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On differences of positive linear operators

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ABSTRACT. We continue our research on the differences of positive linear operators by giving estimates for such differences. Special emphasis is on the Bernstein operators, Beta operators of the second kind as introduced by Lupaş, piecewise linear interpolation at equidistant knots and on certain products of these mappings.

1. INTRODUCTION

One version of Taylor's formula with remainder is given in Theorem 1.6.6 of Davis' book [3] where the remainder term is attributed to Young. This is also known as the Peano form.

Theorem 1.1. *For $n \in \mathbb{N}_0$ let $f(x)$ be n times differentiable at $x = x_0$. Then*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{1}{(n-1)!} f^{(n-1)}(x_0)(x - x_0)^{n-1}$$
$$+ \frac{(x - x_0)^n}{n!}[f^{(n)}(x_0) + \varepsilon(x)],$$

where $\lim\limits_{x \to x_0} \varepsilon(x) = 0.$

If we put $R_n(f; x_0, x) := \frac{(x-x_0)^n}{n!}\,\varepsilon(x)$, then in this remainder term f does not appear explicity. Hence it is not possible to derive from this the Lagrange form of the rest for $(n+1)$-times differentiable functions or bounds thereof. It is therefore desirable to have a bound of $\varepsilon(x)$ which depends on structural properties of f in a vicinity of x_0.

In our recent note [14] we proved the following version of Taylor's theorem with Peano-Young remainder.

Theorem 1.2. *For $n \in \mathbb{N}_0$ let $f \in C^n[a, b]$ and $x, x_0 \in [a, b]$. Then for the remainder in Taylor's formula we have*

$$|R_n(f; x_0, x)| \le \frac{|x - x_0|^n}{n!}\, \tilde{\omega}\left(f^{(n)}; \frac{|x - x_0|}{n+1}\right),$$

where $\tilde{\omega}(f^{(n)}; \cdot)$ is the least concave majorant of the modulus of continuity $\omega(f^{(n)}; \cdot)$.

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Dedicated to Professor **Alexandru Lupaş** on the occasion of his 65th birthday on January 5, 2007.

The modulus of continuity of a function $f \in C[a,b]$ is defined by

$$\omega(f;\varepsilon) = \sup\{|f(x) - f(y)| : x, y \in [a,b], |x-y| \le \varepsilon\}, \varepsilon \ge 0.$$

Its least concave majorant can be written as

$$\tilde{\omega}(f;\varepsilon) = \begin{cases} \sup\limits_{\substack{0 \le x \le \varepsilon \le y \le b-a \\ x \ne y}} \frac{(\varepsilon - x)\omega(f;y) + (y-\varepsilon)\cdot\omega(f;x)}{y-x} & \text{for} \quad 0 \le \varepsilon \le b-a, \\ \tilde{\omega}(f;b-a) = \omega(f;b-a) & \text{for} \quad \varepsilon > b-a. \end{cases}$$

Important for us is the relation between the following K-functional and $\tilde{\omega}$. Define, for $f \in C[a,b]$ and $\varepsilon \ge 0$,

$$K(\varepsilon, f; C[a,b], C^1[a,b]) := \inf\{\|f-g\| + \varepsilon, \|g'\| : g \in C^1[a,b]\}.$$

Then K and $\tilde{\omega}$ are related as in the following result attributed to Brudnyĭ.

Lemma 1.1. *Every continuous function on the compact interval $[a,b]$ satisfies the equality*

$$K(\frac{\varepsilon}{2}, f; C[a,b], C^1[a,b]) = \frac{1}{2}\,\tilde{\omega}(f;\varepsilon), \varepsilon \ge 0.$$

Using the above estimate of the Taylor remainder we proved in [14], among other things, inequalities for the differences of certain positive linear operators. These considerations are continued in the present paper.

2. General inequalities

Theorem 2.3. *Let $A, B : C[0,1] \to C[0,1]$ be positive linear operators such that*

$$(A-B)((e_1 - x)^i; x) = 0 \textit{ for } i = 0, 1, \dots, n \textit{ and } x \in [0,1].$$

Then for $f \in C^n[0,1]$ there holds

$$|(A-B)(f;x)| \le \frac{1}{n!}(A+B)(|e_1 - x|^n; x)\,\tilde{\omega}\left(f^{(n)}; \frac{1}{n+1}\,\frac{(A+B)(|e_1-x|^{n+1};x)}{(A+B)(|e_1-x|^n;x)}\right).$$

Proof. Using the Taylor expansion with quantitative Peano remainder we first have

$$|(A-B)(f;x)| = |(A-B)(f(t);x)| = \left|(A-B)\left(\frac{(t-x)^n}{n!}\,\mu_x(t);x\right)\right|.$$

Here we defined

$$\frac{(t-x)^n}{n!}\mu_x(t) := f(t) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x)\cdot(t-x)^k.$$

