On totally $\tilde{g}s$-continuity, strongly $\tilde{g}s$ -continuity and contra $\tilde{g}s$-continuity

Neelamegarajan Rajesh and Erdal Ekici

ABSTRACT. In this paper, $\tilde{g}s$-closed sets and $\tilde{g}s$-open sets are used to define and investigate a new class of functions. Relationships between this new class and other classes of functions are established.

1. INTRODUCTION

Jain [5], Levine [8] and Dontchev [1] introduced totally continuous functions, strongly continuous functions and contra continuous functions, respectively. Levine [6] also introduced and studied the concepts of generalized closed sets. The notion has been studied extensively in recent years by many topologists. As generalization of closed sets, $\tilde{g}s$-closed sets were introduced and studied by Sundaram et al. in [15]. This notion was further studied by Rajesh and Ekici [12, 13].

In this paper, we will continue the study of some related functions by using $\tilde{g}s$-open and $\tilde{g}s$-closed sets. We introduce and characterize the concepts of totally $\tilde{g}s$-continuous, strongly $\tilde{g}s$-continuous and contra $\tilde{g}s$-continuous functions.

2. PRELIMINARIES

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $\text{cl}(A)$, $\text{int}(A)$ and $A^c$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively. We set $C(X, x) = \{V \in C(X) | x \in V\}$ for $x \in X$, where $C(X)$ denotes the collection of all closed subsets of $(X, \tau)$. The set of all clopen subsets of $(X, \tau)$ is denoted by $\text{CO}(X, \tau)$.

We recall the following definitions, which are useful in the sequel.

**Definition 2.1.** A subset $A$ of a space $(X, \tau)$ is called:

1. semi-open [7] if $A \subseteq \text{cl}(\text{int}(A))$.
2. $\alpha$-open [10] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$.

The complements of the above mentioned sets are called semi-closed and $\alpha$-closed, respectively. The intersection of all semi-closed sets of $X$ containing a subset $A$ is called the semi-closure of $A$ and is denoted by $\text{scl}(A)$.

**Definition 2.2.** A subset $A$ of a space $(X, \tau)$ is called:

1. $\tilde{g}$-closed [17] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$. The complement of a $\tilde{g}$-closed set is called $\tilde{g}$-open.
(2) $^g$-closed set [16] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open in $(X, \tau)$. The complement of a $^g$-closed set is called $^g$-open.

(3) $^g$-semi-closed (briefly $^g$-s-closed) set [18] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$-open in $(X, \tau)$. The complement of a $^g$-s-closed set is called $^g$-open.

(4) $^g$s-closed (briefly $^g$s-closed) set [15] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $^g$s-open in $(X, \tau)$. The complement of a $^g$s-closed set is called $^g$s-open. The class of all $^g$s-open sets of $(X, \tau)$ is denoted by $^gS(X, \tau)$.

(5) $^g$s-clopen if it is both $^g$s-open and $^g$s-closed.

We set $\tilde{G}(X, x) = \{ V \in \tilde{G}(X, \tau) \mid x \in V \}$ for $x \in X$.

**Remark 2.1.** From the Definition 2.1 and 2.2, we have the following diagram,

- $^g$-closed $\rightarrow$ $^g$s-closed $\rightarrow$ semi-closed
- $^g$-open $\rightarrow$ $^g$s-open $\rightarrow$ $^g$s-clopen

where $A \rightarrow B$ (resp. $A \leftrightarrow B$ or $A \rightarrow B$) means $A$ implies $B$ (resp. $A$ and $B$ are independent).

**Definition 2.3.** A function $f: (X, \tau) \to (Y, \sigma)$ is called:

- (1) totally continuous [5] of the inverse image of every open subset of $(Y, \sigma)$ is a clopen subset of $(X, \tau)$;
- (2) strongly continuous [8] if the inverse image of every subset $Y$ is a clopen subset of $(X, \tau)$;
- (3) contra-continuous [1] (resp. contra-semi-continuous [2], contra-$\alpha$-continuous [3]) if the inverse image of every open subset of $Y$ is a closed (resp. semi-closed, $\alpha$-closed) subset of $(X, \tau)$;
- (4) $^g$s-continuous [12] if the inverse image of every open subset of $(Y, \sigma)$ is $^g$s-open in $(X, \tau)$.

