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On p-cluster sets and their application to p-closedness

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ABSTRACT. In this paper, the study of a new type of cluster sets for functions and multifunctions between topological spaces has been initiated by use of preopen sets of [8]. Such a notion of cluster sets is characterized and studied to some extent. In the process, a characterization of Hausdorffness is achieved in terms of the cluster set of a certain class of functions. An important application of the study is exhibited by establishing a characterization of the concept of p-closedness of [7] via the introduced idea of cluster set of multifunctions.

1. INTRODUCTION

The notion of cluster sets of arbitrary functions between topological spaces was first introduced by Weston [17]. Similar theories for different classes of functions and multifunctions have already been studied by good many researchers. Such theories are seen to have many applications specially to the characterizations of different covering properties and certain separation axioms. For instance, Joseph [6] characterized some important covering properties like compactness and H-closedness by means of cluster sets and derived some interesting consequences of such a study. In [10] and [9], δ-cluster sets and S-cluster sets respectively were investigated, which were applied to the characterizations of near compactness and S-closedness respectively among other things.

A new type of covering property under the terminology p-closedness, was rather recently introduced by Abo-Khadra [7] by use of preopen sets of [8], and the concept has already been investigated extensively by many workers (for example, see [4]).

Our aim in this paper is to initiate the study of a new form of cluster sets of functions and multifunctions with preopen sets and allied notions as the supporting tools. Calling it p-cluster set, we show ultimately that such a concept can be applied successfully towards the characterizations of p-closedness. Moreover, we have derived explicit characterizations of p-cluster sets of functions and have found that Hausdorffness of a space can be formulated via the idea of cluster sets of suitable functions. This is a new approach to p-closedness with p-cluster set as an appliance, and surely the latter concept can further be investigated in more detail to arrive at many applications of it.

In what follows, by a space X (or Y) we mean a topological space endowed with a topology τ (say). The closure and interior of a subset of a space will be denoted by cl$A$ and int$A$ respectively. The notation $f : X \to Y$ will stand for a function f from a space X to a space Y. Similarly, for a multifunction F from X to Y (i.e., a function mapping points of X into nonempty subsets of Y) we use

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the notation $F : X \to Y$. A subset A of a space X is called preopen (preclosed) [8] if $A \subseteq$ intcl$A$ (resp. clint$A \subseteq A$). Clearly, A ($\subseteq X$) is preopen iff $(X \setminus A)$ is preclosed. The set of all preopen (resp. open) sets in a space X, each containing a given subset A of X will be denoted by $PO(A)$ (resp. $\tau(A)$). In case A is a singleton, say $A = \{x\}$ (for $x \in X$), we shall write $PO(x)$ (resp. $\tau(x)$) instead of $PO(\{x\})$ (resp. $\tau(\{x\})$). For any subset A of X, the intersection of all preclosed sets containing A is called the preclosure of A [5], denoted by pcl$A$, and A is preclosed iff $A =$ pcl$A$.

For any subset A of a space X, the θ-closure [16] ($p(\theta)$-closure [4]) of A, to be denoted by θ-cl$A$ (resp. $p(\theta)$-cl$A$), is the set of all those points x of X such that for every $U \in \tau(x)$ (resp. $U \in PO(x)$), cl$U \cap A \neq \emptyset$ (resp. pcl$U \cap A \neq \emptyset$). The set A is called θ-closed (resp. $p(\theta)$-closed [4]) if $A = \theta$-cl$A$ (resp. $A = p(\theta)$-cl$A$).

2. p-CLUSTER SETS OF FUNCTIONS AND MULTIFUNCTIONS

Let us begin by recalling the definition of a grill and the notion of its $p(\theta)$-convergence (as considered in [11]).

Definition 2.1. [15] A non-void collection $\mathcal{G}$ of nonempty sets in a topological space X is called a grill on X if (i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \ \Rightarrow\ B \in \mathcal{G}$, and
(ii) $A \cup B \in \mathcal{G}$ $(A, B \subseteq X) \ \Rightarrow\ A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.2. A grill $\mathcal{G}$ on a topological space X is said to be $p(\theta)$-convergent to a point x of X if to each $U \in PO(x)$, there corresponds some $G \in \mathcal{G}$ such that $G \subseteq$ pcl U.

