm:

$$x_1(t_i) = \psi(b) + (t_i - b)\psi'(b) - \frac{b - a}{2nl} [(t_i - b) f(b, x_0(b), x_0(b + \tau)) \tag{3.20}$$

$$+2\sum_{j=i+1}^{q-1}(t_i-t_j)\,f\left(t_j,x_0\left(t_j\right),x_0\left(t_{j+n}\right)\right)+R_{1,i}=\overline{x}_1\left(t_i\right)+R_{1,i},\forall i=\overline{1,q}.$$

(3.21) $$x_2(t_i)=\psi(b)+(t_i-b)\psi'(b)-\frac{b-a}{2nl}[(t_i-b)f(b,\overline{x}_1(b)+R_{1,q},\overline{x}_1(t_{n+q})+R_{1,q+n})$$
$$+\,2\sum_{j=i+1}^{q-1}(t_i-t_j)\,f\left(t_j,\overline{x}_1\left(t_j\right)+R_{1,j},\overline{x}_1\left(t_{j+n}\right)+R_{1,j+n}\right)]+R_{2,i}$$
$$=\psi\left(b\right)+\left(t_i-b\right)\psi'\left(b\right)-\frac{b-a}{2nl}\left[\left(t_i-b\right)f\left(t_q,\overline{x}_1\left(t_q\right),\overline{x}_1\left(t_{q+n}\right)\right)\right.$$
$$\left.+\,2\sum_{j=i+1}^{q-1}(t_i-t_j)\,f\left(t_j,\overline{x}_1\left(t_j\right),\overline{x}_1\left(t_{j+n}\right)\right)\right]+\overline{R}_{2,i}=\overline{x}_2\left(t_i\right)+\overline{R}_{2,i},\ \forall i=\overline{1,q}.$$

By induction, for $m\geq 3$, we obtain:

(3.22) $$x_m\left(t_i\right)=\psi\left(b\right)+\left(t_i-b\right)\psi'\left(b\right)$$
$$-\frac{b-a}{2nl}\left[\left(t_i-b\right)f\left(t_q,\overline{x}_{m-1}\left(t_q\right)+R_{m-1,0},\overline{x}_{m-1}\left(t_{q+n}\right)+R_{m-1,n+q}\right)\right.$$
$$\left.+\,2\sum_{j=i+1}^{q-1}(t_i-t_j)f(t_j,\overline{x}_{m-1}(t_j)+R_{m-1,j},\overline{x}_{m-1}(t_{j+n})+R_{m-1,j+n})\right]+R_{m,i}$$
$$=\psi\left(b\right)+\left(t_i-b\right)\psi'\left(b\right)-\frac{b-a}{2nl}\left[\left(t_i-b\right)f\left(t_q,\overline{x}_{m-1}\left(t_q\right),\overline{x}_{m-1}\left(t_{q+n}\right)\right)\right.$$
$$\left.+2\sum_{j=i+1}^{q-1}(t_i-t_j)f(t_j,\overline{x}_{m-1}(t_j),\overline{x}_{m-1}(t_{j+n}))\right]+\overline{R}_{m,i}=\overline{x}_m(t_i)+\overline{R}_{m,i},\ \forall i=\overline{1,q}.$$

For the remainder estimations we have:

(3.23) $$\left\|R_{1,i}\right\|_X\leq\frac{(b-a)^2\gamma}{4nl},\ \forall\ i=\overline{0,q+n}$$

$$\left\|\overline{R}_{2,i}\right\|_X\leq\left\|R_{2,i}\right\|_X$$
$$+\,\frac{b-a}{2nl}\left[L\left\|R_{1,q}\right\|_X+L\left\|R_{1,q+n}\right\|_X+2\sum_{j=i+1}^{q-1}\left(L\left\|R_{1,j}\right\|_X+L\left\|R_{1,j+n}\right\|_X\right)\right]$$
$$\leq\frac{(b-a)\gamma}{4nl}+\frac{b-a}{2nl}L\left[\|R_{1,q}\|_X+\|R_{1,q+n}\|_X+2\sum_{j=i+1}^{q-1}(\|R_{1,j}\|_X+\|R_{1,j+n}\|_X)\right]$$
$$\leq\frac{(b-a)^2\gamma}{4nl}+\frac{(b-a)\,2L}{2nl}\left[1+2\left(nl-1\right)\right]\frac{(b-a)^2\gamma}{4nl}$$
$$<\frac{(b-a)^2\gamma}{4nl}+(b-a)\,2L\frac{(b-a)^2\gamma}{4nl}=[1+2L\,(b-a)]\,\frac{(b-a)^2\gamma}{4nl},\ \forall\ i=\overline{0,q+n}$$

