CARPATHIAN J. MATH.
22 (2006), No. 1 - 2, 121 - 128

A note on the direct limit of a direct system of multialgebras in a subcategory of multialgebras

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ABSTRACT. We will present some properties of the direct limit of a direct system of multialgebras in a subcategory of the category of multialgebras obtained by considering as morphisms the ideal multialgebra homomorphisms.

1. INTRODUCTION

In a previous paper ([10]) we gave a generalization of the direct limit of a direct system in the case of multialgebras. This construction preserves the identities satisfied on the multialgebras of the given identities of the direct system. This property allows us to obtain many of the existing results on direct limits of particular hyperstructures (see [2, 5, 6, 7, 13]) as consequences of our results. The notion of multialgebra homomorphism follows from the same notion from the relational structure theory. The notion of ideal homomorphism is, somehow, more related to the notion of homomorphism used in the universal algebra theory. In this paper we will prove that the subcategory of multialgebras obtained by considering as morphisms the ideal homomorphisms of multialgebras is closed with respect to the construction of the direct limit of a direct system. We mention without proving it that this makes possible for all the results on identities and direct limits to be formulated for direct systems with ideal homomorphisms (this follows immediately from the results from [10, Section 5] and Section 4). Since the ideal homomorphisms of semihypergroups are the good homomorphisms and the ideal homomorphisms of hypergroups are the very good homomorphisms (see [1]), the properties presented in Section 4 hold for semihypergroups and good homomorphisms as well as for hypergroups and very good homomorphisms.

2. PRELIMINARIES

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence of nonnegative integers (with $o(\tau)$ ordinal number), for any $\gamma < o(\tau)$ let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and we consider the algebra of the n-ary terms (of type τ) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

If A is a set, we denote by $P^*(A)$ the set of the nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$.

Received: 24.10.2005; In revised form: 25.05.2006; Accepted: 01.11.2006
2000 *Mathematics Subject Classification:* 08A05; 08A99.
Key words and phrases: *Multialgebra, direct limit, direct system, ideal homomorphism, fundamental algebra.*

We obtain a universal algebra $\mathfrak{P}^*(A)$ on $P^*(A)$ if we define, for any $\gamma < o(\tau)$ and for any $A_0, \dots, A_{n_\gamma - 1} \in P^*(A)$,

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) = \bigcup \{f_\gamma(a_0, \dots, a_{n_\gamma-1}) \mid a_i \in A_i,\ i \in \{0, \dots, n_\gamma - 1\}\},$$

(see [11]). As in [4], we can construct the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of the n-ary term functions on $\mathfrak{P}^*(A)$, for any $n \in \mathbb{N}$.

For an equivalence relation ρ on A, the factor multialgebra on A/ρ is obtained by defining the multioperations on A/ρ as in [3]:

$$f_\gamma(\rho\langle a_0\rangle, \dots, \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \dots, b_{n_\gamma-1}),\ a_i \rho b_i,\ i \in \{0, \dots, n_\gamma - 1\}\}$$

($\rho\langle x\rangle$ denotes the class of x modulo ρ).

The *fundamental relation* of the multialgebra $\mathfrak{A}$ is the smallest equivalence relation on A for which the factor multialgebra is a universal algebra. The fundamental relation of $\mathfrak{A}$ is the transitive closure α^* of the relation α defined on A as follows:

$$x \alpha y \Leftrightarrow \exists n \in \mathbb{N},\ \exists \mathbf{p} \in \mathbf{P}^{(n)}(\tau),\ \exists a_0, \dots, a_{n-1} \in A:\ x, y \in p(a_0, \dots, a_{n-1}) \tag{2.1}$$

where $p \in P^{(n)}(\mathfrak{P}^*(A))$ is the term function induced by $\mathbf{p}$ on $\mathfrak{P}^*(A)$ (see [8, 9]). The universal algebra $\mathfrak{A}/\alpha^*$ is called the *fundamental algebra* of the multialgebra $\mathfrak{A}$. We will denote by $\overline{\mathfrak{A}}$ the fundamental algebra of $\mathfrak{A}$. We also denote by φ_A the canonical projection of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}$ and by $\overline{a}$ the class of a modulo α^*.

