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Some boundedness properties of certain operators

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ABSTRACT. In this paper we determine some conditions of boundedness for the integral operator (1.1) when it is applied to an analytic function f so that

$$|\arg f'(z)| < \frac{5\pi}{9}, \quad z \in U.$$

In particular, results for the Alexander integral operator, Libera integral operator and Libera generalized integral operator are obtained.

1. INTRODUCTION

Let $H(U)$ denote the class of holomorphic (analytic) functions in the unit disc U of the complex plane $\mathbb{C}$. Let $A \subset H(U)$ denote the set of functions f of the form $f(z) = z + a_2z^2 + \ldots, z \in U$ and $S \subset A$ its subset of univalent functions. Functions in the class $R = \left\{ f \in A : \mathrm{Re}\dfrac{zf'(z)}{f(z)} > 0, \quad z \in U \right\}$ are called functions with bounded turning.

Let $S^* = \left\{ f \in A : \mathrm{Re}\dfrac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$ be the class of starlike functions.

Let β and γ be real numbers, with $\beta > 0$ and $\beta + \gamma > 0$.
Define for $f \in A$,

$$I(z) = I_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}} \int_0^z f(t)t^{\beta+\gamma-2}dt \right]^{\frac{1}{\beta}},\ z \in U. \tag{1.1}$$

In 1994 P. T. Mocanu [6] found a sufficient condition on β and γ so that $I_{\beta,\gamma}(f) \in S^*$ for $f \in R$.

For $\beta = 1$ and $\gamma = 0$, (1.1) is the Alexander integral operator [1]:

$$I_{1,0}(f)(z) = \int_0^z f(t)t^{-1}dt,\ z \in U. \tag{1.2}$$

For $\beta = 1$ and $\gamma = 1$, (1.1) is the Libera integral operator [5]:

$$I_{1,1}(f)(z) = \frac{2}{z} \int_0^z f(t)dt,\ z \in U. \tag{1.3}$$

For $\beta = 1$ and $\gamma = 1, 2, \ldots$, (1.1) is the Libera generalized integral operator:

$$I_{1,\gamma}(f)(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t)t^{\gamma-1}dt,\ z \in U, \tag{1.4}$$

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which has been considered by S. Bernardi [2] in 1969.

In this paper we find the Hardy spaces for the integral operator (1.1) acting on functions f with $|\arg f'(z)| < \dfrac{5\pi}{9}$ in U, and the Hardy spaces for its iterative. Hence we find conditions for boundedness of (1.1). In particular, results for the Alexander integral operator, Libera integral operator and Libera generalized integral operator are given.

2. Preliminaries

For $f \in H(U)$ and $z = re^{i\theta} \in U$, we set

$$M_p(r,f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & 0 < p < \infty \\ \sup\limits_{0\le\theta\le 2\pi} |f(re^{i\theta})|, & p = \infty. \end{cases}$$

A function $f \in H(U)$ is said to be in Hardy space H^p, $0 < p < \infty$, if $M_p(r,f)$ remains bounded as $r \to 1^-$. H^∞ is the class of bounded analytic functions in U.

We shall need the following lemmas to prove our main results.

Lemma 2.1. ([3]). *If $f \in H^p$ and $F(z) = \displaystyle\int_0^z f(t)dt$, $z \in U$, then $f \in H^q$ with $q = \dfrac{1}{1-p}$ for $p < 1$, and $f \in H^\infty$ for $p \geq 1$.*

Lemma 2.2. ([4]). *If $f \in S^* \backslash \{f_0\}$, where $f_0(z) = \dfrac{az}{(1-ze^{i\tau})^2}$, $z \in U$, then there exists $\varepsilon = \varepsilon(f) > 0$ such that $f \in H^{\frac{1}{2}+\varepsilon}$.*

Lemma 2.3. ([7]). *If $f \in A$ and $|\arg f'(z)| < \dfrac{5\pi}{3}$, $z \in U$, then $I_{1,1}(f) \in S^*$.*

3. Main results

Theorem 3.1. *Let β and γ be real numbers with $\beta > 0$, $\beta + \gamma > 0$ and I is defined by (1.1). If $f \in A\backslash\{f_0\}$, where $f_0(z) = \dfrac{az}{(1-\lambda z)^3}$, $z \in U$ and $|\arg f'(z)| < \dfrac{5\pi}{9}$, $z \in U$, then there exists $\varepsilon = \varepsilon(f) > 0$ such that:*

