CARPATHIAN J. MATH.
22 (2006), No. 1 - 2, 137 - 145

On totally $\widetilde{g}s$-continuity, strongly $\widetilde{g}s$ -continuity and contra $\widetilde{g}s$-continuity

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ABSTRACT. In this paper, $\widetilde{g}s$-closed sets and $\widetilde{g}s$-open sets are used to define and investigate a new class of functions. Relationships between this new class and other classes of functions are established.

1. INTRODUCTION

Jain [5], Levine [8] and Dontchev [1] introduced totally continuous functions, strongly continuous functions and contra continuous functions, respectively. Leveine [6] also introduced and studied the concepts of generalized closed sets. The notion has been studied extensively in recent years by many topologists. As generalization of closed sets, $\widetilde{g}s$-closed sets were introduced and studied by Sundaram et al. in [15]. This notion was further studied by Rajesh and Ekici [12, 13].

In this paper, we will continue the study of some related functions by using $\widetilde{g}s$-open and $\widetilde{g}s$-closed sets. We introduce and characterize the concepts of totally $\widetilde{g}s$-continuous, strongly $\widetilde{g}s$-continuous and contra $\widetilde{g}s$-continuous functions.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X, respectively. We set C(X, x) = $\{$V $\in$ C(X) $|$ x $\in$ V$\}$ for x $\in$ X, where C(X) denotes the collection of all closed subsets of (X, τ). The set of all clopen subsets of (X, τ) is denoted by CO(X, τ).

We recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called:

(1) semi-open [7] if A$\subseteq$cl(int(A)).
(2) α-open [10] if A$\subseteq$int(cl(int(A))).

The complements of the above mentioned sets are called semi-closed and α-closed, respectively. The intersection of all semi-closed sets of X containing a subset A is called the semi-closure of A and is denoted by scl(A).

Definition 2.2. A subset A of a space (X, τ) is called:

(1) $\widehat{g}$-closed [17] if cl(A)$\subseteq$U whenever A$\subseteq$U and U is semi-open in (X, τ). The complement of a $\widehat{g}$-closed set is called $\widehat{g}$-open.

Received: 04.05.2006; In revised form: 24.07.2006; Accepted: 01.11.2006
2000 *Mathematics Subject Classification:* 54C10.
Key words and phrases: *Totally continuous functions, totally $\widetilde{g}s$ -continuous, strongly $\widetilde{g}s$-continuous and contra $\widetilde{g}s$ -continuous.*

(2) *g-closed set [16] if cl(A)⊆U whenever A⊆U and U is $\widehat{g}$-open in (X, τ). The complement of a *g-closed set is called *g-open.
(3) $^\#g$-semi-closed (briefly $^\#g$s-closed) set [18] if scl(A)⊆U whenever A⊆U and U is *g-open in (X, τ). The complement of a $^\#g$s-closed set is called $^\#g$s-open.
(4) $\widetilde{g}$-semi-closed (briefly $\widetilde{g}s$-closed) set[15] if scl(A)⊆U whenever A⊆U and U is $^\#g$s-open in (X, τ). The complement of a $\widetilde{g}s$-closed set is called $\widetilde{g}s$-open. The class of all $\widetilde{g}s$-open sets of (X, τ) is denoted by $\widetilde{G}$S(X, τ).
(5) $\widetilde{g}s$-clopen if it is both $\widetilde{g}s$-open and $\widetilde{g}s$-closed.

We set $\widetilde{G}$(X, x) = {V ∈ $\widetilde{G}$S(X, τ) | x ∈ V} for x ∈ X.

Remark 2.1. From the Definition 2.1 and 2.2, we have the following diagram,

$$\begin{tikzcd}
 & \text{closed} \arrow[rr] \arrow[dl] \arrow[dr] & & \alpha\text{-closed} \arrow[rr] \arrow[dl, dash] \arrow[dr, dash] & & \text{semi-closed} \arrow[dl] \arrow[dr] & \\
{}^*g\text{-closed} & — & \widehat{g}\text{-closed} & — & \widetilde{g}s\text{-closed} & — & {}^\#gs\text{-closed}
\end{tikzcd}$$

where A→B (resp. A ↮ B or A — B) means A implies B (resp. A and B are independent).

