CARPATHIAN J. MATH.
22 (2006), No. 1 - 2, 153 - 161

Controlling chaos of a dynamical system with feedback control

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ABSTRACT. The present work is devoted to control chaotic behavior of a three–dimensional differential system introduced in [8]. We stabilize the chaotic dynamics of the system to the unstable equilibrium points. The Lyapunov function method is employed. Using a linear controller, the system is controlled to a stable state. Numerical illustrations are presented to show the control process.

1. INTRODUCTION

Controlling chaos is a topic in nonlinear dynamics which attracted a great deal of work in the last twenty years. Even though there are more definitions of the chaos [4], a distinctive characteristic of the chaotic behavior of a dynamical system is its sensitive dependence on the initial conditions. Generally, chaos is believed to be harmful because in many cases it can lead to disasters. Chaotical behavior is observed in practical applications of many fields, from engineering to biology and economics. Chaos can be suppressed using linear or nonlinear feedback methods [1], [2], [3], [5], [6].

The present work is organized as follows. In Section 1 we record some basic details of the system under study. Section 2 describes the methods to control chaos to the unstable fixed points, while Section 3 presents a linear feedback control of the system. Numerical illustrations are presented in all the Sections.

2. DESCRIPTION OF THE SYSTEM

The three-dimensional differential system which will be controlled in order to suppress chaos is [8]:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - axz \\ \dot{z} &= xy - bz \end{aligned} \tag{2.1}$$

where $a, b, c \in \mathbb{R}$, and $a \neq 0$. Compared with the $L\ddot{u}$ system introduced in [11], in the nonlinear terms, system (2.1) allows a larger possibility in choosing the parameters and consequently a more complex dynamics. The orbit presented in Fig.1 is a chaotic orbit. The Lyapunov exponents are $\lambda_1 = 0.37,\ \lambda_2 = 0.00$ and $\lambda_3 = -3.07$.

Received: 10.02.2006; In revised form: 25.04.2006; Accepted: 01.11.2006
2000 *Mathematics Subject Classification:* 93C10, 93B52.
Key words and phrases: *Controlling chaos, linear feedback control.*

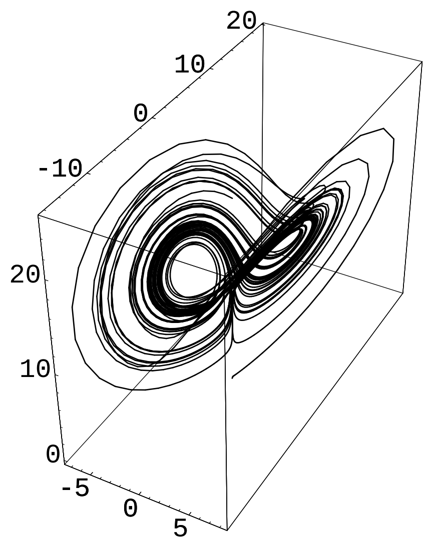

FIGURE 1. Chaotic orbit of the system (2.1), corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$ and initial values $(0.1, -0.3, 0.2)$.

Some basic properties of the system under study are presented in the following. More details can be found in [7],[8],[9]. Denote by X the associated vector field, and by Φ_t its flow. The divergence of the vector field is $\mathrm{div}(X) = -a - b$. Hence, for a, b such that $a + b > 0$, the system is dissipative, with an exponential contraction rate. That is, a volume element V_0 is contracted to $V_t = \Phi_t(V_0) = V_0 e^{(-a-b)t}$. This means that each volume shrinks to zero at an exponential rate, as t tends to infinity. Therefore, all system orbits are ultimately confined to a specific limit set of zero volume, and the system asymptotic motion settles onto an attractor [10]. The system (2.1) is conservative if and only if $a + b = 0$.

If $\frac{b}{a}(c-a) > 0$ the system (2.1) has three isolated equilibria:

$$O(0,0,0), E_1\left(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\right),$$
$$E_2\left(-\sqrt{\frac{b}{a}(c-a)}, -\sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\right)$$

and for $b \neq 0$, $\frac{b}{a}(c-a) \leq 0$ it has only one isolated equilibrium $O(0,0,0)$.

Proposition 2.1. *For $b \neq 0$ the following statements are true:*
a) If $(a > 0, b > 0, c \leq a)$, then $O(0,0,0)$ is asymptotically stable ,
b) If $(b < 0)$ or $(a < 0)$ or $(a > 0, c > a)$, then $O(0,0,0)$ is unstable.