Hence

$$
\begin{aligned}
&|(A-B)(f;x)| \\
&\le (A+B)(\tfrac{|t-x|^n}{n!}\,\tilde\omega(f^{(n)};\tfrac{|t-x|}{n+1});x) \\
&= (A+B)(2\,\tfrac{|t-x|^n}{n!}\,K(f^{(n)};\tfrac{|t-x|}{2(n+1)});x) \\
&\le (A+B)(\tfrac{2\,|t-x|^n}{n!}\,\{\| (f-g)^{(n)} \| + \tfrac{|t-x|}{2(n+1)}\ \| g^{(n+1)} \|\};x), g \in C^{n+1}[0,1] \text{arbitrary}, \\
&= (A+B)(\tfrac{2\,|t-x|^n}{n!};x)\ \| (f-g)^{(n)} \| + (A+B)(\tfrac{|t-x|^{n+1}}{(n+1)!};x)\ \| g^{(n+1)} \| \\
&= (A+B)(\tfrac{2\,|t-x|^n}{n!};x)\,\{\| (f-g)^{(n)} \| + \tfrac{1}{2(n+1)}\,\tfrac{(A+B)(|t-x|^{n+1};x)}{(A+B)(|t-x|^n;x)}\ \| g^{(n+1)} \|\}.
\end{aligned}
$$

Passing back to infimum over $g \in C^{n+1}[0,1]$, and using Brudnyĭ's lemma again shows that

$$
\begin{aligned}
|(A-B)(f;x)| &\le (A+B)(\tfrac{2\,|t-x|^n}{n!};x)\,\tfrac{1}{2}\,\tilde\omega\left(f^{(n)};\tfrac{1}{n+1}\,\tfrac{(A+B)(|t-x|^{n+1};x)}{(A+B)(|t-x|^n;x)}\right) \\
&= \tfrac{1}{n!}(A+B)(|t-x|^n;x)\,\tilde\omega\left(f^{(n)};\tfrac{1}{n+1}\,\tfrac{(A+B)(|t-x|^{n+1};x)}{(A+B)(|t-x|^n;x)}\right).
\end{aligned}
$$

□

Corollary 2.1. *With $L := A+B$ we have for $\underline{n+1 \text{ odd}}$*

$$
\frac{L(|t-x|^{n+1};x)}{L(|t-x|^n;x)} \le \frac{\sqrt{L((t-x)^{2n};x)}\sqrt{L((t-x)^2;x)}}{L((t-x)^n;x)};
$$

so that the bound in Theorem 2.3 can be modified accordingly.

Proof. Write

$$
\begin{aligned}
L(|t-x|^{n+1};x) &= L(|t-x|^n \cdot |t-x|;x) \\
&\le \sqrt{L(|t-x|^{2n};x)}\sqrt{L(|t-x|^2;x)} \\
&= \sqrt{L((t-x)^{2n};x)}\sqrt{L((t-x)^2;x)}
\end{aligned}
$$

which arises from the Cauchy-Schwarz inequality. □

If n is odd the absolute moment $L(|t-x|^n;x)$ appears in the denominator. The operators A and B are such that $A(e_0,x) = B(e_0,x), x \in [0,1]$. We assume now that $A(e_0,x) = B(e_0,x) = 1, x \in [0,1]$.

So $L := \frac{1}{2}(A+B)$ reproduces constant functions. Hence by Hölder's inequality for positive linear operators we have for $1 \le s < r$ that

$$
L(|e_1-x|^s;x)^{\frac{1}{s}} \le L(|e_1-x|^r;x)^{\frac{1}{r}}, \quad \text{and}
$$

thus

$$
(A+B)(|e_1-x|^n;x) = 2L(|e_1-x|^n;x) \ge 2\left\{L((e_1-x)^{n-1};x)^{\frac{n}{n-1}}\right\}.
$$

Thus we have

Corollary 2.2. *If under the assumptions of Theorem 2.3 n is odd, we also get*

$$\begin{aligned} &|(A-B)(f;x)| \\ &\le \tfrac{1}{n!}(A+B)(|e_1-x|^n;x)\tilde{\omega}\left(f^{(n)};\tfrac{1}{2(n+1)}\cdot\frac{(A+B)((e_1-x)^{n+1};x)}{\left\{\frac{1}{2}(A+B)((e_1-x)^{n-1};x)\right\}^{\frac{n}{n-1}}}\right) \\ &= \tfrac{1}{n!}(A+B)(|e_1-x|^n;x)\tilde{\omega}\left(f^{(n)};\frac{2^{\frac{1}{n-1}}}{n+1}\,\frac{(A+B)((e_1-x)^{n+1};x)}{(A+B)((e_1-x)^{n-1};x)^{\frac{n}{n-1}}}\right). \end{aligned}$$

Note that the moments inside $\tilde{\omega}(f^{(n)};\cdot)$ are now both of even order and can thus be evaluated conveniently. The absolute moment in front of $\tilde{\omega}(f^{(n)};\cdot)$ can also be estimated using Hölder's inequality.