Definition 2.3. A function f: $(X, \tau) \to (Y, \sigma)$ is called:
(1) totally continuous [5] of the inverse image of every open subset of (Y, σ) is a clopen subset of (X, τ);
(2) strongly continuous [8] if the inverse image of every subset Y is a clopen subset of (X, τ);
(3) contra-continuous [1] (resp. contra-semi-continuous [2], contra-α-continuous [3]) if the inverse image of every open subset of Y is a closed (resp. semi-closed, α-closed) subset of (X, τ);
(4) $\widetilde{g}s$-continuous [12] if the inverse image of every open subset of (Y, σ) is $\widetilde{g}s$-open in (X, τ).

3. Two classes of functions via $\widetilde{g}s$-clopen sets

We introduce the following definition:

Definition 3.4. A function f: $(X, \tau) \to (Y, \sigma)$ is said to be totally $\widetilde{g}$-semi-continuous (briefly totally $\widetilde{g}s$-continuous) if the inverse image of every open subset of (Y, σ) is a $\widetilde{g}s$-clopen (i.e. $\widetilde{g}$s-open and $\widetilde{g}$-closed) subset of (X, τ).

It is evident that every totally continuous function is totally $\widetilde{g}s$-continuous. But the converse need not be true as shown in the following example.

Example 3.1. Let X = $\{a, b, c\}$, Y = $\{p, q\}$, τ = {∅, $\{a\}$, $\{b\}$, $\{a, b\}$, X} and σ = {∅, $\{p\}$, Y}. Define a function f: $(X, \tau) \to (Y, \sigma)$ such that f(a) = p, f(b) = f(c) = q. Then clearly f is totally $\widetilde{g}s$-continuous, but not totally continuous.

Definition 3.5. A function f: $(X, \tau) \to (Y, \sigma)$ is said to be strongly $\widetilde{g}$-semi-continuous (briefly strongly $\widetilde{g}s$-continuous) if the inverse image of every subset of (Y, σ) is a $\widetilde{g}s$-clopen subset of (X, τ).

It is clear that every strongly $\widetilde{g}s$-continuous function is totally $\widetilde{g}s$-continuous. But the reverse implication is not always true as shown in the following example.

Example 3.2. Let X = {a, b, c} = Y, τ = {$\varnothing$, $\{a\}$, $\{b\}$, $\{a, b\}$, X} and σ = {$\varnothing$, $\{a\}$, Y}. Then the identity function f:$(X, \tau) \to (Y, \sigma)$ is totally $\widetilde{g}s$-continuous but not strongly $\widetilde{g}s$-continuous.

Theorem 3.1. *Every totally $\widetilde{g}s$-continuous function into a T_1-space is strongly $\widetilde{g}s$-continuous.*

Proof. In a T_1-space, singletons are closed. Hence f^{-1}(A) is $\widetilde{g}s$-clopen in (X, τ) for every subset A of Y. □

Remark 3.2. It is clear from the Theorem 3.1 that the classes of strongly $\widetilde{g}s$-continuous functions and totally $\widetilde{g}s$-continuous function coincide when the range is a T_1-space.

Recall that a space (X, τ) is said to be $\widetilde{g}s$-connected [12] if X cannot be expressed as the union of two non-empty disjoint $\widetilde{g}s$-open sets.

Theorem 3.2. *If f is a totally $\widetilde{g}s$-continuous function from a $\widetilde{g}s$-connected space X onto any space Y, then Y is an indiscrete space.*

Proof. Suppose that Y is not indiscrete. Let A be a proper non-empty open subset of Y. Then f^{-1}(A) is a proper non-empty $\widetilde{g}s$-clopen subset of (X, τ), which is a contradiction to the fact that X is $\widetilde{g}s$-connected. □

Definition 3.6. Let A be a subset of X. The intersection of all $\widetilde{g}s$-closed sets containing A is called the $\widetilde{g}s$-closure of A [13] and is denoted by $\widetilde{g}s$cl(A).

Definition 3.7. A space X is said to be $\widetilde{g}s$-T_2 [11] if for any pair of distinct points x, y of X, there exist disjoint $\widetilde{g}s$-open sets U and V such that x $\in$ U and y $\in$ V.