Proposition 2.2. *The equilibrium point* $E_1\left(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\right)$ *is asymptotically stable if and only if* $\big(a+b>0, ab(c-a)>0, b(2a^2+bc-ac)>0\big)$.

Proposition 2.3. *The condition* $\frac{b}{a}(c-a)>0$ *along with that which ensures that the characteristic equation has roots with zero real parts is equivalent to* $(a,b,c)\in\Omega$, *where* $\Omega = \left\{(a,b,c)\in\mathbb{R}^3 \mid b>0, a>b,\ \ 2a^2+bc=ac\right\}$. *In this case the eigenvalues are* $\lambda_1 = \dfrac{2a^2-2ac}{c}$, $\lambda_{2,3} = \pm i\sqrt{ac-2a^2}$.

The system (2.1) displays a Hopf bifurcation at the point E_1.

Proposition 2.4. *If* $b = b_s := \dfrac{ac-2a^2}{c}$, *the characteristic equation has a negative solution* $\lambda_1 = \dfrac{2a^2-2ac}{c} < 0$, *as well as a pair of purely imaginary roots* $\lambda_{2,3} = \pm i\sqrt{ac-2a^2}$ *such that* $Re(\lambda_b'(b_s)) \neq 0$. *Therefore, the system* (2.1) *displays a Hopf bifurcation at the point* E_1.

Because the system is invariant under the transformation $(x,y,z)\rightarrow(-x,-y,z)$, one only needs to consider the stability type and bifurcation process of the equilibrium point E_1.

Proposition 2.5. *If* $c=3a$ *and* $(a,b,c)\in\Omega$ *the equilibrium point* $E_1\left(\sqrt{\frac{2a}{3}}, \sqrt{\frac{2a}{3}}, 2\right)$ *of the system* (2.1) *is unstable and the Hopf bifurcation is subcritical for any* $a>0$.

In the following we consider that the parameters of the system fulfill the conditions: $a,b,c>0$, $c>a$ and $2a^2+bc-ac<0$. By the above Propositions 2.1 and 2.2 we get that the equilibrium points O, E_1 and E_2 are unstable.

3. CONTROLLING CHAOS TO UNSTABLE FIXED POINTS

We introduce the conventional feedback linear control to drag the chaotic trajectory $(x(t),y(t),z(t))$ to a desired unstable equilibrium point (x_0,y_0,z_0). Assume that controlled system (2.1) is given by:

$$\begin{aligned} \dot{x} &= a(y-x)-u_1 \\ \dot{y} &= (c-a)x-axz-u_2 \\ \dot{z} &= xy-bz-u_3 \end{aligned} \tag{3.2}$$

where u_1, u_2 and u_3 are external laws of input. Since for applications is more desirable a simple control, consider here:
$u_1 = k_1(x-x_0), u_2 = k_2(y-y_0)$ and $u_3 = k_3(z-z_0)$. Therefore, system (3.2) leads to:

$$\begin{aligned} \dot{x} &= a(y-x)-k_1(x-x_0) \\ \dot{y} &= (c-a)x-axz-k_2(y-y_0) \\ \dot{x} &= xy-bz-k_3(z-z_0). \end{aligned} \tag{3.3}$$

The controlled system (3.3) has one equilibrium point (x_0, y_0, z_0). Linearizing (3.3) about this equilibrium point, one get:

$$\begin{aligned} \dot{X} &= -(k_1 + a)X + aY \\ \dot{Y} &= (c - a - az_0)X - k_2 Y - ax_0 Z \\ \dot{Z} &= Xy_0 + x_0 Y - (b + k_3)Z. \end{aligned} \tag{3.4}$$

Consider now the first unstable point $(x_0, y_0, z_0) = (0, 0, 0)$. Then system (3.4) leads to:

$$\begin{aligned} \dot{X} &= -(k_1 + a)X + aY \\ \dot{Y} &= (c - a)X - k_2 Y \\ \dot{Z} &= -(b + k_3)Z. \end{aligned} \tag{3.5}$$