$$\left\|\overline{R}_{3,i}\right\|_X\leq\left\{1+2L\left(b-a\right)+\left[2L\left(b-a\right)\right]^2\right\}\frac{(b-a)^2\gamma}{4nl},\ \forall i=\overline{0,q+n}$$

and by induction we infer that

$$\|\overline{R}_{m,i}\|_X \le \left\{1+2L(b-a)+[2L(b-a)]^2+...+[2L(b-a)]^{m-1}\right\}\frac{(b-a)^2\gamma}{4nl} = \frac{1-[2L(b-a)]^m}{1-2L(b-a)}\cdot\frac{(b-a)^2\gamma}{4nl}, \tag{3.24}$$

$\forall\, m\in\mathbb{N}$, $m\ge 2$, $\forall\, i=\overline{0,q+n}$.

**Theorem 3.2.** *In the conditions* $(i)-(iv)$ *and* $(3.14)$*, if* $2L(b-a)<1$ *and* $\|f(t,u,v)\|_X\le M,\ \forall\ t\in[a,b],\ \forall\ u,v\in X$*, then the unique solution* $x^*$ *of the initial value problem (1.1) is approximated on the knots* $t_i$*,* $i=\overline{0,q+n}$ *by the sequence* $(\overline{x}_m(t_i))_{m\in\mathbb{N}}$ *computed in* $(3.20)$*,* $(3.21)$*,* $(3.22)$ *and the apriori error estimation is:*

$$\|x^*(t_i)-\overline{x}_m(t_i)\|_X \le \left(\frac{1}{2}\right)^{m-1}\|x_0-x_1\|_B+\frac{(b-a)^2\gamma}{4nl[1-2L(b-a)]}, \tag{3.25}$$

$\forall\, i=\overline{0,q+n}$, $\forall\, m\in\mathbb{N}^*$.

*Proof.* We see that

$$\|x^*(t_i)-\overline{x}_m(t_i)\|_X \le \|x^*(t_i)-x_m(t_i)\|_X+\|x_m(t_i)-\overline{x}_m(t_i)\|_X$$

and

$$\|x_m(t_i)-\overline{x}_m(t_i)\|_X=\left\|\overline{R}_{m,i}\right\|_X,\ \forall i=\overline{0,q+n}, \forall m\in\mathbb{N}^*.$$

The inequality (3.25) follows now from (2.6) and (3.24). □

## REFERENCES


[1] Bede, B., Gal, S. G., *Quadrature rules for integrals of fuzzy-number-valued functions*, Fuzzy Sets and Systems **145** (2004), 359-380

[2] Bica, A. M., *Metode numerice iterative pentru ecuatii operatoriale*, Ed. Univ. Oradea, 2005.

[3] Bica, A. M., *The Successive Approximations Method and Error Estimation in Terms of at most the First Derivative for Delay Ordinary Differential Equations*, Austral. J. of Math. Anal. and Appl., **2** (2005), issue 2, paper 6, 1-15

[4] Dragomir, S. S., Cerone, P., *Trapezoidal-type rules from an inequalities point of view* in G.A. Anastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman&Hall, CRC Press, Boca Raton, London, New York, Washington DC, 2000 (Chapter 3)

[5] Buse, C., Dragomir, S. S., Roumeliotis, Sofo, A., *Generalized trapezoid type inequalities for vector valued functions and applications*, Math. Inequal. Appl., **5** (2002), no. 3, 435-450

[6] Rus, I. A., *Weakly Picard Operators and Applications*, Seminar on Fixed Point Theory, Cluj-Napoca, **2** (2001), 41-58

[7] Rus, I. A., *Functional-Differential Equations of Mixed Type, via Weakly Picard Operators*, Seminar on Fixed Point Theory Cluj-Napoca, **3** (2002), 335-346



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