Theorem 3.3. [11] *A space X is $\widetilde{g}s$-T_2 if and only if for any pair of distinct points x, y of X there exist $\widetilde{g}s$-open sets U and V such that x $\in$ U, and y $\in$ V and $\widetilde{g}s$cl(U) $\cap$ $\widetilde{g}s$cl(V) = $\varnothing$.*

Theorem 3.4. *If f: $(X, \tau) \to (Y, \sigma)$ be a totally $\widetilde{g}s$-continuous injection and Y is T_0, then X is $\widetilde{g}s$-T_2.*

Proof. Let x and y be any pair of distinct points of X. Then f(x)$\neq$f(y). Since Y is T_0, there exists an open set U containing say, f(x) but not f(y). Then x $\in$ f^{-1}(U) and y $\notin$ f^{-1}(U). Since f is totally $\widetilde{g}s$-continuous, f^{-1}(U) is a $\widetilde{g}s$-clopen subset of X. Also, x $\in$ f^{-1}(U) and y $\in$ X – f^{-1}(U). By Theorem 3.3, it follows that X is $\widetilde{g}s$-T_2. □

Theorem 3.5. *A topological space (X, τ) is $\widetilde{g}s$-connected if and only if every totally $\widetilde{g}s$-continuous function from a space (X, τ) into any T_0-space (Y, σ) is constant.*

Proof. Suppose that X is not $\widetilde{g}s$-connected and every totally $\widetilde{g}s$-continuous function from (X, τ) to (Y, σ) is constant. Since (X, τ) is not $\widetilde{g}s$-connected, there exists a proper non-empty $\widetilde{g}s$-clopen subset A of X. Let Y= $\{a, b\}$ and τ = {$\varnothing$, $\{a\}$, $\{b\}$, Y} be a topology for Y. Let f: $(X, \tau) \to (Y, \sigma)$ be a function such that f(A) = $\{a\}$ and f(Y–A) = $\{b\}$. Then f is non-constant and totally $\widetilde{g}s$-continuous such that Y is T_0, which is a contradiction. Hence X must be $\widetilde{g}s$-connected.
Converse is similar. □

Theorem 3.6. *Let f: $(X,\tau) \to (Y,\sigma)$ be a totally $\widetilde{g}s$-continuous function and Y is a T_1-space. If A is a non-empty $\widetilde{g}s$-connected subset of X, then f(A) is a single point.*

Definition 3.8. Let (X,τ) be a topological space. Then the set of all points y in X such that x and y cannot be separated by a $\widetilde{g}s$-separation of X is said to be the quasi $\widetilde{g}s$-component of X.

Theorem 3.7. *Let f: $(X,\tau) \to (Y,\sigma)$ be a totally $\widetilde{g}s$-continuous function from a topological space (X,τ) into a T_1-space Y. Then f is constant on each quasi $\widetilde{g}s$-component of X.*

Proof. Let x and y be two points of X that lie in the same quasi-$\widetilde{g}s$-component of X. Assume that $f(x) = \alpha \neq \beta = f(y)$. Since Y is T_1, $\{\alpha\}$ is closed in Y and so Y–$\{\alpha\}$ is an open set. Since f is totally $\widetilde{g}s$-continuous, therefore $f^{-1}(\{\alpha\})$ and f^{-1}(Y–$\{\alpha\}$) are disjoint $\widetilde{g}s$-clopen subsets of X. Further, $x \in f^{-1}(\{\alpha\})$ and $y \in f^{-1}$(Y–$\{\alpha\}$), which is a contradiction in view of the fact that y belongs to the quasi $\widetilde{g}s$-component of x and hence y must belong to every $\widetilde{g}s$-open set containing x. □

4. CONTRA-$\widetilde{g}$-SEMI-CONTINUOUS

We introduce the following definition

Definition 4.9. A function f: $(X,\tau) \to (Y,\sigma)$ is called contra-$\widetilde{g}$-semi-continuous (briefly c$\widetilde{g}s$-continuous) if f^{-1}(V) is $\widetilde{g}s$-open in (X,τ) for every closed set V in (Y,σ).

It is clear that every strongly $\widetilde{g}s$-continuous function is c$\widetilde{g}s$-continuous. But the reverse implication is not always true as shown in the following example.

Example 4.3. Let X = Y = $\{a,b,c\}$, τ = $\{\varnothing, \{a\}, \{b,c\}, X\}$ and σ = $\{\varnothing, \{a\}, Y\}$. Then the identity funcion f: $(X,\tau) \to (Y,\sigma)$ is c$\widetilde{g}s$-continuous but it is not strongly $\widetilde{g}s$-continuous.

