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Equations involving arithmetic functions

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ABSTRACT. We solve a few equations concerning arithmetic functions. The proofs in the last section are based on known (but difficult) inequalities.

1. INTRODUCTION

The notations used in the paper are basically the classical ones, namely:

- $\pi(x)$: The number of primes not exceeding x
- p_n: The n^{th} prime (in increasing order)
- c_n: The n^{th} composed number (in increasing order)
- $\varphi(n)$: Euler's totient function
- $\tau(n)$: The number of divisors of n
- $\sigma(n)$: Sum of divisors function
- $\omega(n)$: The number of distinct prime factors of n
- γ: The Euler constant ($\gamma = 0.57721...$)

We will use the following results:

(A) $\pi(x) < \dfrac{x}{\log x - 1.12}$ for all $x \geq 4,$ from [1];

(B) $p_n > n(\log n + \log\log n - 1)$ for all $n \geq 2,$ from [2];

(C) $n\left(1+\dfrac{1}{\log n}+\dfrac{1}{\log^2 n}\right) < c_n < n\left(1+\dfrac{1}{\log n}+\dfrac{3}{\log^2 n}\right)$ for all $n \geq 4,$ from [2];

(D) $\log\tau(n) < \dfrac{\log n \log 2}{\log\log n - 1.39117}$ for all $n \geq 56,$ from [4];

(E) $\varphi(n) = \dfrac{n}{e^{\gamma}\log\log n + \dfrac{2.50637}{\log\log n}}$ for all $n \geq 3,$ from [3];

(F) $\omega(n) \leq \dfrac{\log n}{\log\log n - 1.1714}$ for all $n \geq 26,$ from [5];

The methods we will use in Section 2 are independent of the inequalities (A)–(F) above. For the results presented in Section 3 we will need the mentioned inequalities, standard techniques employed for their application, as well as some computer checking. Although all the proofs are elementary, we will only use the term "elementary" for the equations in Section 2. We choose to do so because the

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methods used in the proofs of the inequalities (A)–(F) are non-elementary, giving the same character to the equations which are solved using them.

2. ELEMENTARY DIOPHANTINE EQUATIONS

Equation 2.1. *The equation* $\sigma(n) = \tau^2(n)$ *has the solutions* 1 *and* 3.

Proof. Let $f(n) = \dfrac{\sigma(n)}{\tau^2(n)}$. The function f is totally multiplicative, so we compute $f(p^\alpha)$ with prime p and integer $\alpha \geq 1$; we have

$$f(p^\alpha) = \frac{\sigma(p^\alpha)}{\tau^2(p^\alpha)} = \frac{p^\alpha + p^{\alpha-1} + \cdots + p + 1}{(\alpha+1)^2}.$$

$f(p^\alpha)$ is obviously increasing with respect to p. We will study its monotony with respect to α. Let $g_p(\alpha) = f(p^\alpha) = \dfrac{p^{\alpha+1} - 1}{(p-1)(\alpha+1)^2}$, $\alpha \geq 1$. The derivative of this function is $g_p'(\alpha) = \dfrac{p^{\alpha+1}((\alpha+1)\ln p - 2) + 2}{(p-1)(\alpha+1)^3}$.

If $p \geq 3$, then $\ln p > 1$. Since $\alpha + 1 \geq 2$, we derive $(\alpha+1)\ln p - 2 > 0$, so $g_p'(\alpha) > 0$. Therefore, g_p is an increasing function for $\alpha \geq 1$.

If $p = 2$ and $\alpha \geq 2$ we have $(\alpha+2)\ln 2 - 2 > 3\ln 2 - 2 > 0$, so $g_2'(\alpha) > 0$. Therefore, g_2 is an increasing function for $\alpha \geq 2$.

Now, Equation 2.1 is equivalent to $f(n) = 1$. This relation is verified by $n = 1$ and $n = 3$. We will study the existence of other solutions, using the above monotony information.

Let m be odd, $m \geq 5$. If $(m, 3) = 1$, we have $f(m) \geq f(5) = 6/4 = 3/2$. If $3^2|m$, then $f(n) \geq f(9) = 13/9$. If $m = 3q, (q, 3) = 1$, then $q \geq 5$ and $f(m) = f(3)f(q) = f(q) \geq 3/2$. Consequently, for odd $m \geq 5$, we have $f(m) \geq 13/9$.

