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On the convergence of the Mann iteration in locally convex spaces

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ABSTRACT. The approximation of fixed points of some quasi contractive operators in arbitrary Banach spaces using the Mann iterative procedure is generalized to complete metrisable locally convex spaces. This turns out to be an extension of a result of Berinde [2] which in turn is an extension of a theorem of Rhoades [12].

1. INTRODUCTION

The Banach Contraction Principle is one of the most important result in fixed point theory. It states thus:

Theorem A. *Let (X, d) be a complete metric space and $T : X \to X$ a strict contraction, i.e. a map satisfying $d(Tx, Ty) \leq ad(x, y)$, for all $x, y \in X$, where $0 \leq a < 1$ is constant. Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{n=\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ converges to p, for any $x_0 \in X$.*

Thus Banach Contraction Principle settles the dual questions in fixed point theory i.e. the existence of fixed points and the approximation of fixed points of an operator when X is a complete metric space and the operator is a strict contraction.

For about four decades, various authors have examined the existence and approximation of fixed points with various generalizations of strict contraction in locally convex spaces, complete metric spaces, Banach spaces and their subspaces. For example see [2], [4-14] and [17]. One of the generalizations of various contractive conditions in [8], [9], and [13] was introduced by Zamfirescu [17].

Theorem B. *Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist the real numbers a,b and c satisfying $0 < a < 1,\ 0 < b,\ c < 1/2$ such that for each pair $x, y \in X$, at least one of the following is true*:

(i) $d(Tx, Ty) \leq ad(x, y)$;
(ii) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;
(iii) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{n=\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ converges to p, for any $x_0 \in X$.

This Zamfirescu's contractive conditions (i)-(iii) in the Theorem B above will be the subject of our work in this paper because it generalises some well known quasi contractions (e.g. see [3]).

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As a result of the limitations of Picard iteration, because it fails to converge to fixed points for many spaces even when the fixed point is unique (e.g. see Chidume [3, p. 102]), many iteration schemes have been introduced. They include the Ishikawa iteration, Mann iteration, Bruck iteration, Schu iteration and Global iteration schemes. For a good review of those schemes including their limitations see, for example, Chidume[3].

Let X be a metrizable topological space space and C be a nonempty subset of X. Let $T : C \to C$ be a mapping. The iteration scheme called $Ishikawa-type\ scheme$ is defined as follows

(1.1) $x_0 \in C,$

(1.2) $y_n = \beta_n T x_n + (1-\beta_n)x_n, \ \ n \geq 0,$

(1.3) $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \ \ n \geq 0$

$\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \leq \alpha_n, \beta_n \leq 1$ for all n, and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For other variants of Ishikawa Iteration scheme see Chidume [3]. If $\beta_n = 0$ for all n, then the Ishikawa Iteration scheme reduces to Mann Iteration scheme. The most general Mann type iterative scheme now studied is the following:

(1.4) $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ \ n \geq 0$

where $\{\alpha_n\}$ satisfy $0 \leq \alpha_n \leq 1$ for all n. We shall also assume that $\sum_{n=0}^{\infty} \alpha_n = \infty$. For other variants of Mann iteration scheme see [3].

Rhoades [12] used Ishikawa iteration scheme to approximate fixed points for operators satisfying conditions (i)-(iii) of Theorem B when X is a uniformly convex space while Berinde [2] generalized the result to arbitrary Banach spaces. Our aim in this paper is to generalize Berinde's result to complete metrisable locally convex spaces by using his approach. Such locally convex spaces which are complete and metrisable abound and are obviously generalizations of Banach spaces (see [15], [16]).

For example, the set of all real (or complex) valued indefinitely differentiable functions on the interval [a, b] becomes a metrisable locally convex space under the topology defined by the seminorms $p_m(f) = \sup_{a \leq t \leq b} |f^{(m)}(t)|$, (m = 0, 1,...). Also consider the set of all real (or complex) valued indefinitely differentiable functions on the interval $(-\infty, \infty)$. Under the topology of compact convergence for all the derivatives defined by the seminorms $p_{mn}(f) = \sup_{-n \leq t \leq n} |f^{(m)}(t)|$, $(m = 0, 1, \ldots; n = 1, 2, \ldots)$, the set is a metrisable locally convex space but not normable (see [15, p. 19]). In fact the duals of those spaces of distributions under their appropriate strong topologies are also metrisable locally convex spaces [15, p. 75]. Those spaces are also complete [15, p. 63]. Complete metrisable locally convex spaces are also called Frechet spaces.