Definition 4.10. Let A be a subset of a topological space (X,τ). The set $\bigcap\{U \in \tau \mid A \subset U\}$ is called the Kernal of A [9] and is denoted by ker(A).

Lemma 4.1. [4] *The following properties hold for subsets A, B of a space X:*

(1) $x \in ker(A)$ *if and only if* $A \cap F \neq \varnothing$ *for any* $F \in C(X,x)$*;*
(2) $A \subset ker(A)$ *and* $A = ker(A)$ *if A is open in* X*;*
(3) If $A \subset B$*, then* $ker(A) \subset ker(B)$*.*

Theorem 4.8. *The following are equivalent for a function f: $(X,\tau) \to (Y,\sigma)$:*

(1) f *is* $c\widetilde{g}s$*-continuous;*
(2) for every closed subset F *of* Y, $f^{-1}(F) \in \widetilde{G}S(X,\tau)$*;*
(3) for each $x \in X$ *and each* $F \in C(Y, f(x))$*, there exists* $U \in \widetilde{G}S(X,\tau)$ *such that* $f(U) \subset F$*;*
(4) $f(\widetilde{g}scl(A)) \subset ker(f(A))$ *for every subset* A *of* X*;*
(5) $\widetilde{g}scl(f^{-1}(B)) \subset f^{-1}(ker(B))$ *for every subset* B *of* Y*.*

Proof. The implications (1)→(2) and (2)→(3) are obvious.

(3)→(2): Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \widetilde{G}S(X,x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \bigcup\{U_x | x \in f^{-1}(F)\} \in \widetilde{G}\mathrm{S}(X,\tau)$ [15].

(2)→(4): Let A be any subset of X. Suppose that $y \notin ker(f(A))$. Then by Lemma 4.1 there exists $F \in C(X,y)$ such that $f(A) \cap F = \varnothing$. Thus, we have $A \cap f^{-1}(F) = \varnothing$ and $\widetilde{g}s\mathrm{cl}(A) \cap f^{-1}(F) = \varnothing$. Therefore, we obtain $f(\widetilde{g}s\mathrm{cl}(A)) \cap F = \varnothing$ and $y \notin f(\widetilde{g}s\mathrm{cl}(A))$. This implies that $f(\widetilde{g}s\mathrm{cl}(A)) \subset ker(f(A))$.

(4)→(5): Let B be any subset of Y. By (4) and Lemma 4.1, we have $f(\widetilde{g}s\mathrm{cl}(f^{-1}(B))) \subset ker(f(f^{-1}(B))) \subset ker(B)$ and $\widetilde{g}s\mathrm{cl}(f^{-1}(B)) \subset f^{-1}(ker(B))$.

(5)→(1): Let V be any open set of Y. Then by Lemma 4.1 we have $\widetilde{g}s\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(ker(V)) = f^{-1}(V)$ and $\widetilde{g}s\mathrm{cl}(f^{-1}(V)) = f^{-1}(V)$. This show that $f^{-1}(V)$ is $\widetilde{g}s$-closed in (X,τ). □

Theorem 4.9. *Every contra semi-continuous function is $c\widetilde{g}s$-continuous.*

Proof. The proof follows from the definitions. □

Remark 4.3. Contra $\widetilde{g}s$-continuous need not be contra semi-continuous in general as shown in the following example.

Example 4.4. Let $X = Y = \{a,b,c\}, \tau = \{\varnothing, \{a,b\}, X\}$ and $\sigma = \{\varnothing, \{b,c\}, Y\}$. Then the identity function $f : (X,\tau) \to (Y,\sigma)$ is $c\widetilde{g}s$-continuous. However, f is not contra-semi continuous, since for the closed set $F = \{a\}$, f^{-1}(F) is $\widetilde{g}s$-open but not semi-open in (X,τ).

Corollary 4.1. *Every contra α-continuous (resp. contra-continuous) function is $c\widetilde{g}s$-continuous.*

Theorem 4.10. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function. Then the following are equivalent:*

(1) The function f is $\widetilde{g}s$-continuous;
(2) For each point x in X and each open set V in (Y,σ) with $f(x) \in V$, there exists a $\widetilde{g}s$-open set U in (X,τ) such that $x \in U$, $f(U) \subset V$.

