CARPATHIAN J. MATH.
22 (2006), No. 1 - 2, 83 - 90

Superpositions of functions involved in nomography

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ABSTRACT. The current paper deals with the conditions by which the functions of more variables, and also the equations which contain such functions, can be represented by the superpositions of functions of fewer variables. We will also give the corresponding nomograms by which these functions are nomographically represented.

1. INTRODUCTION

The decomposition of a function of several variables in superpositions of functions of a smaller number of variables allows its nomographical representation. For the functions that present superpositions, compound nomograms, nomogram with ramified nets or with chain nets can be built.

For the functions of two or of three variables, that present superpositions, the nomographical representation conditions have been studied and the corresponding nomograms have been built [3]. In addition, L. Bal and F. Rado [1], [2] studied the representation conditions of the equations of four and five variables that represent functions with superpositions. For those, there have been built nomograms with ramifications or even with a transparent mobile plan.

In [4] we have presented the systematization of the functions and of the equations with superpositions. We have analyzed the conditions under which a function of two, three and four variables (respectively an equation with three to five variables), can be represented by a function of one or of two variables. We have also presented several types of nomograms, by which such functions can be represented especially on compound nomograms or on ramified ones.

In [5] we have discussed the same issue of finding the conditions of nomographical representation and of building the respective nomograms in the case of functions of five variables.

In this paper we will deal with the study of the conditions by which a function of six variables can be written as a superposition of functions of one, two or of three variables. For each of these cases we will build compound net nomograms or nomograms with a transparent mobile plane. Thus, we will study the functions of five variables, respectively the corresponding equations with seven variables, namely

(1.1) $\quad z_7 = f(z_1, z_2, \ldots, z_6)$

and

(1.2) $\quad F(z_1, z_2, \ldots, z_7) = 0.$

Received: 08.06.2005; In revised form: 05.02.2006; Accepted: 01.11.2006
2000 *Mathematics Subject Classification:* 65S05.
Key words and phrases: *Nomogram, nomographic functions, canonical forms.*

2. Functions of six variables that are the superpositions of functions of one variable

The form of these functions is:

$$z_7 = f(z_1, z_2, \dots, z_6) = \phi[\varphi_1(z_1), \varphi_2(z_2), \dots, \varphi_6(z_6)]. \tag{2.3}$$

There are two important particular cases of the function (2.3) namely:

$$z_7 = f_7^{-1}\left[\sum_{i=1}^{6} f_i(z_i)\right] = f_7^{-1}[f_1(z_1) + f_2(z_2) + \cdots + f_6(z_6)] \tag{2.4}$$

with the corresponding equation

$$\sum_{i=1}^{7} f_i(z_i) = 0 \tag{2.5}$$

and

$$z_7 = f_7^{-1}\left[\prod_{i=1}^{6} f_i(z_i)\right] = f_7^{-1}[f_1(z_1)f_2(z_2)\dots f_6(z_6)] \tag{2.6}$$

with

$$\prod_{i=1}^{7} f_i(z_i) = 1. \tag{2.7}$$

We give the following result.

Theorem 2.1. *The necessary and sufficient conditions for that the equation* (1.2) *admits the form* (2.4) (*respectively* (2.5)) *are:*

$$\frac{\partial^2}{\partial z_1 \partial z_j} \ln\left(\frac{\partial F}{\partial z_1} : \frac{\partial F}{\partial z_j}\right) = 0, \quad j = \overline{2,7}. \tag{2.8}$$

Proof. The necessity is obtained by differentiating both equations (1.2) and (2.5) and considering the proportional coefficients of dz_i, $i = 1, 7$.

The sufficiency is also obtained using certain changing of variables and keeping in account that the equation (1.2) can define the variable z_7 as an implicit function. □

Remark 2.1. The conditions (2.8) and the proof of the above theorem are an extension of Saint-Roberts condition for the equation with three variables.

Remark 2.2. The same conditions as in Theorem 2.1 must be fulfilled so that the equation (1.2) admits the forms (2.6) or (2.7). This remark is due to the fact that by applying the logarithmic function to (2.7) we obtain the equation (2.5).