Corollary 2.3. *If A and B are given as in Theorem 2.3, then for $g \in C^{n+1}[0,1]$, $x \in [0,1]$ there holds*

$$|(A-B)(g;x)| \le \frac{1}{(n+1)!}(A+B)(|t-x|^{n+1};x) \parallel g^{(n+1)} \parallel .$$

The question remains how to estimate the difference for all functions in $C[0,1]$. So we will carry the result over from $C^{n+1}[0,1]$ to $C[0,1]$. In order to do so we use the following from [10] where, for $k \ge 0$ the symbol ω_k denotes the classical k-th order modulus of smoothness.

Lemma 2.2. *Let $I=[0,1]$ and $f \in C^r(I), r \in I\!N_0$. For any $h \in (0,1]$ and $s \in \mathbb{N}$ there exists a function $f_{h,r+s} \in C^{2r+s}(I)$ with*

(i) $\parallel f^{(j)} - f^{(j)}_{h,r+s} \parallel \le c \cdot \omega_{r+s}(f^{(j)};h)$ *for* $0 \le j \le r$,

(ii) $\parallel f^{(j)}_{h,r+s} \parallel \le c \cdot h^{-j} \cdot \omega_j(f;h)$, *for* $0 \le j \le r+s$,

(iii) $\parallel f^{(j)}_{h,r+s} \parallel \le c \cdot h^{-(r+s)} \cdot \omega_{r+s}(f^{(j-r-s)};h)$, *for* $r+s \le j \le 2r+s$.

Here the constant c depends only on r and s.

We will use the above lemma for $r=0, s=n+1$, thus obtaining for $h \in (0,1]$ and $f \in C[0,1]$ functions $f_{h,n+1}$ with

$$\parallel f - f_{h,n+1} \parallel \le c \cdot \omega_{n+1}(f;h), \parallel f^{(n+1)}_{h,n+1} \parallel \le c \cdot h^{-(n+1)} \cdot \omega_{n+1}(f;h).$$

With the aid of Lemma 2.2 we now prove the following

Theorem 2.4. *If A and B are given as in Theorem 2.3, also satisfying $Ae_0 = Be_0 = e_0$, then for all $f \in C[0,1], x \in [0,1]$ we have*

$$|(A-B)(f;x)| \le c_1 \cdot \omega_{n+1}\left(f;\ \sqrt[n+1]{\frac{1}{2}(A+B)(|e_1-x|^{n+1};x)}\right).$$

Here c_1 is an absolute constant independent of f, x and A and B.

Proof. Let $f \in C[0,1]$ be fixed and $g = f_{h,n+1}, 0 < h \le 1$, be given as above. Then, with the constant c from Lemma 2.2,

$$\begin{aligned} &|(A-B)(f;x)| \\ &\le |(A-B)(f-g;x)| + |(A-B)(g;x)| \\ &\le (\parallel A \parallel + \parallel B \parallel) \parallel f-g \parallel + \tfrac{1}{(n+1)!}(A+B)(|e_1-x|^{n+1};x) \parallel g^{(n+1)} \parallel \\ &\le 2 \cdot c \cdot \omega_{n+1}(f;h) + c\tfrac{1}{(n+1)!}(A+B)(|e_1-x|^{n+1};x)\tfrac{1}{h^{n+1}}\omega_{n+1}(f;h). \end{aligned}$$

If $(A+B)(|e_1-x|^{n+1};x)=0$, then – $h>0$ being arbitrary – we also have $|(A-B)(f;x)|=0$.

Otherwise we put $h=\sqrt[n+1]{\frac{1}{2}\,(A+B)(|e_1-x|^{n+1};x)}\le 1$ to arrive at

$$\begin{aligned}&|(A-B)(f;x)|\\&\le c_1\omega_{n+1}(f;\sqrt[n+1]{\tfrac{1}{2}(A+B)(|e_1-x|^{n+1};x)}.\end{aligned}$$

where $c_1=2\cdot c+c\frac{2}{(n+1)!}\,c\,(2+\frac{2}{(n+1)!})$. □

3. Approximation of the identity

Example 3.1. Suppose that $A=I$ is the identity operator on $C[0,1]$ and $B=L$ is a positive linear operator reproducing linear functions. Then the following hold:

(i) $|L(f;x)-f(x)|\le L(|e_1-x|;x)\,\tilde{\omega}\left(f';\frac{1}{2}\frac{L((e_1-x)^2;x)}{L(|e_1-x|;x)}\right),f\in C^1[0,1],x\in[0,1]$.