3. **TWO CLASSES OF FUNCTIONS VIA $^g$s-CLOPEN SETS**

We introduce the following definition:

**Definition 3.4.** A function $f: (X, \tau) \to (Y, \sigma)$ is said to be totally $^g$s-semi-continuous (briefly totally $^g$s-continuous) if the inverse image of every open subset of $(Y, \sigma)$ is a $^g$s-clopen (i.e. $^g$s-open and $^g$s-closed) subset of $(X, \tau)$.

It is evident that every totally continuous function is totally $^g$s-continuous. But the converse need not be true as shown in the following example.

**Example 3.1.** Let $X = \{a, b, c\}$, $Y = \{p, q\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{p\}, Y\}$. Define a function $f: (X, \tau) \to (Y, \sigma)$ such that $f(a) = p$, $f(b) = f(c) = q$. Then clearly $f$ is totally $^g$s-continuous, but not totally continuous.

**Definition 3.5.** A function $f: (X, \tau) \to (Y, \sigma)$ is said to be strongly $^g$s-semi-continuous (briefly strongly $^g$s-continuous) if the inverse image of every subset of $(Y, \sigma)$ is a $^g$s-clopen subset of $(X, \tau)$.

It is clear that every strongly $^g$s-continuous function is totally $^g$s-continuous. But the reverse implication is not always true as shown in the following example.
Example 3.2. Let \( X = \{a, b, c\} = Y, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \). Then the identity function \( f: (X, \tau) \rightarrow (Y, \sigma) \) is totally \( \bar{g}s \)-continuous but not strongly \( \bar{g}s \)-continuous.

**Theorem 3.1.** Every totally \( \bar{g}s \)-continuous function into a \( T_1 \)-space is strongly \( \bar{g}s \)-continuous.

**Proof.** In a \( T_1 \)-space, singletons are closed. Hence \( f^{-1}(A) \) is \( \bar{g}s \)-clopen in \( (X, \tau) \) for every subset \( A \) of \( Y \).

**Remark 3.2.** It is clear from the Theorem 3.1 that the classes of strongly \( \bar{g}s \)-continuous functions and totally \( \bar{g}s \)-continuous function coincide when the range is a \( T_1 \)-space.

Recall that a space \( (X, \tau) \) is said to be \( \bar{g}s \)-connected [12] if \( X \) cannot be expressed as the union of two non-empty disjoint \( \bar{g}s \)-open sets.

**Theorem 3.2.** If \( f \) is a totally \( \bar{g}s \)-continuous function from a \( \bar{g}s \)-connected space \( X \) onto any space \( Y \), then \( Y \) is an indiscrete space.

**Proof.** Suppose that \( Y \) is not indiscrete. Let \( A \) be a proper non-empty open subset of \( Y \). Then \( f^{-1}(A) \) is a proper non-empty \( \bar{g}s \)-clopen subset of \( (X, \tau) \), which is a contradiction to the fact that \( X \) is \( \bar{g}s \)-connected.

**Definition 3.6.** Let \( A \) be a subset of \( X \). The intersection of all \( \bar{g}s \)-closed sets containing \( A \) is called the \( \bar{g}s \)-closure of \( A \) [13] and is denoted by \( \bar{g}scl(A) \).

**Definition 3.7.** A space \( X \) is said to be \( \bar{g}s \)-T_2 [11] if for any pair of distinct points \( x, y \) of \( X \), there exist disjoint \( \bar{g}s \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \).

**Theorem 3.3.** [11] A space \( X \) is \( \bar{g}s \)-T_2 if and only if for any pair of distinct points \( x, y \) of \( X \) there exist \( \bar{g}s \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \) and \( \bar{g}scl(U) \cap \bar{g}scl(V) = \emptyset \).

**Theorem 3.4.** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is a totally \( \bar{g}s \)-continuous injection and \( Y \) is \( T_0 \), then \( X \) is \( \bar{g}s \)-T_2.

**Proof.** Let \( x \) and \( y \) be any pair of distinct points of \( X \). Then \( f(x) \neq f(y) \). Since \( Y \) is \( T_0 \), there exists an open set \( U \) containing \( x \) but not \( y \). Then \( x \in f^{-1}(U) \) and \( y \notin f^{-1}(U) \). Since \( f \) is totally \( \bar{g}s \)-continuous, \( f^{-1}(U) \) is a \( \bar{g}s \)-clopen subset of \( X \). Also, \( x \in f^{-1}(U) \) and \( y \in X - f^{-1}(U) \). By Theorem 3.3, it follows that \( X \) is \( \bar{g}s \)-T_2.