Note 2.1. The above definition of $p(\theta)$-convergence of a grill can equivalently be described as : A grill $\mathcal{G}$ on X is $p(\theta)$-convergent to a point $x(\in X)$ iff $\{$ pcl$U : U \in PO(x)\} \subseteq \mathcal{G}$.

The notion of pre-θ-accumulation point of a filterbase was given in [4] in the following way:

Definition 2.3. A filterbase $\mathcal{F}$ on a topological space X is said to pre-θ-accumulate at a point x of X, written as $x \in p(\theta)$-$ad\mathcal{F}$, if for each $U \in PO(x)$ and each $F \in \mathcal{F}$, $F \cap$ pcl$U \neq \emptyset$.

We now introduce the definition of a p-cluster set as follows.

Definition 2.4. Let $F : X \to Y$ be a multifunction and $x \in X$, where X and Y are any topological spaces. Then the p-cluster set of F at x, denoted by $p(F, x)$ is defined to be the set $\cap\{\theta$-cl $F($pcl$U) : U \in PO(x)\}$.
Similarly for a function $f : X \to Y$, the p-cluster set of f at x, denoted by $p(f, x)$, is defined by $p(f, x) = \cap\{\theta$-cl $f($pcl$U) : U \in PO(x)\}$.

In the next theorem we derive certain characterizing conditions of the above defined concept.

Theorem 2.1. *For any function $f : X \to Y$, the following are equivalent:*
(a) $y \in p(f, x)$.
(b) *The filterbase $f^{-1}($ cl$\tau(y))$ pre-θ-accumulates at x, where cl$\tau(y)$ is the collection of closures of all open neighborhoods of y in Y.*

(c) *There is a grill $\mathcal{G}$ on X such that $\mathcal{G}$ is $p(\theta)$-convergent to $x(\in X)$ and $y \in \cap\{\theta$-cl $f(G) : G \in \mathcal{G}\}$.*

Proof. **(a)** $\Rightarrow$ **(b)**: Let $y \in p(f,x)$. Then for each open neighborhood V of y in Y and each $U \in PO(x)$, $\mathrm{cl}V \cap f(\mathrm{pcl}U) \neq \emptyset$, which gives $f^{-1}(\mathrm{cl}V) \cap \mathrm{pcl}U \neq \emptyset$. Hence $\{f^{-1}(\mathrm{cl}V) : V$ is an open neighbourhood of $y\}$ pre-θ-accumulates at x.

(b) $\Rightarrow$ **(c):** Let $\mathcal{F}$ denote the filterbase $f^{-1}(\mathrm{cl}\tau(y))$, and let $\mathcal{G} = \{G \subseteq X : G \cap F \neq \emptyset$, for all $F \in \mathcal{F}\}$. We first show that $\mathcal{G}$ is a grill on X. Obviously $\mathcal{G}$ is non-void and that $\emptyset \notin \mathcal{G}$. Also, it is easy to see that if $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$. Let now $A \cup B \in \mathcal{G}$ (where $A, B \subseteq X$), i.e., $(A \cup B) \cap F \neq \emptyset$, $\forall F \in \mathcal{F}$. If possible, let $A \notin \mathcal{G}$ and $B \notin \mathcal{G}$. Then for some $F_1, F_2 \in \mathcal{F}$, $A \cap F_1 = B \cap F_2 = \emptyset$. Since $\mathcal{F}$ is a filterbase, there exists $F_3 \in \mathcal{F}$, such that $F_3 \subseteq F_1 \cap F_2$. Then $F_3 \cap (A \cup B) = \emptyset$, a contradiction. Hence $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