A map $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and $a_0, \dots, a_{n_\gamma-1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})).$$

The homomorphism h is ideal if for any $\gamma < o(\tau)$ and $a_0, \dots, a_{n_\gamma-1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) = f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})).$$

A bijective map h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. As it results from [11], the multialgebra isomorphisms can be characterized as being the bijective ideal homomorphisms.

The definition of the multioperations of $\mathfrak{A}/\rho$ allows us to see the canonical map from A to A/ρ as an homomorphism of multialgebras.

Theorem 2.1. [9, Theorem 1] *If $\mathfrak{A}$, $\mathfrak{B}$ are multialgebras and $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ respectively are their fundamental algebras and if $f : A \to B$ is an ideal homomorphism then there exists only one homomorphism of universal algebras $\overline{f} : \overline{A} \to \overline{B}$ such that*

$$\varphi_B \circ f = \overline{f} \circ \varphi_A. \tag{2.2}$$

The proof uses only the fact that f is an homomorphism, so the statement holds if we drop the property of f of being ideal.

Corollary 2.1. [9, Corollaries 1, 2] *If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are multialgebras of the same type τ then $\overline{1_A} = 1_{\overline{A}}$ and if $f : A \to B,\ g : B \to C$ are multialgebra (ideal) homomorphisms, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.*

We can easily construct the category of multialgebras of the same type τ where the morphisms are considered to be the homomorphisms and the composition of two morphisms is the usual map composition. It is known that the universal algebras of the same type τ together with the homomorphisms between them form a category which is, obviously, a full subcategory in the category of the multialgebras introduced above. We denote by $\mathbf{Malg}(\tau)$ the category of multialgebras of type τ and by $\mathbf{Alg}(\tau)$ the category of the universal algebras of type τ mentioned before. It is easy to observe that if we consider as morphisms only the ideal multialgebra homomorphisms then we obtain a subcategory $\mathbf{Malg}_i(\tau)$ of $\mathbf{Malg}(\tau)$ and $\mathbf{Alg}(\tau)$ is also a full subcategory of $\mathbf{Malg}_i(\tau)$.

Remark 2.1. Corollary 2.1 allows us to define a functor $F : \mathbf{Malg}(\tau) \longrightarrow \mathbf{Alg}(\tau)$ as follows: $F(\mathfrak{A}) = \overline{\mathfrak{A}}$, for any multialgebra $\mathfrak{A}$ of type τ, and $F(f) = \overline{f}$ from (2), for any homomorphism f between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ (of type τ).

3. Direct limits of direct systems of multialgebras

Let $\mathcal{A} = ((A_i \mid i \in I), (\varphi_{ij} : A_i \to A_j \mid i, j \in I,\ i \leq j))$ be a direct system of sets having the carrier $(I, \leq)$. Thus, $(I, \leq)$ is a directed preordered set and the maps φ_{ij} $(i, j \in I,\ i \leq j)$ are such that for any $i, j, k \in I$, with $i \leq j \leq k$, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ and $\varphi_{ii} = 1_{A_i}$, for all $i \in I$. On the disjoint union A of the sets A_i one defines the relation $\equiv$ as follows: for any $x, y \in A$ there exist $i, j \in I$, such that $x \in A_i,\ y \in A_j$, and $x \equiv y$ if and only if $\varphi_{ik}(x) = \varphi_{jk}(y)$ for some $k \in I$ with $i \leq k,\ j \leq k$. The relation $\equiv$ is an equivalence relation on A and the factor set $A/_{\equiv} = \{\widehat{x} \mid x \in A\}$ (denoted here by A_∞) is the direct limit of the direct system of sets $\mathcal{A}$ (see [4]).