**Theorem 3.5.** A topological space \( (X, \tau) \) is \( \bar{g}s \)-connected if and only if every totally \( \bar{g}s \)-continuous function from a space \( (X, \tau) \) into any \( T_0 \)-space \( (Y, \sigma) \) is constant.

**Proof.** Suppose that \( X \) is not \( \bar{g}s \)-connected and every totally \( \bar{g}s \)-continuous function from \( (X, \tau) \) to \( (Y, \sigma) \) is constant. Since \( (X, \tau) \) is not \( \bar{g}s \)-connected, there exists a proper non-empty \( \bar{g}s \)-clopen subset \( A \) of \( X \). Let \( Y = \{a, b\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, Y\} \) be a topology for \( Y \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function such that \( f(A) = \{a\} \) and \( f(Y - A) = \{b\} \). Then \( f \) is non-constant and totally \( \bar{g}s \)-continuous such that \( Y \) is \( T_0 \), which is a contradiction. Hence \( X \) must be \( \bar{g}s \)-connected.

Converse is similar.
Theorem 4.8. The following are equivalent for a function $f$: (briefly $c\tilde{g}s$-continuous)

Let $f: (X, \tau) \to (Y, \sigma)$ be a totally $\tilde{g}s$-continuous function and $Y$ is a $T_1$-space. If $A$ is a non-empty $\tilde{g}s$-connected subset of $X$, then $f(A)$ is a single point.

Definition 3.8. Let $(X, \tau)$ be a topological space. Then the set of all points $y$ in $X$ such that $x$ and $y$ cannot be separated by a $\tilde{g}s$-separation of $X$ is said to be the quasi $\tilde{g}s$-component of $X$.

Theorem 3.7. Let $f: (X, \tau) \to (Y, \sigma)$ be a totally $\tilde{g}s$-continuous function from a topological space $(X, \tau)$ into a $T_1$-space $Y$. Then $f$ is constant on each quasi $\tilde{g}s$-component of $X$.

Proof. Let $x$ and $y$ be two points of $X$ that lie in the same quasi-$\tilde{g}s$-component of $X$. Assume that $f(x) = \alpha \neq \beta = f(y)$. Since $Y$ is $T_1$, $\{\alpha\}$ is closed in $Y$ and so $Y-\{\alpha\}$ is an open set. Since $f$ is totally $\tilde{g}s$-continuous, therefore $f^{-1}(\{\alpha\})$ and $f^{-1}(Y-\{\alpha\})$ are disjoint $\tilde{g}s$-clopen subsets of $X$. Further, $x \in f^{-1}(\{\alpha\})$ and $y \in f^{-1}(Y-\{\alpha\})$, which is a contradiction in view of the fact that $y$ belongs to the quasi $\tilde{g}s$-component of $x$ and hence $y$ must belong to every $\tilde{g}s$-open set containing $x$. □

4. CONTRA-$\tilde{g}$-SEMI-CONTINUOUS

We introduce the following definition

Definition 4.9. A function $f: (X, \tau) \to (Y, \sigma)$ is called contra-$\tilde{g}$-semi-continuous (briefly $c\tilde{g}s$-continuous) if $f^{-1}(V)$ is $\tilde{g}s$-open in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.

It is clear that every strongly $\tilde{g}s$-continuous function is $c\tilde{g}s$-continuous. But the reverse implication is not always true as shown in the following example.

Example 4.3. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Then the identity function $f: (X, \tau) \to (Y, \sigma)$ is $c\tilde{g}s$-continuous but it is not strongly $\tilde{g}s$-continuous.

Definition 4.10. Let $A$ be a subset of a topological space $(X, \tau)$. The set $\bigcap \{U \in \tau \mid A \subseteq U\}$ is called the Kernel of $A$ [9] and is denoted by $\ker(A)$.

Lemma 4.1. [4] The following properties hold for subsets $A, B$ of a space $X$:

1. $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$;
2. $A \subseteq \ker(A)$ and $A = \ker(A)$ if $A$ is open in $X$;
3. If $A \subseteq B$, then $\ker(A) \subseteq \ker(B)$.