A locally convex space (X, u) with topology u is a topological vector space which has a local base of convex neighborhoods of zero [16, chap. 7]. It is metrisable if it is Hausdorff and has a countable zero basis. Consequently, it is *metrisable* if u can be described by a countable family of continuous seminorms ([15, p. 9], [16]). Under the topology determined by the set Q of seminorms, X is *Hausdorff* if and only if for each non-zero $x \in X$, there is some $p \in Q$ with $p(x) > 0$ [15, Proposition 8]. To each absolutely convex absorbent subset U of X corresponds a seminorm p, called the *gauge* of U defined by $p(x) = inf\{\lambda : \lambda > 0,\ x \in \lambda U\}$ and with the property that $\{x : p(x) < 1\} \subseteq U \subseteq \{x : p(x) \leq 1\}$, U is a neighborhood of zero if and only if p is continuous. In this case the interior of U is $\{x : p(x) < 1\}$ and the closure of U is $\{x : p(x) \leq 1\}$ [15, p. 13, Propositions 6 and 7]. We shall now state and proof the following theorem which is fundamental to our results in this paper.

Theorem C. [15, Chap. 1, Theorem 4]. *The topology of a metrisable locally convex space can always be defined by a metric which is invariant under translation.*

Proof. If X is metrisable, it is certainly Hausdorff and has a countable neighborhoods of the origin.

If X has a countable base, each neighborhood contains an absolutely convex neighborhood and so there is a base (U_n) of absolutely convex neighborhoods. Let (p_n) be the gauge of (U_n). Put

$$f_c(x) = \sum_{n=1}^{\infty} 2^{-n} \min\{p_n(x), 1\}.$$

Then $f_c(x+y) \leq f_c(x) + f_c(y),\ f_c(-x) = f_c(x)$, and if $f_c(x) = 0,\ p_n(x) = 0$ for all n and so $x = 0$ since X is Hausdorff. Define d by

$$d(x, y) = f_c(x - y)$$

then d is a metric and $d(x + y, y + z) = d(x, y)$, so that d is invariant under translation. In the metric topology, the sets

$$V_n = \{f_c(x) < 2^{-n}\}$$

form a base of neighborhoods. But V_n is open in the original topology, since each p_n, and so f_c, is continuous; also $V_n \subseteq U_n$, for if $x \notin U_n$, then $p_n(x) \geq 1$ and so $f_c(x) \geq 2^{-n}$. Hence d defines the original topology on X.

It should be observed that if X is a normed linear space, then f_c satisfies the triangle inequality and will also be a norm. It is also easy to see that $f_c(x) = 0$ implies that $x = 0$ for any $x \in X$.

Henceforth f_c will denote the function as defined above whenever X is a metrisable locally convex space. □

We now state our main result.

2. Main result

Theorem 2.1. *Let X be a metrisable complete locally convex space, K a closed convex subset of X, and $T : K \to K$ an operator for which there exist the real numbers a, b, c such that satisfying $0 < a < 1$, $0 < b, c < 1/2$ such that for each pair $x, y \in K$, at least one of the following is true:*

(i) $f_c(Tx - Ty) \leq a f_c(x - y)$;
(ii) $f_c(Tx - Ty) \leq b[f_c(x - Tx) + f_c(y - Ty)]$;
(iii) $f_c(Tx - Ty) \leq c[f_c(x - Ty) + f_c(y - Tx)]$.

Let $\{x_n\}_{n=0}^{n=\infty}$ be the Mann iteration as defined in (0.4) above, then $\{x_n\}_{n=0}^{n=\infty}$ converges to the fixed point of T.

Proof. Since K is metrisable and complete, it is clear from Theorem B that T has a unique fixed point in K, which we shall denote p. For any $x, y \in K$, if (ii) holds, then

$$\begin{aligned} f_c(Tx - Ty) &\leq b[f_c(x - Tx) + f_c(y - Ty)] \\ &\leq b\{f_c(x - Tx) + [f_c(y - x) + f_c(x - Tx) + f_c(Tx - Ty)]\}. \end{aligned}$$

Hence

$$(1 - b)f_c(Tx - Ty) \leq b.f_c(x - y) + 2bf_c(x - Tx).$$

Since $0 \leq b < 1$, then

$$f_c(Tx - Ty) \leq \frac{b}{1 - b} f_c(x - y) + \frac{2b}{1 - b} f_c(x - Tx). \tag{2.5}$$

Similarly, if (iii) holds we obtain

$$f_c(Tx - Ty) \leq \frac{c}{1 - c} f_c(x - y) + \frac{2c}{1 - c} f_c(x - Tx). \tag{2.6}$$

Denote

$$\delta = max\{a, \frac{b}{1 - b}, \frac{c}{1 - c}\}. \tag{2.7}$$

Then we have $0 \leq \delta < 1$ and, in view of (i), (2.5) and (2.6) it is clear that the inequality

$$f_c(Tx - Ty) \leq \delta f_c(x - y) + 2\delta f_c(x - Tx) \tag{2.8}$$

holds for all $x, y \in K$.