Proof. (1)→(2): Let $f(x) \in V$. Since f is $\widetilde{g}s$-continuous, we have $x \in f^{-1}(V) \in \widetilde{G}S(X,\tau)$. Let $U = f^{-1}(V)$. Then $x \in V$ and $f(U) \subset V$.

(2)→(1): Let V be an open set in (Y,σ) and let $x \in f^{-1}(V)$. Then, $f(x) \in V$ and thus there exists a $\widetilde{g}s$-open set U_x such that $x \in U_x$ and $f(U_x) \subset V$. Now, $x \in U_x \subset f^{-1}(V)$ and $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$. Since union of $\widetilde{g}s$-open sets is $\widetilde{g}s$-open [12], $f^{-1}(V)$ is $\widetilde{g}s$-open in (X,τ) and therefore f is $\widetilde{g}s$-continuous. □

Theorem 4.11. *If a function $f : (X,\tau) \to (Y,\sigma)$ is $c\widetilde{g}s$-continuous and Y is regular, then f is $\widetilde{g}s$-continuous.*

Proof. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $\mathrm{cl}(W) \subset V$. Since f is $c\widetilde{g}s$-continuous, so by Theorem 4.8 there exists $U \in \widetilde{G}S(X,x)$ such that $f(U) \subset \mathrm{cl}(W)$. Then $f(U) \subset \mathrm{cl}(W) \subset V$. Hence, by theorem 4.10 f is $\widetilde{g}s$-continuous. □

Theorem 4.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : X \to X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then f is $c\widetilde{g}s$-continuous if and only if g is $c\widetilde{g}s$-continuous.*

Proof. Let x $\in X$ and let W be a closed subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is closed in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to Y. Hence $\{y \in$Y$|(x, y) \in W\}$ is a closed subset of Y. Since f is $c\widetilde{g}s$ -continuous, $\bigcup\{f^{-1}(y)|(x, y) \in W\}$ is a $\widetilde{g}s$-open subset of X. Further, $x \in \bigcup\{f^{-1}(y)|(x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is $\widetilde{g}s$-open. Then g is $c\widetilde{g}s$-continuous.
Conversely, let F be a closed subset of Y. Then $X \times F$ is a closed subset of $X \times Y$. Since g is $c\widetilde{g}s$-continuous, $g^{-1}(X \times F)$ is a $\widetilde{g}s$-open subset of X. Also, $g^{-1}(X \times F) = f^{-1}(F)$. Hence f is $c\widetilde{g}s$-continuous. □

Theorem 4.13. *If X is a topological space and for each pair of distinct points x_1 and x_2 in X there exists a map f into a Urysohn topological space Y such that $f(x_1) \neq f(x_2)$ and f is $c\widetilde{g}s$-continuous at x_1 and x_2, then X is $\widetilde{g}s - T_2$.*

Proof. Let x_1 and x_2 be any distinct points in X. Then by hypothesis there is a Urysohn space Y and a function $f : (X, \tau) \to (Y, \sigma)$, which satisfies the conditions of the theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since Y is Urysohn, there exist open neighborhoods U_{y_1} and U_{y_2} of y_1 and y_2 respectively in Y such that cl(U_{y_1})$\cap$ cl(U_{y_2}) $= \varnothing$. Since f is $c\widetilde{g}s$-continuous at x_i, there exists a $\widetilde{g}s$-open neighborhoods W_{x_i} of x_i in X such that $f(W_{x_i}) \subset$ cl(U_{y_i}) for $i = 1, 2$. Hence we get $W_{x_1} \cap W_{x_2} = \varnothing$ because cl(U_{y_1})$\cap$ cl(U_{y_2}) $= \varnothing$. Then X is $\widetilde{g}s$-T_2. □

Corollary 4.2. *If f is a $c\widetilde{g}s$-continuous injection of a topological space X into a Urysohn space Y, then X is $\widetilde{g}s$-T_2.*

Proof. For each pair of distinct points x_1 and x_2 in X, f is a $c\widetilde{g}s$-continuous function of X into Urysohn space Y such that $f(x_1) \neq f(x_2)$ because f is injective. Hence by Theorem 4.13, X is $\widetilde{g}s$-T_2. □