Remark 2.3. If we keep in account that the function f can be an explicit function of six variables, the conditions (2.8) become:

$$\frac{\partial^2}{\partial z_1 \partial z_j} \ln\left(\frac{\partial f}{\partial z_1} : \frac{\partial f}{\partial z_j}\right) = 0. \tag{2.9}$$

The equations (2.5) and (2.7) can be represented by different kind of nomograms, such as:

a) One compound nomogram consisting from five nomograms with alignment point with all parallel scales (both for the variables z_i, $i = \overline{1,7}$ and for the auxiliary variables u_i, $i = \overline{1,4}$). All the support of nomogram's scales are right straight lines. Of course, the scales (z_i), $i = \overline{1,7}$ are marked and the auxiliary scales are "dumb" ones. For this nomographical representation we have the following notation:

$$\begin{gathered} f_1(z_1) + f_2(z_2) = u_1, \quad u_i + f_{i+2}(z_{i+2}) = u_{i+1},\ i = \overline{1,3} \\ u_4 + f_6(z_6) = f_7(z_7). \end{gathered} \tag{2.10}$$

b) A space nomogram with coplanary points consisting of three binary nets and a marked scale.

c) A nomogram with a transparent mobile plane; the fix plane has two binary nets, the mobile plane has a binary net and a marked straight line scale. Each of the nets has two families of marked straight right lines.

3. The function of six variables that are superpositions of three functions

They have the form:

$$z_7 = f(z_1, z_2, \ldots, z_6) = G[\varphi(z_1, z_2), \psi(z_3, z_4), \chi(z_5, z_6)]. \tag{3.11}$$

We can enounce the following

Theorem 3.2. *The necessary and sufficient conditions for that the function of six variables admits the superpositions* (3.11) *are:*

$$\begin{aligned} &\frac{\partial}{\partial z_j}\left(\frac{\partial f}{\partial z_1} : \frac{\partial f}{\partial z_2}\right) = 0, \quad j = 3,4,5,6 \\ &\frac{\partial}{\partial z_k}\left(\frac{\partial f}{\partial z_3} : \frac{\partial f}{\partial z_4}\right) = 0, \quad k = 1,2,5,6 \\ &\frac{\partial}{\partial z_l}\left(\frac{\partial f}{\partial z_5} : \frac{\partial f}{\partial z_6}\right) = 0, \quad l = 1,2,3,4 \end{aligned} \tag{3.12}$$

Proof. The conditions (3.12) are necessary. Let us consider the partial derivatives of the function f (from (3.11)) with respect to each of its variables z_i, $i = \overline{1,6}$ and dividing by two we obtain:

$$\begin{aligned} \frac{\partial f}{\partial z_1} : \frac{\partial f}{\partial z_2} &= \left(\frac{\partial G}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial z_1}\right) : \left(\frac{\partial G}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial z_2}\right) \\ &= \frac{\partial \varphi}{\partial z_1} : \frac{\partial \varphi}{\partial z_2} = h_1(z_1, z_2) \\ \frac{\partial f}{\partial z_3} : \frac{\partial f}{\partial z_4} &= \left(\frac{\partial G}{\partial \psi} \cdot \frac{\partial \psi}{\partial z_3}\right) : \left(\frac{\partial G}{\partial \psi} \cdot \frac{\partial \psi}{\partial z_4}\right) \\ &= \frac{\partial \psi}{\partial z_3} : \frac{\partial \psi}{\partial z_4} = h_2(z_3, z_4) \\ \frac{\partial f}{\partial z_5} : \frac{\partial f}{\partial z_6} &= \left(\frac{\partial G}{\partial \chi} \cdot \frac{\partial \chi}{\partial z_5}\right) : \left(\frac{\partial G}{\partial \chi} \cdot \frac{\partial \chi}{\partial z_6}\right) \\ &= \frac{\partial \chi}{\partial z_5} : \frac{\partial \chi}{\partial z_6} = h_3(z_5, z_6) \end{aligned} \tag{3.13}$$

The function h_1 is one of only two variables z_1 and z_2. So, its partial derivatives with respect to the variables z_3, z_4, z_5, z_6 are zero. Then we obtain the first four conditions of (3.12). Similarly, but considering the partial derivatives of the functions h_2 and h_3 with respect to z_1, z_2, z_5, z_6, respectively to z_1, z_2, z_3, z_4, the last two groups of the four conditions (3.12) are fulfilled.