Theorem 4.8. The following are equivalent for a function $f: (X, \tau) \to (Y, \sigma)$: (briefly $\tilde{g}s$-continuous)

1. $f$ is $c\tilde{g}s$-continuous;
2. For every closed subset $F$ of $Y$, $f^{-1}(F) \in \tilde{g}sF(X, \tau)$;
3. For each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \tilde{g}sF(X, \tau)$ such that $f(U) \subseteq F$;
4. $f(\tilde{g}scl(A)) \subseteq \ker(f(A))$ for every subset $A$ of $X$;
5. $\tilde{g}scl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset $B$ of $Y$. 

Example 4.4. Let $X = Y = \{1, 2, 3\}$, $\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $\sigma = \{\emptyset, \{1\}, Y\}$. Then the identity function $f: (X, \tau) \to (Y, \sigma)$ is $c\tilde{g}s$-continuous.
Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2): Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in GS(X,x)$ such that $f(U_x) \subseteq F$. Therefore, we obtain

$$f^{-1}(F) = \bigcup \{U_x | x \in f^{-1}(F)\} \in GS(X,\tau) \ [15].$$

(2) $\Rightarrow$ (4): Let $A$ be any subset of $X$. Suppose that $y \notin ker(f(A))$. Then by Lemma 4.1 there exists $F \in C(X,y)$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $\bar{\text{gscl}}(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(\bar{\text{gscl}}(A)) \cap F = \emptyset$ and $y \notin f(\bar{\text{gscl}}(A))$. This implies that $f(\bar{\text{gscl}}(A)) \subseteq ker(f(A))$.

(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$. By (4) and Lemma 4.1, we have $f(\bar{\text{gscl}}(f^{-1}(B))) \subseteq ker(f(f^{-1}(B))) \subseteq ker(B)$ and $\bar{\text{gscl}}(f^{-1}(B)) \subseteq f^{-1}(ker(B))$.

(5) $\Rightarrow$ (1): Let $V$ be any open set of $Y$. Then by Lemma 4.1 we have $\bar{\text{gscl}}(f^{-1}(V)) \subseteq f^{-1}(ker(V))$. Since union of $\bar{\text{gscl}}$-open sets is $\bar{\text{gscl}}$-open [12], $f^{-1}(V)$ is $\bar{\text{gscl}}$-open and therefore $f$ is $\bar{\text{gscl}}$-continuous.

Corollary 4.1. Every contra $\alpha$-continuous (resp. contra-continuous) function is $\bar{\text{gs}}$-continuous.

Theorem 4.9. Every contra semi-continuous function is $\bar{\text{gs}}$-continuous.

Proof. The proof follows from the definitions.

Remark 4.3. Contra $\bar{\text{gs}}$-continuous need not be contra semi-continuous in general as shown in the following example.

Example 4.4. Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\bar{\text{gs}}$-continuous. However, $f$ is not contra-semi continuous, since for the closed set $F = \{a\}, f^{-1}(F)$ is $\bar{\text{gs}}$-open but not semi-open in $(X, \tau)$.

Theorem 4.10. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following are equivalent:

1. The function $f$ is $\bar{\text{gs}}$-continuous;
2. For each point $x$ in $X$ and each open set $V$ in $(Y, \sigma)$ with $f(x) \in V$, there exists a $\bar{\text{gs}}$-open set $U$ in $(X, \tau)$ such that $x \in U$, $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let $f(x) \in V$. Since $f$ is $\bar{\text{gs}}$-continuous, we have $x \in f^{-1}(V) \in GS(X,\tau)$. Let $U = f^{-1}(V)$. Then $x \in V$ and $f(U) \subseteq V$.

(2) $\Rightarrow$ (1): Let $V$ be an open set in $(Y, \sigma)$ and let $x \in f^{-1}(V)$. Then, $f(x) \in V$ and thus there exists a $\bar{\text{gs}}$-open set $U_x$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Now, $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$. Since union of $\bar{\text{gs}}$-open sets is $\bar{\text{gs}}$-open [12], $f^{-1}(V)$ is $\bar{\text{gs}}$-open in $(X, \tau)$ and therefore $f$ is $\bar{\text{gs}}$-continuous.

Theorem 4.11. If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\bar{\text{gs}}$-continuous and $Y$ is regular, then $f$ is $\bar{\text{gs}}$-continuous.