If each set A_i is a support set for a multialgebra $\mathfrak{A}_i$ of type τ and each φ_{ij} is a multialgebra homomorphism, the system

$$\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} : A_i \to A_j \mid i, j \in I,\ i \leq j))$$

is a direct system of multialgebras. Sometimes we will refer to $\mathcal{A}$ as the direct system (or the direct family) of multialgebras $(\mathfrak{A}_i \mid i \in I)$.

If $\gamma < o(\tau)$ and $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}} \in A_\infty$ the equality

$$\begin{aligned} f_\gamma(\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}}) \quad &= \quad \{\widehat{x'} \mid \exists m \in I,\ \forall j \in \{0, \ldots, n_\gamma - 1\},\ \exists x'_j \in \widehat{x_j} \cap A_m, \\ &\qquad \text{such that } x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1})\}. \end{aligned}$$

defines a multioperation on A_∞, thus we obtain a multialgebra $\mathfrak{A}_\infty$ of type τ on A_∞.

Lemma 3.1. [10, Lemma 15] *If $\gamma < o(\tau)$ and $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}} \in A_\infty$ and for any $j \in \{0, \ldots, n_\gamma - 1\}$ we take $i_j \in I$ such that $x_j \in A_{i_j}$, the representative x' of a class $\widehat{x'} \in f_\gamma(\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}})$ can be considered such that*

$$\exists\, m \in I,\ i_0 \leq m, \ldots, i_{n_\gamma-1} \leq m \text{ with } x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1}),$$

where $x'_j = \varphi_{i_j m}(x_j)$ $(j \in \{0, \ldots, n_\gamma - 1\})$.

Remark 3.2. [10, Remark 16] If for some $\gamma < o(\tau)$, f_γ is an operation in all the multialgebras $\mathfrak{A}_i$, then f_γ is an operation in $\mathfrak{A}_\infty$. In fact, in order that for a given $\gamma < o(\tau)$, f_γ to be an operation in $\mathfrak{A}_\infty$ it is enough for any two elements from I to have an upper bound $m \in I$ such that in $\mathfrak{A}_m$, f_γ is an operation.

Lemma 3.2. [10, Lemma 28] *Let* $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ *and* $a_0, \dots, a_{n-1} \in A$. *Then we have*

$$p(\widehat{a_0}, \dots, \widehat{a_{n-1}}) = \{\widehat{a} \mid \exists m \in I,\ \forall j \in \{0, \dots, n-1\},\ \exists a'_j \in \widehat{a_j} \cap A_m \text{ such that } a \in p(a'_0, \dots, a'_{n-1})\}.$$

If $i_0, \dots, i_{n-1} \in I$ *are such that* $a_j \in A_{i_j}$ *for all* $j \in \{0, \dots, n-1\}$ *then the set* $p(\widehat{a_0}, \dots, \widehat{a_{n-1}})$ *is equal to*

$$\{\widehat{a} \mid \exists m \in I,\ i_0, \dots, i_{n-1} \leq m,\ a \in p(\varphi_{i_0 m}(a_0), \dots, \varphi_{i_{n-1} m}(a_{n-1}))\}.$$

Remark 3.3. [10, Remark 17] For each $i \in I$, the map $\varphi_{i\infty} : A_i \to A_\infty,\ \varphi_{i\infty}(x) = \widehat{x}$ is a multialgebra homomorphism and for any $i, j \in I$, with $i \leq j$, $\varphi_{j\infty} \circ \varphi_{ij} = \varphi_{i\infty}$.

Theorem 3.2. [10, Theorem 19] *If we consider the category* $\mathcal{I}$ *associated to the preordered set* $(I, \leq)$ *then we can see (as in* [12]*) the direct system consisting of the multialgebras* $(\mathfrak{A}_i \mid i \in I)$ *and the homomorphisms* $(\varphi_{ij} : A_i \to A_j \mid i, j \in I,\ i \leq j)$ *as a covariant functor* $G : \mathcal{I} \longrightarrow \mathbf{Malg}(\tau)$ *and the multialgebra* $\mathfrak{A}_\infty$*, with the homomorphisms* $(\varphi_{i\infty} \mid i \in I)$*, is the direct limit of the functor* G.