A similar inequality appeared in [12], Section 4.

(ii) $|L(g,x)-g(x)|\le\frac{1}{2}\,L((e_1-x)^2;x)\parallel g''\parallel,g\in C^2[0,1],x\in[0,1]$.

This is a well-known inequality in positive linear operator approximation (see, e.g., [4]).

(iii) $|L(f;x)-f(x)|\le c\cdot\omega_2\left(f;\sqrt[2]{\frac{1}{2}\,L((e_1-x)^2;x)}\right),f\in C[0,1],x\in[0,1]$.

To our knowledge such an inequality was first obtained by Esser in [5] and [6]; more precise estimates were given in [9], see also [20]. □

4. Some Fundamental Operators and their moments (up to order 4)

In our recent note [14] we used a special case of Theorem 2.4 (case $n=3$) to prove an inequality for the commutator $[A,B]:=AB-BA$ of certain positive linear operators A and B, thus solving a problem of Lupaş concerning the similarity of the two operator products. To be more specific, the operators A and B considered then were $A=B_n$ and $B=\overline{\mathbb{B}}_n$, defined as follows:

The Bernstein operators are given by

$$\begin{aligned}B_n(f;x)&=\sum_{k=0}^{n}f(\tfrac{k}{n})\,p_{n,k}(x),\\p_{n,k}(x)&=\tbinom{n}{k}x^k(1-x)^{n-k},x\in[0,1].\end{aligned}$$

Moreover, Lupaş' Beta operators $\overline{\mathbb{B}}_n$ (of the second kind) are

$$\overline{\mathbb{B}}_n(f;x)=\begin{cases}f(0), & x=0;\\ \frac{1}{B(nx,n-nx)}\,\int_0^1 t^{nx-1}(1-t)^{n-1-nx}f(t)dt, & 0<x<1;\\ f(1), & x=1.\end{cases}$$

Some properties of $\overline{\mathbb{B}}_n$ are given in [17]. This thesis deals mainly with Beta operators $\mathbb{B}_n$ of the first kind which are defined by

$$\mathbb{B}_n(f;x) = \frac{1}{B(nx+1, n+1-nx)} \int_0^1 t^{nx}(1-t)^{n(1-x)} f(t)dt.$$

The operators $\overline{\overline{\mathbb{B}}}_n$ were considered, for example, in [8], [1], [2]. Further research is needed to understand the $\overline{\overline{\mathbb{B}}}_n$ even better.

The inequality given in [14] is as follows.

Proposition 4.1. (see [14], Prop. 8.6) *If $S_n = \overline{\mathbb{B}}_n \circ B_n$ is a (special) Stancu Operator and $U_n = B_n \circ \overline{\mathbb{B}}_n$ is the genuine Bernstein-Durrmeyer operator, then for $n \in \mathbb{N}$, $f \in C[0,1]$ and $x \in [0,1]$*

$$|(S_n - U_n)(f;x)| \le c_1 \cdot \omega_4\left(f; \sqrt[4]{\frac{3x(1-x)}{n(n+1)}}\right).$$

Here $c_1 \neq c_1(f,n,x)$ is an absolute constant.

In this section we will also consider piecewise linear interpolation at equidistant knots. To be precise, consider the knot sequence

$$\Delta_n : x_{-1} = x_0 = 0 < x_1 < \ldots < x_n = x_{n+1} = 1$$

where $x_i = \frac{i}{n}, 0 \le i \le n$. Then the piecewise linear interpolant at x_i, $0 \le i \le n$, can be written as

$$S_{\Delta_n,1} f(x) = \sum_{j=-1}^{n-1} f\left(\frac{j+1}{n}\right) N_{j,1}(x), \ 0 \le x < 1,$$

and

$$S_{\Delta_n,1} f(1) = \lim_{y \nearrow 1} S_{\Delta_n,1} f(y).$$

Here the normalized B-Splines $N_{j,1}$ of piecewise degree 1 are given by

$$N_{j,1}(x) = (x_{j+2} - x_j)[x_j, x_{j+1}, x_{j+2}](\cdot - x)_+$$

with the usual notation for divided differences and

$$(t-x)_+ = \max\{0, t-x\}.$$

For simplicity we will write

$$S_{\Delta_n}(f;x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) \cdot N_j(x), x \in [0,1].$$

This affects only the subscripts of the normalized splines and the definition of $N_n(1)$.

B_n, $\overline{\mathbb{B}}_n$ and S_{Δ_n} will be the building blocks for the special operators to be considered in subsequent sections. We first collect some known facts in

Proposition 4.2. *If $A_n \in \{B_n, \overline{\mathbb{B}}_n, S_{\Delta_n}\}$, then $A_n : C[0,1] \to C[0,1]$ is a positive linear operator satisfying $A_n e_i = e_i (i = 0, 1)$ where $e_i(x) = x^i, i \in \mathbb{N}_0$.*

In the tables below we collect information concerning the moments of orders 2, 3 and 4.