Corollary 4.3. *If f is a $c\widetilde{g}s$-continuous injection of a topological space X into Ultra Hansdorff space Y, then X is $\widetilde{g}s$-T_2.*

Proof. Let x_1 and x_2 be any distinct points in X. Then since f is injective and Y is Ultra Hansdorff f(x_1) $\neq$ f(x_2) and there exist V_1, $V_2 \in$ CO(Y, σ) such that f(x_1) $\in V_1$, f(x_2) $\in V_2$ and $V_1 \cap V_2$ = $\varnothing$. Then $x_1 \in f^{-1}$(V) $\in \widetilde{G}$S(X, τ) for i = 1, 2 and f^{-1}(V_1) $\cap$ f^{-1}(V_2) = $\varnothing$. Thus, X is $\widetilde{g}s$-T_2. □

Lemma 4.2. *The product of two $\widetilde{g}s$-open sets is $\widetilde{g}s$-open* [11].

Proof. Let $A \in \widetilde{G}SO(X, \tau)$, $B \in \widetilde{G}SO(Y, \sigma)$ and $W = A \times B \subset X \times Y$. Let $F \subset W$ be a $^{\#}gs$-closed set in $X \times Y$, then there exist two $^{\#}gs$-closed sets $F_1 \subset A$, $F_2 \subset B$ and so, $F_1 \subset$sint(A), $F_2 \subset$sint(B). Hence $F_1 \times F_2 \subset$A$\times B$ and $F_1 \times F_2 \subset$sint(B) = sint($A \times B$). Therefore, $A \times B \in \widetilde{G}$SO($X \times Y, \tau \times \sigma$). □

Lemma 4.3. *Let $A \subset Y \subset X$, $Y \in \widetilde{G}S(X, \tau)$ and $A \in \widetilde{G}S(Y, \sigma)$. Then $A \in \widetilde{G}S(X, \tau)$* [15].

Theorem 4.14. *Let f_1: $X_1 \to Y$ and $f_2 : X_2 \to Y$ be two functions where Y is a Urysohn space and f_1 and f_2 are $c\widetilde{g}s$-continuous. Then $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is $\widetilde{g}s$-closed in the product space $X_1 \times X_2$.*

Proof. Let A denote the set $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$. In order to show that A is $\widetilde{g}s$-closed, we show that $(X_1 \times X_2) - A$ is $\widetilde{g}s$-open. Let $(x_1, x_2) \notin A$. Then $f_1(x_1) \neq f_2(x_2)$. Since Y is Urysohn, there exist open V_1 and V_2 of $f_1(x_1)$ and $f_2(x_2)$ such that $\mathrm{cl}(V_1) \cap \mathrm{cl}(V_2) = \varnothing$. Since f_i $(i = 1, 2)$ is $c\widetilde{g}s$-continuous, $f_i^{-1}(\mathrm{cl}(V_i))$ is a $\widetilde{g}s$-open set containing x_i in X_i $(i = 1, 2)$. Hence by Lemma 4.2, $f_1^{-1}(\mathrm{cl}(V_1)) \times f_2^{-1}(\mathrm{cl}(V_2))$ is $\widetilde{g}s$-open. Further $(x_1, x_2) \in f_1^{-1}(\mathrm{cl}(V_1)) \times f_2^{-1}(\mathrm{cl}(V_2)) \subset ((X_1 \times X_2) - A)$. It follows that $X_1 \times X_2 - A$ is $\widetilde{g}s$-open. Thus, A is $\widetilde{g}s$-closed in the product space $X_1 \times X_2$. □

Corollary 4.4. *If $f : (X, \tau) \to (Y, \sigma)$ is $c\widetilde{g}s$-continuous and Y is a Urysohn space, then $A = \{(x_1, x_2)|f(x_1) = f(x_2)\}$ is $\widetilde{g}s$-closed in the product space $X_1 \times X_2$.*

Theorem 4.15. *If $f : (X\tau) \to (Y, \sigma)$ is a contra-$\widetilde{g}s$-continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is a continuous function, then $(g \circ f) : (X, \tau) \to (Z, \eta)$ is $c\widetilde{g}s$-continuous.*