Let now $n = 2^\alpha$. We have

$$f(2) = \frac{3}{4} < 1, f(2^2) = \frac{7}{9} < 1, f(2^3) = \frac{15}{16} < 1, \text{while } f(2^4) = \frac{31}{25} > 1.$$

Using the monotony of g_2, $f(2^\alpha) > 1$ for all $\alpha \geq 4$. Therefore, for $n = 2^\alpha$, we have $f(n) \neq 1$. If $n = 2^\alpha \cdot 3$, $f(n) = f(2^\alpha \cdot 3) = f(2^\alpha) \cdot f(3) = f(2^\alpha) \neq 1$. If $n = 2^\alpha \cdot m$ with odd $m \geq 5$, we have

$$f(n) = f(2^\alpha) \cdot f(m) \geq f(2) \cdot f(m) \geq \frac{3}{4} \cdot \frac{13}{9} > 1.$$

Since we discarded all the other possibilities, the only solutions of Equation 2.1 are 1 and 3. □

Equation 2.2. *The solution of the equation* $n\tau(n) - \sigma(n)\omega(n) = \varphi(n)$ *consists of* $1, 10,$ *and all the prime numbers.*

Proof. $n = 1$ and $n = p$ with prime p are obviously solutions. We will search for other solutions.

If $n = p^\alpha$ with prime p and $\alpha \geq 2$ would be a solution, we would derive

$$p^\alpha(\alpha+1) - (p^\alpha + p^{\alpha-1} + \cdots + 1) = p^\alpha - p^{\alpha-1},$$

whence

$$(\alpha-1)p^\alpha - p^{\alpha-2} - \cdots - p - 1 = 0,$$

and the contradiction $p|1$.

Now let $\omega(n) = k \geq 2$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Equation 2.2 is equivalent to

$$\tau(n) - k\frac{\sigma(n)}{n} = \frac{\varphi(n)}{n}, \tag{2.1}$$

relation that can also be written as

$$\tau(n) - k\prod_{i=1}^{k}\left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i}}\right) = \prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right). \tag{2.2}$$

We have

$$1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i}} < 1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots = \frac{1}{1 - \frac{1}{p_i}} = 1 + \frac{1}{p_i - 1},$$

whence

$$\begin{aligned} \prod_{i=1}^{k}\left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i}}\right) &< \prod_{i=1}^{k}\left(1 + \frac{1}{p_i - 1}\right) \\ &< \left(1 + \frac{1}{2-1}\right)\left(1 + \frac{1}{3-1}\right)\left(1 + \frac{1}{5-1}\right)\cdots\left(1 + \frac{1}{(2k-1)-1}\right) \\ &= 2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2k-1}{2k-2}. \end{aligned} \tag{2.3}$$

One shows by recurrence that for $n \geq 5$ the inequality

$$\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2k-1}{2k-2} < \frac{7}{6}\sqrt{k - \frac{1}{4}},$$

holds true, leading to

$$\frac{\sigma(n)}{n} < \frac{7}{3}\sqrt{k - \frac{1}{4}} < \frac{7}{3}\sqrt{k}. \tag{2.4}$$

Another useful (although simple) relation is

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1) \geq 2^k. \tag{2.5}$$

We will now use the relations established above to determine the other solutions of Equation 2. We will consider the following cases:

Case $a) : k \geq 5$. Using relation (2.4), we get

$$k\frac{\sigma(n)}{n} + \frac{\varphi(n)}{n} < \frac{7}{3}k\sqrt{k} + 1. \tag{2.6}$$

The relation

$$\frac{7}{3}k\sqrt{k} + 1 < 2^k \tag{2.7}$$

holds for all $k \geq 5$ (by recurrence again).

Using relations (2.5), (2.6), and (2.7), we derive that $\tau(n) > k\frac{\sigma(n)}{n} + \frac{\varphi(n)}{n}$, so, taking relation (2.1) into account, Equation 2.2 has no solution with $k \geq 5$.

Case $b) : k = 2$. We have $n = p^\alpha q^\beta$, with $p < q$ primes and $\alpha, \beta \geq 1$.

Case $b1)$: $\alpha = \beta = 1$. Equation 2.2 becomes

$$4 = 2\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)+\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right),$$

so $(p-1)(q-1) = 4$, leading to $p = 2$ and $q = 5$, whence $n = 10$.