Table 4.1.

A_n	$A_n((e_1-x)^2;x)$	$A_n((e_1-x)^3;x)$
B_n	$\frac{x(1-x)}{n}$	$\frac{x(1-x)(1-2x)}{n^2}$
$\overline{\mathbb{B}}_n$	$\frac{x(1-x)}{n+1}$	$\frac{2x(1-x)(1-2x)}{(n+1)(n+2)}$
S_{Δ_n}	$\frac{(nx-[nx])(1+[nx]-nx)}{n^2}$	$\frac{(nx-[nx])(1+[nx]-nx)[1-2(nx-[nx])]}{n^3}$

Table 4.2.

A_n	$A_n((e_1-x)^4;x)$
B_n	$\frac{1}{n^3}\{3(n-2)x^2(1-x)^2+x(1-x)\}$
$\overline{\mathbb{B}}_n$	$\frac{1}{(n+1)(n+2)(n+3)}\{3(n+1)x^2(1-x)^2+6x(1-x)\}$
S_{Δ_n}	$\frac{(nx-[nx])(1+[nx]-nx)[1-3(nx-[nx])(1+[nx]-nx)]}{n^4}$

Proof. The moments of the Bernstein operators can be found in [22], Lema 6.24, for example. Those of the Beta operators are listed on p. 63 / 64 in [17]. Finally, the second and the fourth moments for S_{Δ_n} can be found on p. 46 in [18]. Similarly one can compute the third moments for S_{Δ_n} by using the explicit representation of $S_{\Delta_n}((e_1-x)^3;x)$ on the local support $\left[\frac{k-1}{n},\frac{k}{n}\right]$. For more detailed information see also [7]. $\square$

Knowledge about moments of various orders is useful for the proof of Voronovskaja-type results. In [14] we proved the following

Theorem 4.5. *Let $L: C[0,1]\to C[0,1]$ be a positive linear operator such that $Le_i=e_i,\ i=0,1$. If $f\in C^2[0,1]$ and $x\in[0,1]$, then*

$$|L(f;x)-f(x)-\frac{1}{2}f''(x)L((e_1-x)^2;x)|\le\frac{1}{2}L((e_1-x)^2;x)\,\tilde{\omega}\left(f'';\sqrt{\frac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}}\right).$$

Applications of Theorem 4.5 for B_n and $\bar{\mathbb{B}}_n$ were given in [14]. The application to S_{Δ_n} yields

Proposition 4.3. *Let S_{Δ_n} be given as above, $f\in C^2[0,1],\ x\in[0,1]$. Then*

$$\begin{aligned}(4.1)\qquad |n^2[S_{\Delta_n}(f;x)-f(x)] &- \frac{1}{2}\cdot f''(x)\cdot z_n(x)(1-z_n(x))|\\ &\le \frac{1}{2}z_n(x)(1-z_n(x))\cdot\tilde{\omega}\left(f'';\frac{1}{3n}\right).\end{aligned}$$

Here $z_n(x)=nx-[nx]$, where $[nx]$ denotes the integer part of nx.

Proof. Write $z_n(x):=nx-[nx]$. Then from Tables 4.1 and 4.2 we see that

$$\begin{aligned}S_{\Delta_n}((e_1-x)^2;x) &= \frac{1}{n^2}z_n(x)(1-z_n(x)),\ \text{and}\\ S_{\Delta_n}((e_1-x)^4;x) &= \frac{1}{n^2}z_n(x)(1-z_n(x))[1-3z_n(x)(1-z_n(x))].\end{aligned}$$

Substituting these into the inequality of Theorem 4.5 yields the result once we take into account that

$$\frac{S_{\Delta_n}((e_1-x)^4;x)}{S_{\Delta_n}((e_1-x)^2;x)} = \frac{1}{n^2}[1-3z_n(x)(1-z_n(x))] \le \frac{1}{n^2} \text{ for } x \in [0,1].$$

□

As can be noted from Table 4.1 the second moments of B_{n+1} and the Lupaş operators $\bar{\mathbb{B}}_n$ agree. Thus Corollary 2.1 and Theorem 2.4 can be applied in this case with $n=2$. An application of the corollary mentioned gives

Proposition 4.4.

$$\begin{aligned} |(B_{n+1}-\bar{\mathbb{B}}_n)(f;x)| &\le \frac{x(1-x)}{n+1}\,\tilde{\omega}\left(f'';\sqrt{\frac{(n+1)(6nx(1-x)+7)}{18n^2}}\right),\ f\in C^2[0,1] \\ &\le \frac{x(1-x)}{3n\sqrt{n+1}}\sqrt{\frac{6nx(1-x)+7}{2n}}\|f'''\|,\ f\in C^3[0,1]. \end{aligned}$$