By(b), for each $U \in PO(x)$ in X and each open neighbourhood V of y in Y, $f^{-1}(\mathrm{cl}V) \cap \mathrm{pcl}U \neq \emptyset$, i.e., $F \cap \mathrm{pcl}U \neq \emptyset$, for each $F \in \mathcal{F}$ and each $U \in PO(x)$ in X. Hence for each $U \in PO(x)$ in X, $\mathrm{pcl}U \in \mathcal{G}$. So $\mathcal{G}$ is $p(\theta)$-convergent to x. Now, the definition of $\mathcal{G}$ implies that $f(G) \cap \mathrm{cl}W \neq \emptyset$, for each $G \in \mathcal{G}$ and each open neighbourhood W of y in Y. Thus $y \in \cap\{\theta\text{-cl}f(G) : G \in \mathcal{G}\}$.

(c) $\Rightarrow$ **(a):** Let $\mathcal{G}$ be a grill on X such that $\mathcal{G}$ is $p(\theta)$-convergent to some point x of X and let $y \in \cap\{\theta\text{-cl}f(G) : G \in \mathcal{G}\}$. Then $\{\mathrm{pcl}U : U \in PO(x)$ in $X\} \subseteq \mathcal{G}$ and $y \in \theta\text{-cl}f(G)$, for all $G \in \mathcal{G}$, i.e., $y \in \cap\{\theta\text{-cl}f(\mathrm{pcl}U) : U \in PO(x)\}$. Hence $y \in p(f,x)$.

It is clear from the definition of p-cluster set of a function (or multifunction) $f : X \to Y$ at any point x of X that $f(x) \in p(f,x)$. In the couple of theorems that follow, we shall investigate certain situations where $p(f,x)$ (resp. $p(F,x)$, for a multifunction $F : X \to Y$) becomes degenerate, i.e., $p(f,x) = \{f(x)\}$ (resp. $p(F,x) = F(x)$). □

Theorem 2.2. *A topological space Y is Hausdorff if for some space X and some surjection $f : X \to Y$, $p(f,x)$ is degenerate for each $x \in X$.*

Proof. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. As f is onto, there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now $p(f,x)$ being degenerate for each $x \in X$, $y_2 = f(x_2) \notin p(f,x_1)$. Thus there exist $V \in \tau(y_2)$ in Y and $U \in PO(x_1)$ in X such that $f(\mathrm{pcl}U) \cap \mathrm{cl}V = \emptyset$, whence we get $f(\mathrm{pcl}U) \subseteq Y \backslash \mathrm{cl}V$. Then $Y \backslash \mathrm{cl}V \in \tau(y_1)$ and $V \in \tau(y_2)$ such that $V \cap (Y \backslash \mathrm{cl}V) = \emptyset$, proving Y to be Hausdorff. □

A weak version of the converse of the above theorem is now proved.

Theorem 2.3. *If a topological space (Y,τ) is Urysohn then for some space X and some function $f : X \to Y$, $p(f,x)$ is degenerate for each $x \in X$.*

Proof. Let Y be a Urysohn space and if possible, let for each space X and each function $f : X \to Y$, $p(f,x)$ be not degenerate for at least one $x \in X$. Consider the identity map $i : (Y,\tau) \to (Y,\tau)$. Then for some $x \in X$, there exists $y \in Y$ with $y \neq x(= i(x))$ such that $y \in p(i,x)$. Then for all $U \in PO(x)$ and all $V \in \tau(y)$, $\mathrm{pcl}U \cap \mathrm{cl}V \neq \emptyset$ and hence in particular, $\mathrm{cl}U \cap \mathrm{cl}V = \mathrm{pcl}U \cap \mathrm{cl}V \neq \emptyset, \forall U \in \tau(x)$ and $\forall V \in \tau(y)$ (note that for an open set A, $\mathrm{pcl}A = \mathrm{cl}A$ [5]). Thus Y is not Urysohn. □

To ascertain for the type of functions for which the converse of Theorem 2.2 holds, we recall the definition of a specific class of functions:

Definition 2.5. [12] A function $f : X \to Y$ is said to be strongly θ-precontinuous if for each $x \in X$ and each open set V containing $f(x)$ in Y, there exists $U \in PO(x)$ in X such that $f(\text{ pcl}U) \subseteq V$.