Case $b2)$: $\alpha + \beta = 3$. We have $n = pq^2$ or $n = p^2q$, and therefore,

$$\begin{aligned}\frac{k\sigma(n)}{n}+\frac{\varphi(n)}{n} &= 2\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}+\frac{1}{q^2}\right)+\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\\ &= 3+\frac{1}{p}+\frac{1}{q}+\frac{3}{pq}+\frac{2}{q^2}\left(1+\frac{1}{p}\right)\\ &\le 1+\frac{1}{2}+\frac{1}{3}+\frac{3}{6}+\frac{2}{2^2}\left(1+\frac{1}{3}\right) < 6 \le \tau(n),\end{aligned}$$

so Equation 2.2 has no solutions in this case.

Case $b3)$: $\alpha + \beta \ge 4$. Then we have

$$\tau(n) = (\alpha+1)(\beta+1) \ge 8. \tag{2.8}$$

Moreover,

$$\begin{aligned}\frac{k\sigma(n)}{n}+\frac{\varphi(n)}{n} &< 2\left(1+\frac{1}{p-1}\right)\left(1+\frac{1}{q-1}\right)+\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\\ &= 3+\frac{2}{p-1}-\frac{1}{p}+\frac{2}{q-1}-\frac{1}{q}+\frac{2}{(p-1)(q-1)}+\frac{1}{pq}\\ &= 3+\frac{p+1}{p(p-1)}+\frac{q+1}{q(q-1)}+\frac{2}{(p-1)(q-1)}+\frac{1}{pq}.\end{aligned} \tag{2.9}$$

Since $p \ge 2$, $q \ge 3$, and the function $f(x) = \dfrac{x+1}{x(x-1)}$ decreases for $x \ge 2$, taking relations (2.8) and (2.9) into account, we get

$$\frac{k\sigma(n)}{n}+\frac{\varphi(n)}{n} < 3+\frac{3}{2}+\frac{2}{3}+1+\frac{1}{6} < 8 \le \tau(n),$$

so this case does not provide solutions for Equation 2.2.

Case c): $k = 3$. We have $n = p^\alpha q^\beta r^\gamma$.

Case c1): $\alpha = \beta = \gamma = 1$. Equation 2.2 becomes

$$8 = 3\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)\left(1+\frac{1}{r}\right)+\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right).$$

If $p, q, r > 2$, we obtain

$$\begin{aligned}\frac{k\sigma(n)}{n}+\frac{\varphi(n)}{n} &< 3\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)\left(1+\frac{1}{r}\right)+1\\ &\le 3\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)+1 < 6+1 < 8 \le \tau(n),\end{aligned}$$

so Equation 2.2 has no solutions.

If $p = 2$, we will have $\frac{\varphi(n)}{n} < \frac{1}{2}$, leading to

$$\frac{k\sigma(n)}{n} + \frac{\varphi(n)}{n} < 3\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right) + \frac{1}{2} < 8 \le \tau(n),$$

so Equation 2.2 has no solutions.

Case c2): $\alpha+\beta+\gamma \ge 4$. We then have $\tau(n) \ge 12$.

If $p, q, r > 2$, we get

$$\begin{aligned}\frac{k\sigma(n)}{n} + \frac{\varphi(n)}{n} &< 3\left(1+\frac{1}{p-1}\right)\left(1+\frac{1}{q-1}\right)\left(1+\frac{1}{r-1}\right) + 1\\ &\le 3\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} + 1 < 7 + 1 < 12 \le \tau(n),\end{aligned}$$

so Equation 2.2 has no solutions.

If $p = 2$,

$$\frac{k\sigma(n)}{n} + \frac{\varphi(n)}{n} \le 3\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4} + \frac{1}{2} = \frac{47}{4} < 12 \le \tau(n).$$

We conclude that Equation 2.2 has no solution in *Case* c).

Case d): $k = 4$. We have $n = p^\alpha q^\beta r^\gamma s^\delta$ with p, q, r, s distinct primes and $\alpha, \beta, \gamma, \delta \ge 1$, so $\tau(n) \ge 16$.