The multialgebra $\mathfrak{A}_\infty$ is the *direct limit* of the (direct system of) multialgebras $(\mathfrak{A}_i \mid i \in I)$ and it will be denoted by $\varinjlim_{i \in I} \mathfrak{A}_i$.

In what follows we will consider that $(I, \leq)$ is a directed partially ordered set (unless we will specify something else). Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and let us consider $J \subseteq I$ such that $(J, \leq)$ is also a directed partially ordered set. We will denote by $\mathcal{A}_J$ the direct system consisting of the multialgebras $(\mathfrak{A}_i \mid i \in J)$ whose carrier is $(J, \leq)$ and the homomorphisms are $(\varphi_{ij} \mid i, j \in J,\ i \leq j)$.

Proposition 3.1. [10, Proposition 22] *Let* $\mathcal{A}$ *be a direct system of multialgebras with the carrier* $(I, \leq)$ *and let us consider* $J \subseteq I$ *such that* $(J, \leq)$ *is a directed partially ordered set cofinal with* $(I, \leq)$. *Then the multialgebras* $\varinjlim \mathcal{A}$ *and* $\varinjlim \mathcal{A}_J$ *are isomorphic.*

Let us consider that the support set I of the carrier $(I, \leq)$ of the direct system $\mathcal{A} = ((\mathfrak{A}_p \mid p \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ of multialgebras can be written as $I = \bigcup_{p \in P} I_p$, where $(I_p, \leq)$ is a directed partially ordered subset of $(I, \leq)$ for each $p \in P$ and $(P, \leq)$ is also a directed partially ordered set such that $I_p \subseteq I_q$, whenever $p, q \in P,\ p \leq q$. Denote $\varinjlim \mathcal{A} = \mathfrak{A}_\infty = (A_\infty, (f_\gamma)_{\gamma < o(\tau)})$ and $\varinjlim \mathcal{A}_{I_p} = \mathfrak{A}^p_\infty = (A^p_\infty, (f_\gamma)_{\gamma < o(\tau)})$ if $p \in P$. Then for any $p, q \in P,\ p \leq q$ we obtain the map

$$\psi_{pq} : A^p_\infty \to A^q_\infty,\ \psi_{pq}(\widehat{x}_{I_p}) = \widehat{x}_{I_q},$$

(where $x \in A_i$, for some $i \in I_p$). In this way we obtain a direct system of sets denoted by $\mathcal{A}/P$ consisting of $(P, \leq)$, the multialgebras $\mathfrak{A}^p_\infty$, and the maps ψ_{pq}.

Theorem 3.3. [10, Theorem 23] $\mathcal{A}/P$ *is a direct system of multialgebras and the multialgebras* $\varinjlim \mathcal{A}$ *and* $\varinjlim \mathcal{A}/P$ *are isomorphic.*

We will use the term of algebraic class for those classes of multialgebras which are closed under the formation of isomorphic images.

Theorem 3.4. [10, Theorem 24] *If* K *is an algebraic class of multialgebras then* K *is closed under the formation of direct limits of arbitrary direct systems if and only if* K *is closed under the formation of direct limits of well-ordered direct systems.*

Theorem 3.5. [10, Theorem 25] *The functor* $F : \mathbf{Malg}(\tau) \longrightarrow \mathbf{Alg}(\tau)$ *is a left adjoint for the inclusion functor* $U : \mathbf{Alg}(\tau) \longrightarrow \mathbf{Malg}(\tau)$.