Theorem 2.4. *Let $f : X \to Y$ be strongly θ-precontinuous with Y a Hausdorff space. Then $p(f, x)$ is degenerate for each $x \in X$.*

Proof. Let $x \in X$. As f is strongly θ-precontinuous, for each open set V containing $f(x)$ in Y, there exists $U \in PO(x)$ in X such that $f(\text{ pcl}U) \subseteq V$.
Now $p(f, x) = \cap\{\theta\text{-cl}f(\text{ pcl}U) : U \in PO(x)\} \subseteq \cap\{\theta\text{-cl}V : V \in \tau(f(x))\} = \cap\{\text{ cl}V : V \in \tau(f(x))\} \cdots (1)$. Let $y \in Y$ with $y \neq f(x)$. As Y is Hausdorff, there are disjoint open sets G and H in Y such that $y \in G$ and $f(x) \in H$. Then obviously $G\cap \text{ cl}H = \emptyset$. So $y \notin \text{ cl}H$ and then by (1), $y \notin p(f, x)$. Hence $p(f, x) = \{f(x)\}$. □

It now follows from Theorem 2.2 and 2.4 that

Corollary 2.1. *Let $f : X \to Y$ be a strongly θ-precontinuous surjection. Then Y is Hausdorff iff $p(f, x)$ is degenerate for each $x \in X$.*

Definition 2.6. [3] A function $f : X \to Y$ is called θ-closed if images of θ-closed sets of X are θ-closed in Y.

Definition 2.7. [14] A topological space X is said to be almost regular if for every regular closed set A in X and each point $x \in X \setminus A$, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$ (a set A is regular closed if $A = \text{ clint}A$).

It is known [14] that in an almost regular space X, θ-cl$A$ is θ-closed for each $A \subseteq X$.

Theorem 2.5. *Let $f : X \to Y$ be a θ-closed map from an almost regular space X into a space Y. Let $f^{-1}(y)$ is θ-closed in X for all $y \in Y$, then $p(f, x)$ is degenerate for each $x \in X$.*

Proof. We have

$$\begin{aligned} p(f, x) &= \cap\{\theta - \text{cl}f(\text{ pcl}U) : U \in PO(x)\} \\ &\subseteq \cap\{\theta - \text{cl}f(\theta - \text{cl}U) : U \in PO(x)\}. \end{aligned}$$

As X is an almost regular space, θ-cl$U$ is θ-closed for each $U \in$PO(x). As f is θ-closed map, θ-cl $f(\theta\text{-cl}U) = f(\theta\text{-cl}U)$. Hence

$$p(f, x) \subseteq \cap\{f(\theta\text{-cl}U) : U \in \text{PO}(x)\} \tag{2.1}$$

Let $y \in Y$ and $y \neq f(x)$. Then by hypothesis, $f^{-1}(y)$ is θ-closed and $x \notin f^{-1}(y)$. Hence there exists an open set G containing x in X such that cl$G \cap f^{-1}(y) = \emptyset$ and then $y \notin f(\text{cl}G) = f(\theta\text{-cl}G)$ (as G is open). So by (2.1), $y \notin p(f, x)$. Thus $p(f, x)$ is degenerate. □

Theorem 2.6. *Let $f : X \to Y$ be a θ-closed injection, where X is an almost regular Hausdorff space. Then $p(f, x)$ is degenerate.*