Let $\{x_n\}_{n=0}^{n=\infty}$ be the Mann iteration defined by (1.4) and $x_0 \in K$ arbitrary. Then

$$\begin{aligned} f_c(x_{n+1} - p) &= f_c((1 - \alpha_n)x_n + \alpha_n Tx_n - (1 - \alpha_n + \alpha_n)p) \\ &= f_c((1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p)) \\ &\leq (1 - \alpha_n)f_c(x_n - p) + \alpha_n f_c(Tx_n - p)). \end{aligned} \tag{2.9}$$

If $x = p$ and $y = x_n$, from (2.8) we obtain

$$f_c(Tx_n - p) \leq \delta f_c(x_n - p) \tag{2.10}$$

and hence by (2.9) and 2.10) we obtain

$$f_c(x_{n+1} - p) \leq [1 - (1 - \delta)\alpha_n]f_c(x_n - p), \quad n = 0, 1, 2, \ldots . \tag{2.11}$$

Since $1 - (1 - \delta)\alpha_n < 1$, then $\{x_n\}$ converges to p. □

Corollary 2.1. *Let X be a Banach space, K a closed convex subset of X, and $T : K \to K$ an operator for which there exist the real numbers a,b and c satisfying $0 < a < 1$, $0 < b$, $c < 1/2$ such that for each pair $x, y \in K$, at least one of the following is true:*

(i) $\|Tx - Ty\| \leq a\|x - y\|$;

(ii) $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$;

(iii) $\|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{n=\infty}$ be the Mann iteration as defined in (1.4) above, then $\{x_n\}_{n=0}^{n=\infty}$ converges strongly to the fixed point of T.

We now proceed to generalize the theorem by allowing a, b, c, d, e, f to be monotonically decreasing functions of $f_c(x-y)$.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist monotonically decreasing functions f = f(d(x,y)), g = g(d(x,y)) and h = h(d(x,y)), for all $x, y \in X$, from $[0, \infty)$ to $[0, 1)$ satisfying $0 < f(t) < 1$, $0 < g(t)$, $h(t) < 1/2$ for all $t \in d(x, y)$. If for each pair $x, y \in X$, at least one of the following is true:*

(i) $d(Tx, Ty) \leq f d(x, y)$;

(ii) $d(Tx, Ty) \leq g[d(x, Tx) + d(y, Ty)]$;

(iii) $d(Tx, Ty) \leq h[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{n=\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ converges to p, for any $x_0 \in X$.

Proof. The proof of the following theorem follows essentially the same method used by Hardy and Rogers in [7, Theorem 2] to generalize [7, Theorem 1]. For if $a = b = c = 0$ in [7, Theorem 2] we have (i); if $c = e = f = 0$ in [7, Theorem 2] we have (ii) and if $a = b = f = 0$ in [7, Theorem 2] we have (iii). □

By substituting $a = f(f_c(x, y))$, $b = g(f_c(x, y))$ and $c = h(f_c(x, y))$ in the proof of Theorem 2.1 and in view of Theorem 2.2 we now have the following generalization of Theorem 1.1.

Theorem 2.3. *Let X be a metrisable complete locally convex space, K a closed convex subset of X, and $T : K \to K$ an operator for which there exist monotonically decreasing functions $f = f(f_c(x-y))$, $g = g(f_c(x-y))$ and $h = h(f_c(x-y))$, for all $x,\ y \in X$, from $[0, \infty)$ to $[0, 1)$ satisfying $0 < f(t) < 1$, $0 < g(t)$, $h(t) < 1/2$ for all $t \in f_c(x-y)$. If for each pair $x, y \in K$, at least one of the following is true:*

(i) $f_c(Tx - Ty) \leq f f_c(x-y)$;

(ii) $f_c(Tx - Ty) \leq g[f_c(x - Tx) + f_c(y - Ty)]$;

(iii) $f_c(Tx - Ty) \leq h[f_c(x - Ty) + f_c(y - Tx)]$,

then the Mann iteration $\{x_n\}_{n=0}^{n=\infty}$ as defined in (1.4) above, converges to the fixed point of T.

Consequently, we have the following result which we will show later to be another generalization of Berinde's [2, Theorem 2] result.