Let $(I, \leq)$ be a directed preordered set. Since any functor which has a right adjoint preserves the direct limits, we have:

Corollary 3.2. [10, Corollary 26] *If* $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ *is a direct system of multialgebras of type* τ *then* $\overline{\mathcal{A}} = ((\overline{\mathfrak{A}_i} \mid i \in I), (\overline{\varphi_{ij}} \mid i, j \in I,\ i \leq j))$ *is a direct system of universal algebras of type* τ *and the universal algebra* $\overline{\varinjlim \mathcal{A}}$ *is isomorphic to the universal algebra* $\varinjlim \overline{\mathcal{A}}$.

4. Direct limits of direct systems with ideal homomorphisms

The properties presented in Section 3 hold in the subcategory of $\mathbf{Malg}(\tau)$ which has the same objects and for which the morphisms are the ideal homomorphisms. In other words, the results established in Section 3 hold if we replace 'homomorphism' by 'ideal homomorphism'. As we will see, the definition of the multioperations in the direct limit will be easier in this case.

Lemma 4.3. *Let* $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ *be a direct system of multialgebras and* $\mathfrak{A}_\infty = \varinjlim \mathcal{A}$. *If all the homomorphisms* φ_{ij} *are ideal homomorphisms then the multioperations from* $\mathfrak{A}_\infty$ *can be defined as follows: for any* $\gamma < o(\tau)$ *and for any* $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}} \in A_\infty$ *with* $x_0 \in A_{i_0}, \ldots, x_{n_\gamma-1} \in A_{i_{n_\gamma-1}}$ *we consider an element* $m \in I$, $i_0, \ldots, i_{n_\gamma-1} \leq m$ *and we define*

$$f_\gamma(\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}}) = \{\widehat{x} \mid x \in f_\gamma(\varphi_{i_0 m}(x_0), \ldots, \varphi_{i_{n_\gamma-1} m}(x_{n_\gamma-1}))\}. \tag{4.3}$$

Proof. From Lemma 3.1 it results that it is enough to prove that the definition the set in the right side of (5) does not depend on $m \in I$. Indeed, taking any other $m' \in I$, with $i_j \leq m'$, for all $j \in \{0, \ldots, n_\gamma - 1\}$ and $x'_j = \varphi_{i_j m}(x_j)$, $x''_j = \varphi_{i_j m'}(x_j)$ we have

$$\{\widehat{x'} \mid x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1})\} = \{\widehat{x''} \mid x'' \in f_\gamma(x''_0, \ldots, x''_{n_\gamma-1})\}.$$

For, if $q \in I$ such that $m \leq q$, $m' \leq q$ then

$$\begin{aligned}
\varphi_{mq}(f_\gamma(x'_0, \ldots, x'_{n_\gamma-1})) &= f_\gamma(\varphi_{mq}(x'_0), \ldots, \varphi_{mq}(x'_{n_\gamma-1})) \\
&= f_\gamma(\varphi_{i_0 q}(x_0), \ldots, \varphi_{i_{n_\gamma-1} q}(x_{n_\gamma-1})) \\
&= f_\gamma(\varphi_{m'q}(x''_0), \ldots, \varphi_{m'q}(x''_{n_\gamma-1})) \\
&= \varphi_{m'q}(f_\gamma(x''_0, \ldots, x''_{n_\gamma-1}));
\end{aligned}$$

thus for each $x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1})$ there exists $x'' \in f_\gamma(x''_0, \ldots, x''_{n_\gamma-1})$ such that $x' \equiv x''$ and conversely, for each $x'' \in f_\gamma(x''_0, \ldots, x''_{n_\gamma-1})$ there exists $x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1})$ such that $x' \equiv x''$. □

Remark 4.4. In this case is easier to observe that if for some $\gamma < o(\tau)$ we have for any two elements from I an upper bound $m \in I$ such that f_γ is an operation in $\mathfrak{A}_m$ then f_γ is an operation in $\mathfrak{A}_\infty$.

From Lemma 3.2 and Lemma 4.3 we deduce the following result.