The conditions (3.12) are sufficient. From (3.12) results that the system of equations with partial derivatives (3.13) has for each of them the general solutions of the following forms:

$$\begin{aligned} & f = H_1[\varphi(z_1, z_2), z_3, z_4, z_5, z_6] \\ & f = H_2[z_1, z_2, \psi(z_3, z_4), z_5, z_6] \\ & f = H_3[z_1, z_2, z_3, z_4, \chi(z_5, z_6)] \end{aligned} \tag{3.14}$$

Then the general solution of the system (3.13) is

$$f = G[\varphi(z_1, z_2), \psi(z_3, z_4), \chi(z_5, z_6)] \tag{3.15}$$

□

Remark 3.4. If we consider the conditions (3.12) written for the equation (1.1) we obtain an equivalent form of the Theorem 3.1.

Theorem 3.3. *For that the equation* (1.2) *can be written in the form* (3.11) *it is necessary and sufficient that the conditions*

$$\begin{aligned} \frac{1}{\frac{\partial F}{\partial z_3}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_3}\right)}{D(z_1, z_2)} &= \frac{1}{\frac{\partial F}{\partial z_4}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_4}\right)}{D(z_1, z_2)} = \frac{1}{\frac{\partial F}{\partial z_5}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_5}\right)}{D(z_1, z_2)} \\ &= \frac{1}{\frac{\partial F}{\partial z_6}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_6}\right)}{D(z_1, z_2)} = \frac{1}{\frac{\partial F}{\partial z_7}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_7}\right)}{D(z_1, z_2)} \end{aligned}$$

$$\begin{aligned} \frac{1}{\frac{\partial F}{\partial z_1}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_1}\right)}{D(z_3, z_4)} &= \frac{1}{\frac{\partial F}{\partial z_2}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_2}\right)}{D(z_3, z_4)} = \frac{1}{\frac{\partial F}{\partial z_5}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_5}\right)}{D(z_3, z_4)} \\ &= \frac{1}{\frac{\partial F}{\partial z_6}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_6}\right)}{D(z_3, z_4)} = \frac{1}{\frac{\partial F}{\partial z_7}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_7}\right)}{D(z_3, z_4)} \end{aligned} \tag{3.16}$$

$$\frac{1}{\frac{\partial F}{\partial z_1}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_1}\right)}{D(z_5, z_6)} = \frac{1}{\frac{\partial F}{\partial z_2}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_2}\right)}{D(z_5, z_6)} = \frac{1}{\frac{\partial F}{\partial z_3}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_3}\right)}{D(z_5, z_6)}$$
$$= \frac{1}{\frac{\partial F}{\partial z_4}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_4}\right)}{D(z_5, z_6)} = \frac{1}{\frac{\partial F}{\partial z_7}} \cdot \frac{D\left(F, \frac{\partial F}{\partial z_7}\right)}{D(z_5, z_6)}$$

are hold.

We will study now the nomographical representation of the function (3.11) (respectively, equation (1.2)).

The functions (3.11) can be nomographically represented by a compound plane nomogram with ramifications. For this aim, with the notations

(3.17) $\varphi(z_1, z_2) = u, \quad \psi(z_3, z_4) = v, \quad \chi(z_5, z_6) = w$

the function (3.11) become

(3.18) $z_7 = F(u, v, w)$

where u, v, w are the auxiliary variables. (3.18) can be written in an equivalent form

(3.19) $g(z_7, w) = h(u, v)$.

With a new auxiliary variable, s, (3.19) becomes

(3.20) $g(z_7, w) = s = h(u, v)$.

For each of the equations (3.17) and (3.20) we can build a net nomogram with three marked families of curves, two of them being straight line families. The variables z_i, $i = \overline{1,7}$ are represented by straight line families (z_i), and the auxiliary variables by families of unmarked curves, e.g. (u), (v), (w), (s).

So, we represent in fig.1 the compound plane nomogram with ramifications, for the function of six variables (3.11) (or for the equation with seven variables (1.2)).