Proof. Let $x$ be an arbitrary point of $X$ and $V$ an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $W$ in $Y$ containing $f(x)$ such that $cl(W) \subseteq V$. Since $f$ is $\bar{\text{gs}}$-continuous, so by Theorem 4.8 there exists $U \in GS(X,x)$ such that $f(U) \subseteq cl(W)$. Then $f(U) \subseteq cl(W) \subseteq V$. Hence, by theorem 4.10 $f$ is $\bar{\text{gs}}$-continuous.
**Theorem 4.12.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a function and \( g : X \to X \times Y \) the graph function, given by \( g(x) = (x, f(x)) \) for every \( x \in X \). Then \( f \) is \( cgs \)-continuous if and only if \( g \) is \( cgs \)-continuous.

**Proof.** Let \( x \in X \) and let \( W \) be a closed subset of \( X \times Y \) containing \( g(x) \). Then \( W \cap \{(x) \times Y \} \) is closed in \( \{x \} \times Y \) containing \( g(x) \). Also \( \{x \} \times Y \) is homeomorphic to \( Y \). Hence \( \{y \in Y | (x, y) \in W \} \) is a closed subset of \( Y \). Since \( f \) is \( cgs \)-continuous, \( \bigcup \{f^{-1}(y) | (x, y) \in W \} \) is a \( gs \)-open subset of \( X \). Further, \( x \in \bigcup \{f^{-1}(y) | (x, y) \in W \} \). Hence \( g^{-1}(W) \) is \( gs \)-open. Then \( g \) is \( cgs \)-continuous.

Conversely, let \( F \) be a closed subset of \( Y \). Then \( X \times F \) is a closed subset of \( X \times Y \). Since \( g \) is \( cgs \)-continuous, \( g^{-1}(X \times F) \) is a \( gs \)-open subset of \( X \). Also, \( g^{-1}(X \times F) = f^{-1}(F) \). Hence \( f \) is \( cgs \)-continuous. \( \square \)

**Theorem 4.13.** If \( X \) is a topological space and for each pair of distinct points \( x_1 \) and \( x_2 \) in \( X \) there exists a map \( f \) into a Urysohn topological space \( Y \) such that \( f(x_1) \neq f(x_2) \) and \( f \) is \( cgs \)-continuous at \( x_1 \) and \( x_2 \), then \( X \) is \( gs-T_2 \).

**Proof.** Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \). Then by hypothesis there is a Urysohn space \( Y \) and a function \( f : (X, \tau) \to (Y, \sigma) \), which satisfies the conditions of the theorem. Let \( y_i = f(x_i) \) for \( i = 1, 2 \). Then \( y_1 \neq y_2 \). Since \( Y \) is Urysohn, there exist open neighborhoods \( U_{y_1} \) and \( U_{y_2} \) of \( y_1 \) and \( y_2 \) respectively in \( Y \) such that \( \text{cl}(U_{y_1}) \cap \text{cl}(U_{y_2}) = \emptyset \). Since \( f \) is \( cgs \)-continuous at \( x_i \), there exists a \( gs \)-open neighborhoods \( V_{x_i} \) of \( x_i \) in \( X \) such that \( f(V_{x_1}) \subset \text{cl}(U_{y_1}) \) for \( i = 1, 2 \). Hence we get \( W = V_{x_1} \cap V_{x_2} = \emptyset \) because \( f(V_{x_1}) \cap f(V_{x_2}) = \emptyset \). Then \( X \) is \( gs-T_2 \). \( \square \)

**Corollary 4.2.** If \( f \) is a \( cgs \)-continuous injection of a topological space \( X \) into a Urysohn space \( Y \), then \( X \) is \( gs-T_2 \).

**Proof.** For each pair of distinct points \( x_1 \) and \( x_2 \) in \( X \), \( f \) is a \( cgs \)-continuous function of \( X \) into Urysohn space \( Y \) such that \( f(x_1) \neq f(x_2) \) because \( f \) is injective. Hence by Theorem 4.13, \( X \) is \( gs-T_2 \). \( \square \)

**Corollary 4.3.** If \( f \) is a \( cgs \)-continuous injection of a topological space \( X \) into Ultra Hansdorff space \( Y \), then \( X \) is \( gs-T_2 \).