Proof. As X is almost regular and f is a θ-closed map, we have θ-cl $f(\theta\text{-cl}U) = f(\theta\text{-cl}U)$, for any $U \subseteq X$. Hence $p(f,x) = \cap\{\theta\text{-cl } f(\text{pcl}U) : U \in PO(x)\} \subseteq \cap\{\theta\text{-cl } f(\theta\text{-cl}U) : U \in PO(x)\}$ (since pcl$U \subseteq \theta$-cl$U) = \cap\{f(\theta\text{-cl}U) : U \in PO(x)\}$). Let $x, x_1 \in X$ with $x \neq x_1$. As f is injective, $f(x) \neq f(x_1)$. By Hausdorffness of X, there are disjoint open sets G and H in X such that $x \in G$, $x_1 \in H$. Then cl$H \cap G = \emptyset$ and hence $x_1 \notin \theta$-cl G, so that $f(x_1) \notin f(\theta\text{-cl } G)$. Because $G \in \tau(x) \subseteq PO(x)$, from (2.1) it follows that $f(x_1) \notin p(f,x)$. Hence $p(f,x)$ is degenerate. □

Theorem 2.7. *For a multifunction $F : X \to Y$, if F has θ-closed graph $G(F)$, then $p(F,x) = F(x)$, for each $x \in X$, where $G(F) = \{(x,y) \in X \times Y : y \in F(x)\}$.*

Proof. It is obvious that $F(x) \subseteq p(F,x)$, for each $x \in X$. Let $y \in p(F,x)$. Then for each $V \in \tau(y)$ in Y and $U \in PO(x)$ in X, cl$V \cap F($ pcl $U) \neq \emptyset$, i.e., $F^-($ cl $V) \cap$ pcl $U \neq \emptyset$ for each $V \in \tau(y)$ in Y and each $U \in PO(x)$ in X, where for any $B \subseteq Y$ the lower inverse F^- of F is defined in the usual way as $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. Then for any basic open set $M \times N$ in $X \times Y$ containing (x,y), $F^-($ cl $N) \cap$ pcl $M \neq \emptyset$. So for any basic open set $M \times N$ in $X \times Y$ containing (x,y), $($ cl$M \times$ cl$N) \cap G(F) \neq \emptyset$. Hence cl$(M \times N) \cap G(F) \neq \emptyset$, where $M \times N$ is a basic open set in $X \times Y$ containing (x,y). Hence $(x,y) \in \theta$-cl $G(F) = G(F)$ (as $G(F)$ is θ-closed). Hence $(x,y) \in [G(F) \cap (\{x\} \times Y)]$, so that $y \in p_2\,[G(F) \cap (\{x\} \times Y)] = F(x)$, where $p_2 : X \times Y \to Y$ is the second projection map. Hence $p(F,x) = F(x)$, $\forall\, x \in X$. □

Towards the converse of the above theorem, we now show that the degeneracy of $p(F,x)$ implies the $p(\theta)$-closedness of the graph of F; this serves as a weak converse, because it is easy to see that a θ-closed set is $p(\theta)$-closed but not conversely. To that end, we need the following results quoted from [13].

Lemma 2.1. *Let X, Y be topological spaces, $A \subseteq X$, $B \subseteq Y$, $x \in X$ and $y \in Y$. Then (a) pcl$(A \times B) \in$ pcl $A \times$ pcl B, (b) $U \in PO(x)$ and $V \in PO(y) \Rightarrow U \times V \in PO((x,y))$.*

Theorem 2.8. *For a multifunction $F : X \to Y$, if $p(F,x) = F(x)$ for each $x \in X$, then the graph $G(F)$ of F is $p(\theta)$-closed.*

Proof. Let $(x,y) \in X \times Y \setminus G(F)$. Now, $y \notin F(x) = p(F,x) \Rightarrow$ there exist $V \in \tau(y)$ in Y and $W \in PO(x)$ in X such that cl $V \cap F(\text{pcl } W) = \emptyset \Rightarrow ($ pcl$V \times$pcl $W) \cap G(F) = \emptyset$ (since pcl$V \subseteq$ cl$V) \Rightarrow$ pcl$(V \times W) \cap G(F) = \emptyset$ (by Lemma 2.1 (a)) $\Rightarrow (x,y) \notin p(\theta)$-cl $G(F)$ (using Lemma 2.1 (b)). Hence $G(F) = p(\theta)$-cl $G(F)$ so that $G(F)$ is $p(\theta)$-closed. □

3. p-CLOSEDNESS VIA p-CLUSTER SETS

As already proposed, in this section we shall show an application of the introduced concept of p-cluster set to the characterizations of the well known notion of p-closedness. The idea of p-closedness, initiated by Abo-khadra [7] followed by its extensive study by many others, goes as follows.