For an easily refer to the curves of the families which form net nomogram, we will note those three families of curves (or straight lines) by a triplet. This triplet is formed with the parameters of curves and straight lines families. So, to those five net nomograms which form the compound nomogram from fig.1 correspond the triplets (z_1, z_2, u), (z_3, z_4, v), (z_5, z_6, w), (u, v, s), (s, w, z_7).

We assume that the values of the variables z_i, $i = \overline{1,6}$ are known (e.g. z_i^0, $i = \overline{1,6}$) and we search with the help of the nomogram from fig.1, the values of the variables z_7. All values z_i^0 of the variables z_i, $i = \overline{1,7}$ satisfy the equation (3.11).

The crossings of the straight lines with marks z_i^0 and z_{i+1}^0, $i = 1, 3$, determine in the respective net nomograms the points P_1 and P_2. Through these points pass the curves u^0 and v^0 from the auxiliary unmarked families of curves (u) respectively (v). The crossing of curves u^0 and v^0 determines in the nomogram (u, v, s) the point P, through which passes the straight line s^0 from the family (s) of this last nomogram. Through the point P_3, determined by the straight lines z_5^0

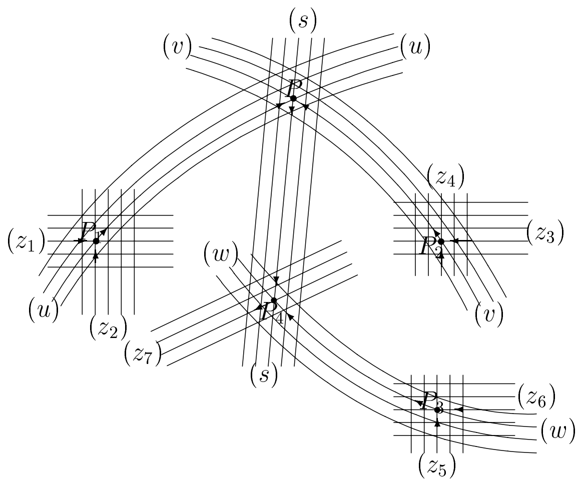

FIGURE 1

and z_6^0 passes the curve w^0 from the auxiliary family (w). The crossing of straight line s^0 with the curve w^0 determines the point P_4. Through this last point of the nomogram (s, w, z_7) passes one straight line from the family (z_7) and his mark z_7^0 will give the value we have searched for the variable z_7.

4. THE FUNCTION OF SIX VARIABLES THAT ARE SUPERPOSITIONS OF TWO FUNCTIONS

They present the form:

$$z_7 = f(z_1, z_2, \ldots, z_6) = G[\varphi(z_1, z_2, z_3), \psi(z_4, z_5, z_6)]. \tag{4.21}$$

We have

Theorem 4.4. *The necessary and sufficient conditions for that the function* (1.1) *admits the superpositions* (4.21) *are:*

$$\frac{\partial}{\partial z_j}\left(\frac{\partial f}{\partial z_i} : \frac{\partial f}{\partial z_{i+1}}\right) = 0, \quad i = 1, 2,\ j = 4, 5, 6 \tag{4.22}$$

and

$$\frac{\partial}{\partial z_k}\left(\frac{\partial f}{\partial z_i} : \frac{\partial f}{\partial z_{i+1}}\right) = 0, \quad i = 4, 5,\ k = 1, 2, 3. \tag{4.23}$$

Proof. It is easy to prove the necessity, remarking that the quotients of partial derivatives with respect to z_i and z_{i+1} for $i = 1, 2$ are functions only of three variables, z_1, z_2, z_3, namely:

$$\frac{\partial f}{\partial z_i} : \frac{\partial f}{\partial z_{i+1}} = q_i(z_1, z_2, z_3), \quad i = 1, 2. \tag{4.24}$$

So, their partial derivatives (of (4.24)) with respect to other three variables of the function f are zero; and we obtain the relations (4.22). The conditions (4.23) are obtained analogously.

Conversely, assuming that (4.22) hold, then the system of two equations (4.24) hold. The general solution of the first equation (4.24) is

(4.25) $f = G[\varphi_1(z_1, z_2, z_3), z_3, z_4, z_5, z_6]$.