**Proof.** Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \). Then since \( f \) is injective and \( Y \) is Ultra Hansdorff \( f(x_1) \neq f(x_2) \) and there exist \( V_1, V_2 \subset \text{CO}(Y, \sigma) \) such that \( f(x_1) \in V_1, f(x_2) \in V_2 \) and \( V_1 \cap V_2 = \emptyset \). Then \( x_1 \in f^{-1}(V_1) \subset \bar{S}(X, \tau) \) for \( i = 1, 2 \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \). Thus, \( X \) is \( gs-T_2 \). \( \square \)

**Lemma 4.2.** The product of two \( gs \)-open sets is \( gs \)-open [11].

**Proof.** Let \( A \in \bar{GS}(X, \tau), B \in \bar{GS}(Y, \sigma) \) and \( W = A \times B \subset X \times Y \). Let \( F \subset W \) be a \#gs-closed set in \( X \times Y \), then there exist two \#gs-closed sets \( F_1 \subset A, F_2 \subset B \) and so, \( F_1 \subset \text{int}(A), F_2 \subset \text{int}(B) \). Hence \( F_1 \times F_2 \subset A \times B \) and \( F_1 \times F_2 \subset \text{int}(AB) = \text{int}(A \times B) \). Therefore, \( A \times B \in \bar{GS}(X \times Y, \tau \times \sigma) \). \( \square \)

**Lemma 4.3.** Let \( A \subset Y \subset X, Y \in \bar{GS}(X, \tau) \) and \( A \in \bar{GS}(Y, \sigma) \). Then \( A \in \bar{GS}(X, \tau) \) [15].
Theorem 4.14. Let $f_1: X_1 \to Y$ and $f_2 : X_2 \to Y$ be two functions where $Y$ is a Urysohn space and $f_1$ and $f_2$ are $\tilde{g}s$-continuous. Then $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is $\tilde{g}s$-closed in the product space $X_1 \times X_2$.

Proof. Let $A$ denote the set $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$. In order to show that $A$ is $\tilde{g}s$-closed, we show that $(X_1 \times X_2) - A$ is $\tilde{g}s$-open. Let $(x_1, x_2) \notin A$. Then $f_1(x_1) \neq f_2(x_2)$. Since $Y$ is Urysohn, there exist open $V_1$ and $V_2$ of $f_1(x_1)$ and $f_2(x_2)$ such that $\text{cl}(V_1) \cap \text{cl}(V_2) = \emptyset$. Since $f_i$ $(i = 1, 2)$ is $\tilde{g}s$-continuous, $f_i^{-1}(\text{cl}(V_i))$ is a $\tilde{g}s$-open set containing $x_i$ in $X_i$ $(i = 1, 2)$. Hence by Lemma 4.2, $f_i^{-1}(\text{cl}(V_i)) \times f_j^{-1}(\text{cl}(V_j))$ is $\tilde{g}s$-open. Further $(x_1, x_2) \in f_1^{-1}(\text{cl}(V_1)) \times f_2^{-1}(\text{cl}(V_2)) \subset ((X_1 \times X_2) - A)$. It follows that $X_1 \times X_2 - A$ is $\tilde{g}s$-open. Thus, $A$ is $\tilde{g}s$-closed in the product space $X_1 \times X_2$.

Corollary 4.4. If $f : (X, \tau) \to (Y, \sigma)$ is $\tilde{g}s$-continuous and $Y$ is a Urysohn space, then $A = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is $\tilde{g}s$-closed in the product space $X_1 \times X_2$.

Theorem 4.15. If $f : (X, \tau) \to (Y, \sigma)$ is a contra-$\tilde{g}s$-continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is a continuous function, then $(g \circ f) : (X, \tau) \to (Z, \eta)$ is $\tilde{g}s$-continuous.

Theorem 4.16. Let $f : (X, \tau) \to (Y, \sigma)$ be surjective $\tilde{g}$s-irresolute and $\tilde{g}s$-open and $g : (Y, \sigma) \to (Z, \eta)$ be any function. Then $(g \circ f) : (X, \tau) \to (Z, \eta)$ is $\tilde{g}s$-continuous if and only if $g$ is $\tilde{g}s$-continuous.