Proof. Using again the moments listed in Tables 4.1 and 4.2 we arrive at

$$\begin{aligned} (B_{n+1}+\bar{\mathbb{B}}_n)((t-x)^2;x) &= \frac{2x(1-x)}{n+1}, \\ (B_{n+1}+\bar{\mathbb{B}}_n)((t-x)^4;x) &= \left(\frac{3(n-1)}{(n+1)^3}+\frac{3}{(n+2)(n+3)}\right)x^2(1-x)^2 \\ &+ \left(\frac{1}{(n+1)^3}+\frac{6}{(n+1)(n+2)(n+3)}\right)x(1-x) \\ &\le \left(\frac{3}{n^2}+\frac{3}{n^2}\right)x^2(1-x)^2+\left(\frac{1}{n^3}+\frac{6}{n^3}\right)x(1-x) \\ &= \frac{x(1-x)}{n^2}\cdot\frac{6nx(1-x)+7}{n}. \end{aligned}$$

Using the above mentioned corollary and properties of $\tilde{\omega}$ we obtain the desired inequalities. □

For all $f \in C[0,1]$ Theorem 2.4 implies the following

Proposition 4.5.

$$\begin{aligned} |(B_{n+1}-\bar{\mathbb{B}}_n(f;x))| &\le c\cdot\omega_3\left(f;\sqrt[3]{\frac{1}{2}(B_{n+1}+\bar{\mathbb{B}}_n)(|e_1-x|^3;x)}\right) \\ &\le c\cdot\omega_3\left(f;\sqrt[6]{\frac{x^2(1-x)^2}{n^3}\cdot\frac{6nx(1-x)+7}{n}}\right). \end{aligned}$$

Proof. The first inequality is a direct consequence of Theorem 2.4. The second one can be obtained via Cauchy-Schwarz:

$$L(|e_1-x|^3;x)^{\frac{1}{3}} \le L((e_1-x)^2;x)^{\frac{1}{6}}\cdot L((e_1-x)^4;x)^{\frac{1}{6}},$$

where L can be replaced in this case by $L := \frac{(B_{n+1}+\bar{\mathbb{B}}_n)}{2}$. Involving parts of the proof of the previous proposition we get to the desired result. □

5. Higher order moments of (some composite) operators

As we observed above, information about higher order moments is needed in order to arrive at estimates concerning the difference of two positive linear operators.

Some information concerning $A_n \in \{B_n, \overline{\mathbb{B}}_n, S_{\Delta_n}\}$ was given in the previous section.

Normally the computation of higher order moments is tedious, but rather mechanical work. It is a recursive job, as we will see to some extent in this section: once we know the moments of low orders, we know the ones of higher orders.

Another aspect to be considered is the computation of moments (of a fixed order) of *composite operators* where information about the factors is available. We mostly exclude this aspect from our considerations here, but refer the reader to [15] and [11] for relevant results concerning this direction.

It was already mentioned in Section 4 that certain compositions of the building blocks discussed there play a prominent role in approximation theory. We will discuss these and some others in the sequel.

Of particular interest are the mappings $U_n = B_n \circ \overline{\mathbb{B}}_n$, the so-called genuine Bernstein-Durrmeyer operators, the explicit form of which is

$$\begin{aligned} U_n(f;x) &= (B_n \circ \overline{\mathbb{B}}_n)(f;x) \\ &= f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt. \end{aligned}$$

The opposite operator product, namely $S_n := \overline{\mathbb{B}}_n \circ B_n$ is a special Stancu operator and is given in explicit form by

$$S_n(f;x) = (\overline{\mathbb{B}}_n \circ B_n)(f;x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(nx)_k(n-nx)_{n-k},$$

where $(a)_0 = 1$, $(a)_b = \prod_{k=0}^{b-1}(a-k)$, $a \in \mathbb{R}$, $b \in \mathbb{N}$.

This is a special case of the Stancu operator introduced in [21], the case $\alpha = \frac{1}{n}$. More information on S_n can be found in [19].

Another interesting object is

$$D_n := B_n \circ B_{n+1},$$

the composition of two Bernstein operators. This is a mapping having some similarity with U_n from above. This will also be discussed below.

The second moments of composite operators can be computed as follows.

Lemma 5.3. [15] *For two linear operators P, Q where $Qe_i = e_i, i \in \{0,1\}$, one has*

$$(PQ)((e_1 - x)^2; x) = P^u(Q((e_1-u)^2;u);x) + P((e_1-x)^2;x).$$

The superscript u indicates that P is applied to functions in the variable u.

Lemma 5.3 can be used to derive the information in the following table. The moments of U_n can also be found in [16], and those of S_n in [19]. It is an easy computation to find those of D_n.