This solution satisfies the second equation (4.24), namely

(4.26) $$\frac{\partial G}{\partial \varphi_1} \cdot \frac{\partial \varphi_1}{\partial z_2} = \left[\frac{\partial G}{\partial \varphi_1} \cdot \frac{\partial \varphi_1}{\partial z_3} + \frac{\partial G}{\partial z_3}\right] q_2(z_1, z_2, z_3).$$

But and the function of three variables $\varphi_1(z_1, z_2, z_3) = f[\varphi_1(z_1, z_2, z_3), 0, 0, 0]$ is a solution of the system (4.24), so we have

(4.27) $$\frac{\partial \varphi_1}{\partial z_1} : \frac{\partial \varphi_1}{\partial z_2} = q_1(z_1, z_2, z_3), \quad \frac{\partial \varphi_1}{\partial z_2} : \frac{\partial \varphi_1}{\partial z_3} = q_2(z_1, z_2, z_3).$$

So, if we take in account the second equation from (4.27), the relation (4.26) become

(4.28) $$\frac{\partial G}{\partial z_3} q_2(z_1, z_2, z_3) = 0.$$

From (4.28), considering $q_2(z_1, z_2, z_3) \neq 0$ results $\dfrac{\partial G}{\partial z_3} = 0$. If $q_2(z_1, z_2, z_3)$ should be zero, it results that the function G (so and f) should be independent of variables z_2; but this fact is falsely. Therefore the function f has the form

$$f = G[\varphi_1(z_1, z_2, z_3), z_4, z_5, z_6]$$

or

(4.29) $z_7 = f[z_1, \ldots, z_6] = G[\varphi_1(z_1, z_2, z_3), \psi_1(z_4, z_5, z_6)]$.

But (4.29) is exactly the form (4.21) of the function f.

Having in view the nomographic representation of the function f from (4.21) we built his nomogram by the help of auxiliary variables u, v

(4.30) $u = \varphi(z_1, z_2, z_3), \quad v = \psi(z_4, z_5, z_6)$.

If the functions φ and ψ satisfy the Goursat's condition of separation variables namely

(4.31) $$\frac{D\left(\varphi, \dfrac{\partial \varphi}{\partial z_3}\right)}{D(z_1, z_2)} = 0, \quad \frac{D\left(\psi, \dfrac{\partial \psi}{\partial z_6}\right)}{D(z_4, z_5)} = 0$$

we can built for each of them a compound net nomogram by means of other two auxiliary variables, s and t

(4.32) $s = \varphi_1(z_1, z_2) = \varphi_2(z_3, u), \quad t = \psi_1(z_4, z_5) = \psi_2(z_6, v)$.

Each pair of these four equations can be represented by a compound net nomogram, with the auxiliary families of curves s, respective t.

Finally, replacing (4.30) in (4.21) we obtain

(4.33) $z_7 = f(u, v)$.

For the equation (4.33) we can also built a net nomogram, where two families are straight lines and the third one is a family of curve.

Finally, if we consider (4.30), (4.32), (4.33) we conclude that the function (4.21) can be nomographically represented by a compound plane nomogram with ramification. This compound nomogram is formed from five net nomograms which correspond to the triplets (a triplet consist up of three parameters of families net's nomogram) namely (z_1, z_2, s), (z_3, u, s), (z_4, z_5, t), (z_6, v, t), (u, v, z_7).

The compound nomogram and his use is given in fig.2.

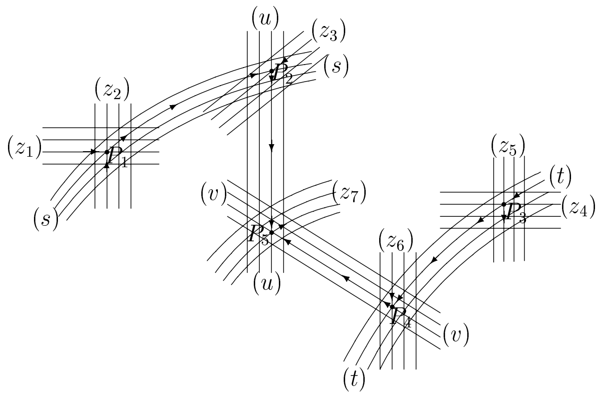

FIGURE 2

□

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