If $\alpha = \beta = \gamma = \delta = 1$, then

$$\begin{aligned}\frac{k\sigma(n)}{n} + \frac{\varphi(n)}{n} &< 4\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right)\left(1+\frac{1}{r}\right)\left(1+\frac{1}{s}\right) + 1\\ &\le 4\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7} + 1 < 11 + 1 < 16 \le \tau(n).\end{aligned}$$

If $\alpha+\beta+\gamma+\delta \ge 5$, then $\tau(n) \ge 2\cdot2\cdot2\cdot3 = 24$, and

$$\begin{aligned}\frac{k\sigma(n)}{n} + \frac{\varphi(n)}{n} &< 4\left(1+\frac{1}{p-1}\right)\left(1+\frac{1}{q-1}\right)\left(1+\frac{1}{r-1}\right)\left(1+\frac{1}{s-1}\right) + 1\\ &\le 4\cdot\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} + 1 = \frac{35}{2} + 1 < 24 \le \tau(n).\end{aligned}$$

Therefore, *Case* d) does not provide solutions for Equation 2.2.

We conclude that the solution of Equation 2.2 consists of 1,10, and all the prime numbers. □

3. NON-ELEMENTARY DIOPHANTINE EQUATIONS

Equation 3.3. *The solutions of the equation* $\varphi(n)\tau(n) = c_n$ *are* 11 *and* 14.

Proof. We first show that for $n \ge 130$ we have $\varphi(n)\tau^2(n) > c_n$.

Let $f(n) = \frac{\varphi(n)\tau(n)}{n}$. The function f is obviously multiplicative.

Case a): $n = p^\alpha$. We have

$$f(n) = \frac{p^{\alpha-1}(p-1)(\alpha+1)}{p^\alpha} = \frac{(p-1)(\alpha+1)}{p} \ge 2\left(1-\frac{1}{p}\right) \ge 1,$$

equality holding only if $p = 2$ and $\alpha = 1$, i.e., $n = 2$; for the other values of n we have $f(n) > 4/3$.

Case b): $\omega(n) = k \geq 2$.
Then we have $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $p_1 < p_2 < \ldots < p_k$. Since $f(p^\alpha) = \left(1 - \dfrac{1}{p}\right)(\alpha + 1)$ increases with respect to both p and α,

$$f(n) = f(p_1^{\alpha_1}) f(p_2^{\alpha_2}) \cdots f(p_k^{\alpha_k}) \geq f(p_k^{\alpha_k}) \geq f(3) = \frac{4}{3}.$$

We conclude that for every $n \geq 3$ we have $f(n) \geq 4/3$.

Now, according to inequality (**C**), for all $n \geq 2$ we have

$$c_n < n\left(1 + \frac{1}{\log n} + \frac{3}{\log^2 n}\right).$$

On the other hand, for all $n \geq 130$ we have $1 + \dfrac{1}{\log n} + \dfrac{3}{\log^2 n} < \dfrac{4}{3} \leq f(n)$. Since Equation 3.3 can be written as $f(n) = \dfrac{c_n}{n}$, the last three relations show that no value $n \geq 130$ can verify Equation 3.3. Computer checking for $n < 130$ shows that the solution of Equation 3.3 consists of 11 and 14. □

Equation 3.4. *The solutions of equation* $c_n = n + 2\tau(n)$ *are* $2, 10, 16, 20, 48$, *and* 60.

Proof. We will first prove that for $n \geq 46103$ we have

(3.10) $\quad c_n > n + 2\tau(n)$.

Using inequalities (**C**) and (**D**), for $n \geq 56$ relation (3.10) is equivalent to

$$n\left(1 + \frac{1}{\log n} + \frac{1}{\log^2 n}\right) > n + 2 \cdot 2^{\frac{\log n}{\log\log n - 1.39177}},$$

otherwise written as

(3.11) $\quad \dfrac{n}{\log n} + \dfrac{n}{\log^2 n} > 2 \cdot 2^{\frac{\log n}{\log\log n - 1.39177}}.$

Relation (3.11) would obviously be true if $\dfrac{n}{\log n} > 2 \cdot 2^{\frac{\log n}{\log\log n - 1.39177}}$. By taking logarithms of both sides, the last relation is equivalent to

$$\log n - \log\log n > \frac{\log n \log 2}{\log\log n - 1.39177} + \log 2,$$

otherwise written as

(3.12) $\quad \log n > \dfrac{(\log\log n + \log 2)(\log\log n - 1.39177)}{\log\log n - 2.085}.$

Now, for $n > 46103$ we have

(3.13) $\quad \dfrac{(\log\log n + \log 2)(\log\log n - 1.39177)}{\log\log n - 2.085} < 3.4\,.$

On the other hand, for $x \geq 46103$ the function $f(x) = \log x - 3.4(\log\log x + \log 2$ is increasing, and $f(46103) > 10.73 - 10.43 > 0$. Therefore,

(3.14) $\quad \log n > 3.4(\log\log n + \log 2)$for all $n \geq 46103$.