Proof. The “if” part is easy to prove. To prove the “only if” part, let $(g \circ f) : (X, \tau) \to (Z, \eta)$ be $\tilde{g}s$-continuous. Let $F$ be a closed subset of $Z$. Then $(g \circ f)^{-1}(F)$ is a $\tilde{g}s$-open subset of $X$. That is $f^{-1}(g^{-1}(F))$ is $\tilde{g}s$-open. Since $f$ is $\tilde{g}s$-open, $f(f^{-1}(g^{-1}(F)))$ is a $\tilde{g}s$-open subset of $Y$. So $g^{-1}(F)$ is $\tilde{g}s$-open in $Y$. Hence $g$ is $\tilde{g}s$-continuous.

Theorem 4.17. Let $\{X_i \mid i \in \Lambda\}$ be any family of topological spaces. If $f : X \to \Pi_{i \in \Lambda} X_i$ is a $\tilde{g}s$-continuous function. Then $\pi_i \circ f : X \to X_i$ is $\tilde{g}s$-continuous for each $i \in \Lambda$, where $\pi_i$ is the projection of $\Pi_{i \in \Lambda} X_i$ onto $X_i$.

Definition 4.11. The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\tilde{g}s$-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \tilde{GS}(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.4. The graph $f : (X, \tau) \to (Y, \sigma)$ is $\tilde{g}s$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exists $U \in \tilde{GS}(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

Proof. The proof follows from the definition.

Theorem 4.18. If $f : (X, \tau) \to (Y, \sigma) \tilde{g}$s-continuous and $Y$ is Urysohn, then $G(f)$ is contra-$\tilde{g}s$-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets $V, W$ such that $f(x) \in V, y \in W$ and $\text{cl}(U) \cap \text{cl}(W) = \emptyset$. Since $f$ is $\tilde{g}s$-continuous, there exists $U \in \tilde{GS}(X, x)$ such that $f(U) \subset \text{cl}(V)$. Therefore, we obtain $f(U) \cap \text{cl}(W) = \emptyset$. This shows that $G(f)$ is contra-$\tilde{g}s$-closed.
Theorem 4.19. A $\tilde{g}$-continuous image of a $\tilde{g}$-connected space is connected.

Proof. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a contra-$\tilde{g}$s-continuous function of a $\tilde{g}$s-connected space $X$ onto a topological space $Y$. Let $Y$ be disconnected. Let $A$ and $B$ form a disconnected of $Y$. Then $A$ and $B$ are clopen and $Y = A \cup B$ where $A \cap B = \emptyset$. Since $f$ is a contra-$\tilde{g}$s-continuous function $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty $\tilde{g}$s-open sets in $X$. Also $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Hence $X$ is non $\tilde{g}$s-connected which is a contradiction. Therefore $Y$ is connected. \hfill $\square$

Theorem 4.20. Let $X$ be $\tilde{g}$s-connected and $Y$ a $T_1$ space. If $f$ is $\tilde{g}$s-continuous, then $f$ is constant.

Proof. Since $Y$ is $T_1$ space, $\wedge = \{f^{-1}(\{y\}) : y \in Y\}$ is a disjoint $\tilde{g}$s-open partition of $X$. If $|\wedge| \geq 2$, then $X$ is the union of two non-empty $\tilde{g}$s-open sets. Since $X$ is $\tilde{g}$s-connected, $|\wedge| = 1$. Hence, $f$ is constant. \hfill $\square$

Definition 4.12. A topological space $(X, \tau)$ is said to be $\tilde{g}$s-normal if each pair of non-empty disjoint closed sets can be separated by disjoint $\tilde{g}$s-open sets.

Definition 4.13. [14] A topological space $(X, \tau)$ is said to be ultra normal if each pair of non-empty disjoint closed sets can be separated by disjoint $\tilde{g}$s-open sets.

Theorem 4.21. If $f : (X, \tau) \rightarrow (y, \sigma)$ is a $c\tilde{g}$s-continuous, closed injection and $Y$ is ultra-normal, then $X$ is $\tilde{g}$s-normal.

Proof. Let $F_1$ and $F_2$ be a disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra normal $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets $V_1$ and $V_2$ respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in GS(X, \tau)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, $X$ is $\tilde{g}$s-normal. \hfill $\square$

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Department of Mathematics
Kongu Engineering College
Perundurai, Erode-638 052
Tamil Nadu, India
E-mail address: nraje@topology@yahoo.co.in

Department of Mathematics
Canakkale Onsekiz Mart University
Terzioglu Campus
17020 Canakkale, Turkey
E-mail address: eekici@comu.edu.tr