Table 5.1.

A_n	$A_n((e_1-x)^2;x)$
$U_n := B_n \circ \overline{\mathbb{B}}_n$	$\frac{2x(1-x)}{n+1}$
$S_n := \overline{\mathbb{B}}_n \circ B_n$	$\frac{2x(1-x)}{n+1}$
$D_n := B_n \circ B_{n+1}$	$\frac{2x(1-x)}{n+1}$

A further way to derive information on all moments of linear operators (not necessarily positive) is given in

Proposition 5.6. *For a linear operator L and $k \in \mathbb{N}_0$ one has*

$$L((e_1-x)^k;x) = L(e_k;x) - \sum_{l=0}^{k-1} \binom{k}{l} x^{k-l} L((e_1-x)^l;x).$$

Proof. Write

$$\begin{aligned} L(e_k;x) &= L((e_1-x+x)^k;x) \\ &= L\left(\sum_{l=0}^{k} \binom{k}{l} x^{k-l} \cdot (e_1-x)^l;x\right) \\ &= \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \cdot L((e_1-x)^l;x) \\ &= L((e_1-x)^k;x) + \sum_{l=0}^{k-1} \binom{k}{l} x^{k-l} \cdot L((e_1-x)^l;x), \end{aligned}$$

which implies the representation of the k-th moment. □

Remark 5.1. (i) Note that the equality of Proposition 5.6 holds without the assumption $Le_i = e_i, i \in \{0,1\}$.

(ii) The proposition means that $L((e_1-x)^k;x)$ can be computed if we know $L(e_k;x)$ and the lower order moments $L((e_1-x)^l;x), 0 \le l \le k-1$.

Corollary 5.4. *For a linear operator L with $Le_i = e_i, i \in \{0,1\}$, we have*

$$L((e_1-x)^3;x) = L(e_3;x) - x^3 - 3xL((e_1-x)^2;x).$$

Proof. Immediate consequence of Proposition 5.6. □

Using the latter equality we derive the following table.

Table 5.2.

A_n	$A_n((e_1-x)^3;x)$
$U_n := B_n \circ \overline{\mathbb{B}}_n$	$\frac{6x(1-x)(1-2x)}{(n+1)(n+2)}$
$S_n := \overline{\mathbb{B}}_n \circ B_n$	$\frac{6x(1-x)(1-2x)}{(n+1)(n+2)}$
$D_n := B_n \circ B_{n+1}$	$\frac{(5n-1)x(1-x)(1-2x)}{n(n+1)^2}$

Proof. (i) The third moments of U_n are given in [16], Proposition 3.5.

(ii) For the Stancu operators S_n we refer again to [19], p. 68, where it is shown that

$$S_n(e_3;x) = x^3 + \frac{6x(1-x)}{(n+1)(n+2)} + \frac{6nx^2(1-x)}{(n+1)(n+2)}$$

Hence, by Corollary 5.4 we find that

$$\begin{aligned} S_n((e_1-x)^3;x) &= \frac{6x(1-x)}{(n+1)(n+2)} + \frac{6nx^2(1-x)}{(n+1)(n+2)} - 3x\,\frac{2x(1-x)}{n+1} \\ &= \frac{6x(1-x)(1-2x)}{(n+1)(n+2)}. \end{aligned}$$

(iii) The representation of $D_n((e_1-x)^3;x)$ can be derived from Corollary 5.4 using properties of the Bernstein operators. It can also be found in the forthcoming note [13].

□

Corollary 5.5. *For a linear operator L with $Le_i = e_i, i \in \{0,1\}$, the fourth moments can be computed as*

$$L((e_1-x)^4;x) = L(e_4;x) - x^4 - \{4x \cdot L((e_1-x)^3;x) + 6x^2 \cdot L((e_1-x)^2;x)\}.$$

Below is a list of the fourth moments of the composite operators.

Table 5.3.

A_n	$A_n((e_1-x)^4;x)$
$U_n := B_n \circ \overline{\mathbb{B}}_n$	$\frac{1}{(n+1)(n+2)(n+3)}\{12(n-7)x^2(1-x)^2 + 24x(1-x)\}$
$S_n := \overline{\mathbb{B}}_n \circ B_n$	$\frac{1}{n(n+1)(n+2)(n+3)}\{12(n^2-7n)x^2(1-x)^2 + (26n-2)x(1-x)\}$
$D_n := B_n \circ B_{n+1}$	$\frac{1}{n^2(n+1)^3}\{12(n^3-6n^2+4n-1)x^2(1-x)^2+(15n^2-9n+2)x(11-x)\}$

Proof. The moments of U_n were computed in [16], Proposition 3.5, those of S_n can be found in [19], p. 68, and the ones of D_n are also computed in [13]. □

6. Estimates for the differences of some composite operators

We first give one further application of Theorem 2.4 for the case $n = 1$.