Corollary 4.3. *Let $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$. If $i_0, \ldots, i_{n-1} \in I$ are such that $a_j \in A_{i_j}$ for all $j \in \{0, \ldots, n-1\}$ and $m \in I$ with $i_0, \ldots, i_{n-1} \leq m$ then*

$$p(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) = \{\widehat{a} \mid a \in p(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n-1} m}(a_{n-1}))\}.$$

Lemma 4.4. *Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and $\mathfrak{A}_\infty = \varinjlim \mathcal{A}$. If all the homomorphisms φ_{ij} are ideal homomorphisms then the homomorphisms $\varphi_{i\infty} : A_i \to A_\infty$, $\varphi_{i\infty}(x) = \widehat{x}$ are ideal homomorphisms.*

Proof. For any $\gamma < o(\tau)$ and for all $x_0, \ldots, x_{n_\gamma - 1} \in A_i$ we have

$$\begin{aligned}
\varphi_{i\infty}(f_\gamma(x_0, \ldots, x_{n_\gamma-1})) &= \{\varphi_{i\infty}(x) \mid x \in f_\gamma(x_0, \ldots, x_{n_\gamma-1})\} \\
&= \{\widehat{x} \mid x \in f_\gamma(x_0, \ldots, x_{n_\gamma-1})\} \\
&= f_\gamma(\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}}) \\
&= f_\gamma(\varphi_{i\infty}(x_0), \ldots, \varphi_{i\infty}(x_{n_\gamma-1})),
\end{aligned}$$

thus the homomorphism $\varphi_{i\infty}$ is ideal. □

Theorem 4.6. *The subcategory $\mathbf{Malg}_i(\tau)$ of $\mathbf{Malg}(\tau)$ is closed under the formation of the direct limit of a direct system.*

Proof. Let us consider the following diagrams:

$$\begin{tikzcd}
A_\infty & \\
A_i \arrow[u, "\varphi_{i\infty}"] \arrow[r, "\varphi_{ij}"] & A_j \arrow[ul, "\varphi_{j\infty}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& A' \\
A_i \arrow[ur, "\alpha_i"] \arrow[r, "\varphi_{ij}"] & A_j \arrow[u, "\alpha_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_\infty \arrow[r, "\mu"] & A' \\
A_i \arrow[u, "\varphi_{i\infty}"] \arrow[ur, "\alpha_i"'] &
\end{tikzcd}
.$$

The first diagram is commutative (see 3.3) and whenever a multialgebra $\mathfrak{A}'$ of type τ, together with a family $(\alpha_i : A_i \to A' \mid i \in I)$ of homomorphisms make the second diagram commutative, there exists a unique homomorphism $\mu : A_\infty \to A'$ such that the third diagram is commutative. The unique homomorphism μ which make the third diagram commutative is defined as follows: for $\widehat{x} \in A_\infty$ there exists $i \in I$ such that $x \in A_i$, and $\mu(\widehat{x}) = \mu(\varphi_{i\infty}(x)) = \alpha_i(x)$.

If all the homomorphisms φ_{ij} and α_i are ideal homomorphisms then, as we have seen in Lemma 4.4 all the homomorphisms $\varphi_{i\infty}$ are ideal homomorphisms, and μ is an ideal homomorphism, too. Indeed, for any $\gamma < o(\tau)$ and $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}} \in A_\infty$ with $x_0 \in A_{i_0}, \ldots, x_{n_\gamma-1} \in A_{i_{n_\gamma-1}}$, let $m \in I$ be such that $i_0, \ldots, \imath_{n_\gamma-1} \leq m$ and $x'_0 = \varphi_{i_0 m}(x_0), \ldots, x'_{n_\gamma-1} = \varphi_{i_{n_\gamma-1} m}(x_{n_\gamma-1})$; we have