Definition 3.8. A nonempty subset A of a topological space X is said to be p-closed relative to X if for each cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by preopen sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{$ pcl $U_\alpha : \alpha \in \Lambda_0\}$. If, in addition $A = X$, then X is called a p-closed space.

For any subset B of a topological space X, we now introduce a notion $\mathcal{C}_B$ to denote the collection of subsets of X given by

$$\mathcal{C}_B = \{\cup_{i=1}^n \text{ pcl } U_i : U_i \in PO(X) \text{ and } \cup_{i=1}^n \text{ pcl } U_i \supseteq B,\ n \in I\!N\}.$$

The following result, giving incidentally a characterization of the p-closedness of a set, is required for our desired result.

Lemma 3.2. *A subset A of a topological space (X,τ) is p-closed relative to X iff whenever for any filterbase $\mathcal{F}$ on X, $F \cap C \neq \emptyset$ holds for each $F \in \mathcal{F}$ and each $C \in \mathcal{C}_A$, one has $A \cap p(\theta)$-ad $\mathcal{F} \neq \emptyset$.*

Proof. Let A be p-closed relative to X and $\mathcal{F}$ be a filterbase on X satisfying the given hypothesis. If possible, let $A \cap p(\theta)$-ad $\mathcal{F} = \emptyset$. Then for each $a \in A$, there are $U(a) \in PO(a)$ and $F(a) \in \mathcal{F}$ such that $F(a) \cap$ pcl $U(a) = \emptyset$. Since $\{U(a) : a \in A\}$ is a cover of A by preopen sets of X, by p-closedness of A relative to X, there exists a finite subset A_0 of A such that $A \subseteq \cup_{a \in A_0}$ pcl $U(a)$. Let $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap_{a \in A_0} F(a)$. Then $F_0 \cap (\cup_{a \in A_0}$ pcl $U(a)) = \emptyset$. But $\cup_{a \in A_0}$ pcl $U(a) = C$ (say) $\in \mathcal{C}_A$ and $F_0 \in \mathcal{F}$ with $F_0 \cap C = \emptyset$, a contradiction. □

Conversely, suppose A is not p-closed relative to X. Then there exists a cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by preopen sets of X such that $A \not\subseteq \cup\{$ pcl $U_\alpha : \alpha \in \Lambda_0\}$ for any finite subset Λ_0 of Λ. Then $\mathcal{F} = \{A \setminus \cup_{\alpha \in \Lambda_0}$ pcl $U_\alpha : \Lambda_0$ is a finite subset of $\Lambda\}$ is a filterbase on X [in fact, $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 = A \setminus \cup_{\alpha \in \Lambda_1}$ pcl U_α and $F_2 = A \setminus \cup_{\alpha \in \Lambda_2}$ pcl U_α, for some finite subsets Λ_1, Λ_2 of $\Lambda \Rightarrow F_1 \cap F_2 = A \setminus \cup_{\alpha \in \Lambda_1 \cup \Lambda_2}$ pcl $U_\alpha \in \mathcal{F}$] such that $F \cap C \neq \emptyset$ for each $F \in \mathcal{F}$ and each $C \in \mathcal{C}_A$ (since $\emptyset \neq F \subseteq A$ and $C \supseteq A$). But $A \cap p(\theta)$-ad $\mathcal{F} = \emptyset$. For $a \in A \Rightarrow \exists U_\alpha(\alpha \in \Lambda)$ such that $a \in U_\alpha$, and then $A \setminus$ pcl $U_\alpha \in \mathcal{F}$ such that (pcl $U_\alpha) \cap (A \setminus$ pcl $U_\alpha) = \emptyset$.