(i) $I(f) \in H^{\beta p}$, where $p = \dfrac{1}{2} + \varepsilon$;

(ii) if $(\beta + \beta^2 + \cdots + \beta^{n-1})p < 1$, then $I^n(f) \in H^q$, where

$$q = \frac{\beta^n p}{1 - (\beta + \beta^2 + \cdots + \beta^{n-1})p}, \; p = \frac{1}{2} + \varepsilon \text{ and } I^n = \underbrace{I \circ I \circ \cdots \circ I}_{n};$$

(iii) if $(\beta + \beta^2 + \cdots + \beta^{n-1})p \geq 1$, then $I^n(f) \in H^\infty$.

Proof. (i). Observe that

(3.5) $\quad I = F \circ G,$

where $F = F(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}} f(z)\right]^{\frac{1}{\beta}}$, $z \in U$, and $G = G(f)(z) = \int_0^z f(t)t^{\beta+\gamma-2}dt$, $z \in U$. We will determine the Hardy spaces for F and G.

Let f be analytic and $f \in H^q$. For $F(f)$ we have:

$$M_p^p(r,F) = \frac{1}{2\pi}\int_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^p d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\left[\frac{\beta+\gamma}{r^{\gamma}e^{i\gamma\theta}} f(re^{i\theta})\right]^{\frac{1}{\beta}}\right|^p d\theta =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{(\beta+\gamma)^{\frac{1}{\beta}}}{r^{\frac{\gamma}{\beta}} e^{i\frac{\gamma\theta}{\beta}}} f^{\frac{1}{\beta}}(re^{i\theta})\right|^p d\theta =$$
$$= \frac{(\beta+\gamma)^{\frac{p}{\beta}}}{r^{\frac{\gamma p}{\beta}}}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\frac{p}{\beta}} d\theta\right) = \frac{(\beta+\gamma)^{\frac{p}{\beta}}}{r^{\frac{\gamma p}{\beta}}} \cdot M_{\frac{p}{\beta}}^{\frac{p}{\beta}}(r,f).$$

Hence we obtain $M_p(r,F) = \frac{(\beta+\gamma)^{\frac{1}{\beta}}}{r^{\frac{\gamma}{\beta}}} \cdot M_{\frac{p}{\beta}}^{\frac{1}{\beta}}(r,f)$ and

$$\lim_{r\to 1^-} M_p(r,F) = \lim_{r\to 1^-} \frac{(\beta+\gamma)^{\frac{1}{\beta}}}{r^{\frac{\gamma}{\beta}}} M_{\frac{p}{\beta}}^{\frac{1}{\beta}}(r,f) = (\beta+\gamma)^{\frac{1}{\beta}} \lim_{r\to 1^-} M_{\frac{p}{\beta}}^{\frac{1}{\beta}}(r,f).$$

Because $\lim_{r\to 1^-} M_{\frac{p}{\beta}}^{\frac{1}{\beta}}(r,f) < \infty$, if $\lim_{r\to 1^-} M_{\frac{p}{\beta}}(r,f) < \infty$ and from $f \in H^q$ we have $\lim_{r\to 1^-} M_{\frac{p}{\beta}}(r,f) < \infty$ for $\frac{p}{\beta} = q$. Hence we have that

(3.6) $\quad F(f) \in H^{\beta q}$.

For G we will deduce that if $f \in H^q$ then $G(f)$ and $I_{1,0}(f)$ have the same Hardy space. Indeed, if $f \in H^q$ then f is holomorphic and $G(f)$ is holomorphic and $[G(f)(z)]' = f(z)z^{\beta+\gamma-2}$. Analog $[I_{1,0}(f)(z)]' = f(z)\cdot z^{-1}$.
We can easily deduce that $f(z)z^{-1}$ and $f(z)\cdot z^{\beta+\gamma-2}$ have the same Hardy space. Hence we obtain that $G(f)$ and $I_{1,0}(f)$ have the same Hardy space.