$$\begin{aligned}
\mu(f_\gamma(\widehat{x_0}, \ldots, \widehat{x_{n_\gamma-1}})) &= \{\mu(x') \mid x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1}) (\subseteq A_m)\} \\
&= \{\alpha_m(x') \mid x' \in f_\gamma(x'_0, \ldots, x'_{n_\gamma-1}) (\subseteq A_m)\} \\
&= \alpha_m(f_\gamma(x'_0, \ldots, x'_{n_\gamma-1})) \\
&= f_\gamma(\alpha_m(x'_0), \ldots, \alpha_m(x'_{n_\gamma-1})) \\
&= f_\gamma(\mu(\widehat{x'_0}), \ldots, \mu(\widehat{x'_{n_\gamma-1}})) \\
&= f_\gamma(\mu(\widehat{x_0}), \ldots, \mu(\widehat{x_{n_\gamma-1}})),
\end{aligned}$$

and the theorem is proved. □

Corollary 4.4. *Let* $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ *and* $a_0, \ldots, a_{n-1} \in A$. *If* $i_0, \ldots, i_{n-1} \in I$ *are such that* $a_j \in A_{i_j}$ *for all* $j \in \{0, \ldots, n-1\}$ *and* $m \in I$ *with* $i_0, \ldots, i_{n-1} \leq m$ *then*

$$p(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) = \{\widehat{a} \mid a \in p(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n-1} m}(a_{n-1}))\}.$$

Remark 4.5. If all the homomorphisms from $\mathcal{A}$ are ideal homomorphisms then all the homomorphisms ψ_{pq} from Theorem 3.3 are ideal homomorphisms, so Theorem 3.3 holds in $\mathbf{Malg}_i(\tau)$. Since in the proof of Proposition 3.1 will be no important changes if the homomorphisms of $\mathcal{A}$ are ideal and Theorem 3.4 follows from Proposition 3.1 and Theorem 3.3 we conclude that Theorem 3.4 also holds in $\mathbf{Malg}_i(\tau)$.

Indeed, let us take $\gamma < o(\tau)$, $p, q \in P$, $p \leq q$ and $(\widehat{x_j})_{I_p} \in A^p_\infty$, $j \in \{0, \ldots, n_\gamma - 1\}$; we can consider that $x_j \in A_m$ with $m \in I_p$, for all $j \in \{0, \ldots, n_\gamma - 1\}$ and so we have

$$\begin{aligned}
\psi_{pq}(f_\gamma((\widehat{x_0})_{I_p}, \ldots, (\widehat{x_{n_\gamma - 1}})_{I_p})) &= \{\psi_{pq}(\widehat{x}_{I_p}) \mid x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\} \\
&= \{\widehat{x}_{I_q} \mid x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\} \\
&= f_\gamma((\widehat{x_0})_{I_q}, \ldots, (\widehat{x_{n_\gamma - 1}})_{I_q}) \\
&= f_\gamma(\psi_{pq}((\widehat{x_0})_{I_p}), \ldots, \psi_{pq}((\widehat{x_{n_\gamma - 1}})_{I_p})).
\end{aligned}$$

Remark 4.6. Since any homomorphism of universal algebras is ideal, it is easy to observe that, in the case of universal algebras, the results in Section 3 lead us to the results presented in [4, §21].

Remark 4.7. The restriction of $F : \mathbf{Malg}(\tau) \longrightarrow \mathbf{Alg}(\tau)$ to $\mathbf{Malg}_i(\tau)$ is a left adjoint for the inclusion functor $U : \mathbf{Alg}(\tau) \longrightarrow \mathbf{Malg}_i(\tau)$. So, if $(I, \leq)$ is a directed preordered set and $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ is a direct system from $\mathbf{Malg}_i(\tau)$ then $\overline{\mathcal{A}} = ((\overline{\mathfrak{A}_i} \mid i \in I), (\overline{\varphi_{ij}} \mid i, j \in I,\ i \leq j))$ is a direct system of universal algebras of type τ and the universal algebras $\overline{\varinjlim \mathcal{A}}$ and $\varinjlim \overline{\mathcal{A}}$ are isomorphic.

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