Definition 3.9. For a point x of a topological space X and any family $\mathcal{F}$ of subsets of X, we shall write $x \in \theta$-ad $\mathcal{F}$ if for each open set V in X containing x and each $F \in \mathcal{F}$, $F \cap$ cl $V \neq \emptyset$.

We need to recall from [4] the following results as the final prerequisites.

Lemma 3.3. (a) *A $p(\theta)$-closed subset A of a p-closed space X is p-closed relative to X.*

(b) *A topological space X is p-closed iff every filterbase on X pre-θ-accumulates at some point of X.*

We are now equipped enough to prove the main desired theorem as follows.

Theorem 3.9. *A topological space X is p-closed iff for each space Y, each $p(\theta)$-closed set $A \subseteq X$ and each multifunction $F : X \to Y$, $p(F, A) \supseteq \theta$-ad $F(\mathcal{C}_A)$.*

Proof. Let X be p-closed, and suppose that for a space Y, a multifunction $F : X \to Y$ and for some $p(\theta)$-closed subset A of X, $z \in \theta$-ad $F(\mathcal{C}_A)$. Then for each $C \in (\mathcal{C}_A)$ and each open neighborhood V of z in Y, cl $V \cap F(C) \neq \emptyset$ i.e., $F^-($ cl $V) \cap C \neq \emptyset$. Now, $\{F^-(\text{cl} V) : V$ is an open neighborhood of z in $Y\}(= \mathcal{F}$, say) forms a filterbase on X. As A is a $p(\theta)$-closed subset of a p-closed space X, A becomes p-closed relative to X (by Lemma 3.3 (a)). Hence it follows from Lemma 3.2 that $(p(\theta)$-ad $\mathcal{F}) \cap A \neq \emptyset$. Let $x \in (p(\theta)$-ad $\mathcal{F}) \cap A$. Then for all $U \in$PO(x) and all open neighborhood V of z in Y, we have $F^-($ cl $V) \cap$ pcl $U \neq \emptyset \Rightarrow$ cl $V \cap$

$F(\text{ pcl } U) \neq \emptyset \Rightarrow z \in \theta$-cl $F(\text{ pcl } U), \forall U \in PO(x) \Rightarrow z \in p(F,x) \subseteq p(F,A)$ (since $x \in A$).

Conversely, let $\mathcal{F}$ be a filterbase on X and $y_0 \notin X$. Let us construct $Y = X \cup \{y_0\}$ and $\sigma = \{U \subseteq Y : y_0 \notin U\} \cup \{U \subseteq Y : y_0 \in U$ and $\exists F \in \mathcal{F}$ such that $F \subseteq U\}$. Clearly σ is a topology on Y. We then consider the inclusion map $f : (X,\tau) \rightarrow (Y,\sigma)$. We first show that θ-cl$_Y X \subseteq \theta$-ad $f(\mathcal{C}_X)$, where θ-cl$_Y X$ stands for the θ-closure of the subset $X(\subseteq Y)$ in (Y,σ) (similar notation cl$_Y A$ will stand for the closure of a set A in Y in (Y,σ)). In fact, $y \in \theta$-cl$_Y X \Rightarrow$ for each σ-open neighborhood $U(y)$ of y in Y, cl$_Y U(y) \cap X \neq \emptyset$, i.e., cl$_Y U(y) \cap f(X) \neq \emptyset$.