Theorem 4.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be surjective $\widetilde{g}s$-irresolute and $\widetilde{g}s$-open and $g : (Y, \sigma) \to (Z, \eta)$ be any function. Then $(g \circ f) : (X, \tau) \to (Z, \eta)$ is $c\widetilde{g}s$-continuous if and only if g is $c\widetilde{g}s$-continuous.*

Proof. The "if "part is easy to prove. To prove the "only if "part, let $(g \circ f) : (X, \tau) \to (Z, \eta)$ be $c\widetilde{g}s$-continuous. Let F be a closed subset of Z. Then $(g \circ f)^{-1}(F)$ is a $\widetilde{g}s$-open subset of X. That is $f^{-1}(g^{-1}(F))$ is $\widetilde{g}s$-open. Since f is $\widetilde{g}s$-open, $f(f^{-1}(g^{-1}(F)))$ is a $\widetilde{g}s$-open subset of Y. So $g^{-1}(F)$ is $\widetilde{g}s$-open in Y. Hence g is $c\widetilde{g}s$-continuous. □

Theorem 4.17. *Let $\{X_i| \ i \in \wedge\}$ be any family of topological spaces. If $f : X \to \Pi X_i$ is a $c\widetilde{g}s$-continuous function. Then $\pi_i \circ f : X \to X_i$ is $c\widetilde{g}s$-continuous for each $i \in \wedge$, where π_i is the projection of ΠX_i onto X_i.*

Definition 4.11. The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be $c\widetilde{g}s$-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \widetilde{G}S(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \varnothing$.

Lemma 4.4. *The graph $f : (X, \tau) \to (Y, \sigma)$ is $c\widetilde{g}s$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exists $U \in \widetilde{G}S(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \varnothing$.*

Proof. The proof follows from the definition. □

Theorem 4.18. *If $f : (X, \tau) \to (Y, \sigma)$ $c\widetilde{g}s$ -continuous and Y is Urysohn, then $G(f)$ is contra-$\widetilde{g}s$-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets V, W such that $f(x) \in V$, $y \in W$ and $\mathrm{cl}(U) \cap \mathrm{cl}(W) = \varnothing$. Since f is $c\widetilde{g}s$-continuous, there exists $U \in \widetilde{G}S(X, x)$ such that $f(U) \subset \mathrm{cl}(V)$. Therefore, we obtain $f(U) \cap \mathrm{cl}(W) = \varnothing$. This shows that $G(f)$ is contra-$\widetilde{g}s$-closed. □

Theorem 4.19. *A $c\widetilde{g}s$-continuous image of a $\widetilde{g}s$-connected space is connected.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a contra-$\widetilde{g}s$-continuous function of a $\widetilde{g}s$-connected space X onto a topological space Y. Let Y be disconnected. Let A and B form a disconnected of Y. Then A and B are clopen and $Y = A \cup B$ where $A \cap B = \varnothing$. Since f is a contra-$\widetilde{g}$s-continuous function $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty $\widetilde{g}s$-open sets in X. Also $f^{-1}(A) \cap f^{-1}(B) = \varnothing$. Hence X is non $\widetilde{g}s$-connected which is a contradiction. Therefore Y is connected. □

Theorem 4.20. *Let X be $\widetilde{g}s$-connected and Y a T_1 space. If f is $c\widetilde{g}s$-continuous, then f is constant.*

Proof. Since Y is T_1 space, $\wedge = \{f^{-1}(\{y\}) : y \in Y\}$ is a disjoint $\widetilde{g}s$-open partition of X. If $| \wedge | \geq 2$, then X is the union of two non-empty $\widetilde{g}s$-open sets. Since X is $\widetilde{g}s$-connected, $| \wedge | = 1$. Hence, f is constant. □

Definition 4.12. A topological space (X, τ) is said to be $\widetilde{g}s$-normal if each pair of non-empty disjoint closed sets can be separated by disjoint $\widetilde{g}s$-open sets.

Definition 4.13. [14] A topological space (X, τ) is said to be ultra normal if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 4.21. *If $f : (X, \tau) \to (y, \sigma)$ is a $c\widetilde{g}s$-continuous, closed injection and Y is ultra-normal, then X is $\widetilde{g}s$-normal.*

Proof. Let F_1 and F_2 be a disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is ultra normal $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2 respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in \widetilde{G}S(X, \tau)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$. Thus, X is $\widetilde{g}s$-normal. □

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