Corollary 2.2. *Let X be a Banach space, K a closed convex subset of X, and $T : K \to K$ an operator for which there exist monotonically decreasing functions $f = f(\|x-y\|)$, $g = g(\|x-y\|)$ and $h = h(\|x-y\|)$, for all $x,\ y \in X$, from $[0, \infty)$ to $[0, 1)$ satisfying $0 < f(t) < 1,\ 0 < g(t),\ h(t) < 1/2$ for all $t \in \|x-y\|$, such that for each pair $x, y \in X$, at least one of the following is true:*

(i) $\|Tx - Ty\| \leq f[\|x-y\|]$;

(ii) $\|Tx - Ty\| \leq g[\|x - Tx\| + \|y - Ty\|]$;

(iii) $\|Tx - Ty\| \leq h[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{n=\infty}$ be the Mann iteration as defined in (1.4) above, then $\{x_n\}_{n=0}^{n=\infty}$ converges strongly to the fixed point of T.

Remark 2.1. Berinde proved Corollary 2 in [2, Theorem 2] when the iteration scheme is Ishikawa-type sequence described by (1.1) - (1.3). This was a generalization of RhoadesTheorem [12]. In the Ishikawa-type sequence, which has been studied extensively for over two decades, it is assumed that α_n and β_n are independent and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Chidume [3, p. 172] rightly pointed out that in this case, if the Ishikawa-type sequence converges, then the Mann's sequence defined by $x_0 \in K$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$, $n > 0$, converges. One simply sets $\beta_n = 0$ for all integers $n \geq 0$. Thus, since the Mann iteration scheme we used is simpler and converges faster than Ishikawa iteration scheme, the use of Ishikawa iteration scheme in [2, Theorem 2] and [12] appears unnecessary.

Remark 2.2. Our Theorem 1 improves on Berinde [2] and consequently on Rhoades' Theorem [12], by extending it from Banach spaces to complete metrisable locally convex spaces. The fact that there are complete metrisable spaces, including many useful function spaces, that are not normable makes our Theorem a needed generalization of Berinde's Theorem. Also, our Theorem [3] is an improvement of Berinde's Theorem in the sense that the constants a, b and c are respectively generalized in our Corollary 2 to monotonically decreasing functions $f = f(\|x - y\|)$, $g = g(\|x - y\|)$ and $h = h(\|x - y\|)$.

REFERENCES

[1] Berinde, V., Iterative Approximation of Fixed Points, Editura Efemeride, Baia Mare, 2002

[2] Berinde, V., *On the Convergence of the Ishikawa Iteration in the class of Quasi contractive Operators*, Acta Math. Univ. Comeniaca, **73** (**1**) (2004), 119-126

[3] Chidume, C. E., Geometric Properties of Banach Spaces and Nonlinear Iterations, Monograph, International Centre for Theoretical Physics, Italy, In print

[4] Derrick, W. R., Nova, L., *Fixed Point Theorems for Discontinuous operators*, Glasnik Mat. **24** (1989), 339-348

[5] Derrick, W. R., Nova, L., *Interior Properties and Fixed Points of Certain Discontinuous Operators*, Progress In Functional Analysis, Proceedings of International Functional Analysis meeting held in Spain in 22-27 October, 1990 editted by K. D. Bierstedt et-al and published by Elsevier Science Publishers, 1992

[6] Goebel, K., Kirk, W. A. and Shimi, T. N., *A Fixed Point Theorem in Uniformly Convex Spaces*, Boll. Un. Mat. Ital., (**4**) **7** (1973), 63-65

[7] Hardy, G., Rogers, T., *A Generalization of a Fixed Point Theorem of Reic*, Canad. Math. Bull. (**16**) **2**, (1973), 201-206

[8] Kannan, R., *Some Results on Fixed Points II*, Amer. Math. Monthly, **76** (1969), 405 - 408

[9] Kannan, R., *Some Results on Fixed Points III*, Fund. Math., **70** (1971), 169-177

[10] Nova, L., *Fixed Point Theorems for Discontinuous operators*, Pacific Journal of Mathematics, **123** (1986), 189-196

[11] Olaleru, J. O., *Extensions of Some Fixed Point Theorems to Locally Convex spaces*, Submitted

[12] Rhoades, B. E., *Comments on Two Fixed Points Iteration Methods*, J. Math. Anal. Appl. **56** (**2**) (1976), 741-750

[13] Rhoades, B. E., *A Comparism of Various definitions of Contractive Mappings*, Trans. Amer. Math. Soc., **226** (1977), 257-290

[14] Rhoades, B. E., *Contractive Definitions Revisited*, Contemporary Mathematics, **21** (1983), 189-205

[15] Robertson, A. P., Robertson, W. J., Topological vector spaces, Cambridge University Press, 1980

[16] Schaffer, H. H., Topological Vector Spaces, Springer- Verlag, 1999

[17] Zamfirescu, T., *Fixed Point Theorems in Metric Spaces*, Arch. Math. (Basel), **23** (1972), 292-298

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