Let $C \in \mathcal{C}_X$ (in X), then $C = \cup_{i=1}^{n}$ pcl U_i (for some $n \in I\!N$) with $U_i \in PO(X)$ and $X \subseteq \cup_{i=1}^{n}$ pcl U_i. It is then clear that $C = X$. Hence cl$_Y U(y) \cap f(C) \neq \emptyset$, $\forall C \in \mathcal{C}_X$ (in X) and for all open neighborhood $U(y)$ of y in Y, and so $y \in \theta$-ad $f(\mathcal{C}_X)$. Thus θ-cl$_Y X \subseteq \theta$-ad $f(\mathcal{C}_X) \subseteq p(F,X)$ (by hypothesis). Now, $y_0 \in \theta$-cl$_Y X \Rightarrow y_0 \in p(f,x) \Rightarrow y_0 \in p(f,x)$, for some $x \in X \Rightarrow y_0 \in \theta$-cl$_Y f(\text{ pcl } U)$, $\forall U \in PO(x) \Rightarrow$ cl$_Y(F \cup \{y_0\}) \cap f(\text{ pcl } U) \neq \emptyset$, $\forall U \in PO(x)$ and $\forall F \in \mathcal{F} \Rightarrow (F \cup \{y_0\}) \cap f(\text{ pcl } U) \neq \emptyset$, $\forall U \in PO(x)$ and $\forall F \in \mathcal{F}$ (as $F \cup \{y_0\}$ is closed in Y) $\Rightarrow F \cap (\text{ pcl } U) \neq \emptyset$, $\forall U \in PO(x)$ and $\forall F \in \mathcal{F}$ (since $f(\text{ pcl } U) = \text{pcl } U \subseteq X$) $\Rightarrow x \in p(\theta)$-ad $\mathcal{F}$.

Hence by Lemma 3.3 (b), X becomes p-closed. □

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REFERENCES

[1] Choquet, G., *Sur Les Notions De Et Grille*, Comples Rendus Acad. Sci Paris, **224** (1947), 171–173

[2] Collingwood, E. F. and Lohwater, J. A., *The theory of cluster sets*, Cambridge University Press, Cambridge, 1968

[3] D'Aristotile, J. A., *On θ-perfect mappings*, Boll. Un. Mat. Ital. **4(9)** (1974), 655–661

[4] Dontchev, J., Ganster, M. and Noiri, T., *On p-closed spaces*, Internat. Jour. Math. Math. Sci. **24**(**3**)(2000), 203–212

[5] El-Deeb, N., Hasanein, I.A., Mashhour, A.S, and Noiri, T., *On p-regular spaces*, Bull.Math. De la Soc. Sci. Math. Math. de la R.S. de Roumanie Tome. 27 (**75**) no. 4, (1983), 311–315

[6] Joseph, J. E., *Multifunctions and cluster sets*, Proc. Amer. Math. Soc. **74**, No.2 (1979), 329–337

[7] Khadra - Abo Abd. El. Aziz Ahmed, *On generalised forms of compactness*, Master's Thesis, Faculty of Science, Taunta University, Egypt(1989)

[8] Mashhour, A. S., Abd El-Monsef and El. Deeb, S. N., *On pre-continuous and weak pre-continuous mappings*, Math. Phys. Soc. Egypt **53**(1982), 47–53

[9] Mukherjee, M. N. and Debray, A., *On S-cluster sets and S-closed spaces*, Internat. J. Math. and Math. Sc. **23**, No. 9 (2000), 597–603

[10] Mukherjee, M. N. and Raychaudhuri, S., *A new type of cluster sets and its applications*, Tamkang Jr. Math. **24** (**4**) (1995), 327–336

[11] Mukherjee, M. N. and Roy, B., *p-closed topological space in terms of grills* (communicated)

[12] Noiri, T., *Strongly θ-precontinuous functions*, Acta. Math. Hungar. **90**(**4**) (2001), 307–316

[13] Paul, R. and Bhattacharya, P., *Quasi-precontinuous functions*, J. Indian Acad. Math. **14**, No. 2 (1992), 115–126

[14] Singal, M. K. and Arya, S. P., *On almost regular spaces* , Glasnik Mat. (**4**) **24** (1969), 89–99

[15] Thron, W. J., *Proximity structures and grills*, Math. Ann. **206** (1973), 35–62

[16] Velicko, N. V., *H-closed topological spaces*, Amer. Math. Soc. Transl. **78** (1968), 103–118

[17] Weston, J. D., *Some theorems on cluster sets*, J. London Math. Soc. **33** (1958), 453–441

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