Proposition 6.7.

$$|(B_n - U_n)\left(f;x)\right| \leq c \cdot \omega_2(f; \sqrt[2]{\frac{3x(1-x)}{2n}}\Bigg).$$

Proof. From Tables 4.1 and 5.1 we see that $B_n((e_1-x)^2;x) = \frac{x(1-x)}{n}$, $U_n((e_1-x)^2;x) = \frac{2x(1-x)}{n+1}$. Hence

$$\frac{1}{2} \cdot (B_n + U_n)((e_1-x)^2;x) \leq \frac{3}{2n}\,x(1-x),$$

which implies the claim. □

We investigate the difference

$$D_n - U_n := B_n \circ B_{n+1} - U_n = B_n \circ B_{n+1} - B_n \circ \overline{\mathbb{B}}_n = B_n \circ (B_{n+1} - \overline{\mathbb{B}}_n)$$

next in order to give further applications of Theorems 2.3 and 2.4 for $n = 2$.

All operators involved reproduce linear functions, so

$$(D_n - U_n)((e_1 - x)^i; x) = 0 \text{ for } i = 0, 1.$$

Moreover, $B_{n+1}((e_1 - x)^2; x) = \overline{\mathbb{B}}_n((e_1 - x)^2; x) = \frac{x(1-x)}{n+1}$, and hence also $(D_n - U_n)((e_1 - x)^2; x) = 0$. Thus Theorem 2.3 is applicable with $n = 2$, once we have estimated

$$(D_n + U_n)(|e_1 - x|^3; x) \le \sqrt{(D_n + U_n)((e_1 - x)^2; x)} \cdot \sqrt{(D_n + U_n)((e_1 - x)^4; x)},$$

which follows from the Cauchy-Schwarz inequality. All moments needed were given in Table 5.1 and Table 5.3. Adding them entails

$$\begin{aligned}
(D_n + U_n)((t - x)^2; x) &= \frac{4x(1 - x)}{n + 1}, \\
(D_n + U_n)((t - x)^4; x) &= \left(\frac{12(n^3 - 6n^2 + 4n - 1)}{n^2(n+1)^3} + \frac{12(n-7)}{(n+1)(n+2)(n+3)}\right) x^2(1-x)^2 \\
&\quad + \left(\frac{15n^2 - 9n + 2}{n^2(n + 1)^3} + \frac{24}{(n + 1)(n + 2)(n + 3)}\right) x(1 - x) \\
&\le \left(\frac{12}{n^2} + \frac{12}{n^2}\right) x^2(1 - x)^2 + \left(\frac{15}{n^3} + \frac{24}{n^3}\right) x(1 - x) \\
&= \frac{x(1 - x)}{n^2} \frac{24nx(1 - x) + 39}{n}.
\end{aligned}$$

This leads to

Proposition 6.8.

$$\begin{aligned}
|(D_n - U_n)(f; x)| &\le \frac{2x(1-x)}{n+1}\, \tilde{\omega}\left(f'', \sqrt{\frac{(n+1)(8nx(1-x)+13)}{12n^3}}\right), f \in C^2[0, 1], \\
&\le \frac{x(1 - x)}{n\sqrt{n + 1}} \sqrt{\frac{8nx(1 - x) + 13}{3n}} \parallel f''' \parallel, f \in C^3[0, 1].
\end{aligned}$$

An application of Theorem 2.4 yields

Proposition 6.9.

$$\begin{aligned}
|(D_n - U_n)(f; x)| &\le c \cdot \omega_3 \left(f; \sqrt[3]{\frac{1}{2}(D_n + U_n)(|e_1 - x|^3; x)}\right) \\
&\le c \cdot \omega_3 \left(f; \sqrt[6]{\frac{x^2(1 - x)^2}{(n + 1)n^3} \cdot (24nx(1 - x) + 39)}\right).
\end{aligned}$$

Remark 6.2. *For the difference $D_n - S_n$ similar estimates can be given, since the second moments of both operators are the same (see Table 5.1) and the structures of the second and fourth moments are analogous to the cases considered before.*

7. Concluding remarks

One further interesting mapping to be considered is also motivated by the previous work of Lupaş.

We put $\mathbb{L}_n := \overline{\mathbb{B}}_n \circ S_{\Delta_n}$, i.e., for $f \in C[0,1]$, $x \in [0,1]$, consider

$$\mathbb{L}_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \overline{\mathbb{B}}_n(N_k;x).$$

We observed that the operator product $\mathbb{L}_n = \bar{\mathbb{B}}_n \circ S_{\Delta_n}$ seems to be a very good approximate and non-trivial decomposition of the Bernstein operator B_n. We suggest further research concerning this matter.

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