Relations (3.13) and (3.14) show that relation (3.12) - and, consequently, relation (3.10) - holds true for all $n \geq 46103$. Therefore, Equation 3.3 will have no solution $n \geq 46103$.

Computer checking for $n \leq 46102$ shows that the solution of Equation 3.4 consists of 2, 10, 16, 20, 48, and 60. □

Equation 3.5. *The solutions of the equation* $\varphi(n) = \pi(n) + \omega(n)$ *are* $5, 15, 22, 54$, *and* 78.

Proof. Let $n \geq 31$. We then have $(\log\log n)^2 > 1.50334$, so $1.6672(\log\log n)^2 > 2.50637$, leading to $\left(\dfrac{1}{0.29} - e^\gamma\right)(\log\log n)^2 > 2.50637$, otherwise written as

$$\frac{1}{e^\gamma \log\log n + \dfrac{2.50637}{\log\log n}} > \frac{0.29}{\log\log n};$$

taking (**E**) into account, we find that

$$\text{(3.15)} \quad \varphi(n) > \frac{0.29n}{\log\log n} \quad \text{for all} \quad n \geq 20.$$

If $n \geq 32000$, we have $\dfrac{n}{\log n} > 3000$ and $\dfrac{1}{\log\log n - 1.1714} < \dfrac{2.1}{\log\log n}$, so $\dfrac{\log n}{\log\log n - 1.1714} < \dfrac{0.01n}{\log\log n}$; taking (**F**) into account, we derive that

$$\text{(3.16)} \quad \omega(n) < \frac{0.01n}{\log\log n} \quad \text{for all} \quad n \geq 329.$$

Finally, for $n \geq 32000$ we have $\log n - \dfrac{25}{7}\log\log n - 1.12 > 0$, so

$$\frac{1}{\log\log n - 1.12} < \frac{0.28}{\log\log n}.$$

Using relation (**A**), we obtain

$$\text{(3.17)} \quad \pi(n) < \frac{0.28n}{\log\log n} \quad \text{for all} \quad n \geq 800.$$

Relations (3.15), (3.16), and (3.17) imply

$$\pi(n) + \omega(n) < \frac{0.29n}{\log\log n} < \varphi(n) \quad \text{for all} \quad n \geq 32000.$$

Computer checking for $n < 32000$ allow us to list the solutions of Equation 3.5: 5, 15, 22, 54, and 78. □

Equation 3.6. *The solutions of the equation* $p_n = c_n\omega(n) + 1$ *are* 5 *and* 42. *The equation* $p_n = c_n\omega(n) - 1$ *has no solution.*

Proof. Using relations (**C**) and (**F**), we have

$$c_n\omega(n) < n\left(1 + \frac{1}{\log n} + \frac{3}{\log^2 n}\right)\frac{\log n}{\log\log n - 1.1714}.$$

For $n \geq 12239$, we have $\log\log n - 1.1714 > 1.07$, so

$$c_n\omega(n) < \frac{n\log n\left(1 + \dfrac{1}{\log n} + \dfrac{3}{\log^2 n}\right)}{1.07}. \tag{3.18}$$

On the other hand, for $n \geq 12239$ we have $\log n > 6.55$, so $0.07\log n > \dfrac{3}{\log n}$; this relation and $\log\log n - 1.2 > 1$ imply

$$\log n + 1 + \frac{3}{\log n} < 1.07\log n + 1.07(\log\log n - 1.2),$$

otherwise written as

$$n\left(\log n + 1 + \frac{3}{\log n}\right) < 1.07\left(n(\log n + \log\log n - 1) - \frac{n}{5}\right). \tag{3.19}$$

Relations (3.18), (3.19), and (**B**) imply

$$p_n > c_n\omega(n) + \frac{n}{5} \quad \text{for all} \quad n \geq 12239.$$

Therefore, Equations 3.6 have no solution $n \geq 12239$.

Computer checking made for $n < 12239$ led to:

The solution of the equation $p_n = c_n\omega(n) + 1$ consists of 5 and 42.

The equation $p_n = c_n\omega(n) - 1$ has no solutions. □

Remark 3.1. The solution of the equation $p_n = n\omega(n) + 1$ consists of 2 and 6. The equation $p_n = n\omega(n) - 1$ has no solutions.

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