Now, from Lemma 2.3 we have that if $f \in A$ and $|\arg f'(z)| < \frac{5\pi}{9}$ then $I_{1,1}(f) \in S^*$. From Lemma 2.2 we deduce that there exists $\varepsilon = \varepsilon(f)$ such that $I_{1,1}(f) \in H^{\frac{1}{2}+\varepsilon}$. On the other hand we can easily deduce that $I_{1,0}(f)$ and $I_{1,1}(f)$ have the same Hardy space. Hence and from (3.6) we obtain that $F(G(f)) \in H^{\beta(\frac{1}{2}+\varepsilon)}$ namely $I(f) \in H^{\beta p}$, where $p = \frac{1}{2}+\varepsilon$.

(ii) and (iii). From Lemma 2.1 and from (i) we deduce that if $I(f) \in H^{\beta p}$, then $\int_0^z I(f)(t)^{\beta+\gamma-2}dt \in H^{\frac{\beta p}{1-\beta p}}$ and hence

$$I^2(f)(z) = I(I(f))(z) = \left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z I(f)(t)t^{\beta+\gamma-2}dt\right]^{\frac{1}{\beta}} \in H^{\lambda},$$

where $\lambda = \beta\frac{\beta p}{1-\beta p} = \frac{\beta^2 p}{1-\beta p}$, for $\beta p < 1$, and $\lambda = \infty$ for $\beta p \geq 1$.

We suppose that $I^{n-1}(f) \in H^\lambda$ where $\lambda = \dfrac{\beta^{n-1}p}{1-(\beta+\beta^2+\cdots+\beta^{n-2})p}$ for $(\beta+\beta^2+\cdots+\beta^{n-2})p < 1$ and $\lambda = \infty$ for $(\beta+\beta^2+\cdots+\beta^{n-2})p \geq 1$.

From Lemma 2.1 and from assertion (i) we have $I^n(f) \in H^\lambda$ where $\lambda = \dfrac{\beta^n p}{1-(\beta+\beta^2+\cdots+\beta^{n-1})p}$, for $(\beta+\beta^2+\cdots+\beta^{n-1})p < 1$ and $\lambda = \infty$ for $(\beta+\beta^2+\cdots+\beta^{n-1})p \geq 1$, $p = \dfrac{1}{2}+\varepsilon$. □

Corollary 3.1. *Let β and γ be real numbers with $\beta > 2$, $\beta+\gamma > 0$ and I is defined by* (1.1)*. If $f \in A$ and $|\arg f'(z)| < \dfrac{5\pi}{9}$ then $I^n(f)$ are bounded for all $n \geq 2$.*

Indeed, is known that if $f \in S^*$ then $f \in A^p$, for all $p < \dfrac{1}{2}$. If $\beta > 2$ then $\beta p \geq 1$ and from Theorem 3.1 we obtain $I^2(f) \in H^\infty$ and hence $I^2(f)$ is bounded. Analog, $(\beta+\beta^2+\cdots+\beta^{n-1})p \geq 1$ for $n \geq 2$ and hence $I^n(f) \in H^\infty$, namely $I^n(f)$ is bounded. For $\beta = 1$ and $\gamma = 0$ the integral operator I becomes $I_{1,0}$ defined by (1.2). For the Alexander integral operator $I_{1,0}$ we have the following result.

Corollary 3.2. *If $f \in A\backslash\{f_0\}$, where $f_0(z) = \dfrac{az}{(1-\lambda z)^3}$, $z \in U$ and $|\arg f'(z)| < \dfrac{5\pi}{9}$, then $I^n(f)$ are bounded for all $n \geq 3$.*

For $\beta = 1$ and $\gamma = 0$ the integral operator I becomes $I_{1,1}$ defined by (1.3).

For the Libera integral operator $I_{1,1}$ we have the following result.

Corollary 3.3. *If $f \in A\backslash\{f_0\}$, $|\arg f'(z)| < \dfrac{5\pi}{9}$, then $I^n_{1,1}(f)$ are bounded for all $n \geq 3$.*

For $\beta = 1$ and γ the real number so that $1+\gamma > 0$ the integral operator I becomes the Libera generalized integral operator defined by (1.4).

Corollary 3.4. *If $f \in A\backslash\{f_0\}$, $|\arg f'(z)| < \dfrac{5\pi}{9}$, then $I^n_{1,\gamma}(f)$ is bounded for all $n \geq 3$.*

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