In order to prove the asymptotic stability of the solution $(0, 0, 0)$ for (3.5), we use the Lyapunov function method. Define the Lyapunov function for (3.5) by:

$$V(X, Y, Z) = \frac{\frac{1}{a}X^2 + \frac{1}{c-a}Y^2 + Z^2}{2}. \tag{3.6}$$

The function V satisfies:
i) $V(0, 0, 0) = 0$
ii) $V(X, Y, Z) > 0$ for X, Y, Z in the neighborhood of the origin, therefore $V(X, Y, Z)$ is positive definite. In addition, we have that the time orbital derivative of the function V is:

$$\begin{aligned} \frac{dV}{dt} &= \frac{1}{a}X\left(-(k_1+a)X+aY\right)+\frac{1}{c-a}Y\left((c-a)X-k_2Y\right)+\left(-(b+k_3)Z\right)Z = \\ &- \frac{1}{a}X^2k_1 + 2XY - \frac{1}{c-a}Y^2k_2 - Z^2b - Z^2k_3 - X^2 = \\ &- \left(\sqrt{\frac{k_1}{a}}X - \sqrt{\frac{k_2}{c-a}}Y\right)^2 - X^2 - Z^2\left(b + k_3\right) \end{aligned}$$

Therefore the derivative $\dfrac{dV}{dt} < 0$ whenever,

$$k_1 k_2 = a\left(c - a\right) \quad \text{and} \quad b + k_3 > 0, \tag{3.7}$$

i.e $\dfrac{dV}{dt}$ is negative definite under condition (3.7). Consequently, we have the theorem:

Theorem 3.1. *If the feedbacks k_1, k_2, k_3 satisfy $k_1 k_2 = a\left(c - a\right)$ and $b + k_3 > 0$ then the equilibrium solution $(0, 0, 0)$ of the controlled system (3.3) is asymptotically stable.*

For the second unstable point $E_1\left(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\right)$, system (3.4) leads to:

$$\begin{aligned}\dot{X} &= -(k_1+a)X + aY\\ \dot{Y} &= -k_2 Y - \sqrt{ab(c-a)}Z\\ \dot{Z} &= X\sqrt{\frac{b}{a}(c-a)} + Y\sqrt{\frac{b}{a}(c-a)} - (b+k_3)Z.\end{aligned} \tag{3.8}$$

We chose the Lyapunov function for the system (3.8) given by:

$$V(X,Y,Z) = \frac{\frac{2}{a}X^2 + \frac{2}{c-a}Y^2 + Z^2}{2}. \tag{3.9}$$

The function V satisfies:
i) $V(0,0,0)=0$
ii) $V(X,Y,Z)>0$ for X,Y,Z in the neighborhood of the origin, therefore $V(X,Y,Z)$ is positive definite. In addition, we have that the time orbital derivative of the function V is:

$$\frac{dV}{dt} = -\left(\sqrt{\frac{k_1}{a}}X - \sqrt{\frac{k_2}{c-a}}Y\right)^2 - \left(\sqrt{\frac{k_1}{a}}X - Z\sqrt{b}\right)^2 - \left(\sqrt{\frac{k_2}{c-a}}Y + Z\sqrt{k_3}\right)^2 - 2X^2.$$

Therefore the derivative $\frac{dV}{dt} < 0$ whenever,

$$k_1 = \frac{1}{4}\left(c-a\right), \qquad k_2 = 4a, \qquad k_3 = \frac{1}{16}b\frac{\left(-3a+c\right)^2}{a^2} \tag{3.10}$$

i.e $\frac{dV}{dt}$ is negative definite under condition (3.10). Therefore we get the second theorem:

Theorem 3.2. *If the feedbacks k_1,k_2,k_3 satisfy $k_1 = \frac{1}{4}(c-a)$, $k_2 = 4a$ and $k_3 = \frac{1}{16}b\frac{(-3a+c)^2}{a^2}$, then the equilibrium solution $E_1\left(\sqrt{\frac{b}{a}(c-a)}, \sqrt{\frac{b}{a}(c-a)}, \frac{c-a}{a}\right)$ of the controlled system (3.3) is asymptotically stable.*

4. Linear feedback control

Consider a simple controller $u_1(t) = -kx$. Adding it to the second equation of the system (2.1), it leads to:

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= (c-a)x - axz - kx\\ \dot{z} &= xy - bz\end{aligned} \tag{4.11}$$

The first equilibrium point of the controlled system (4.11) is the origin $O(0,0,0)$. The Jacobian matrix associated to this system is:

$$\begin{pmatrix} -a & a & 0\\ c-a-k & 0 & 0\\ 0 & 0 & -b\end{pmatrix} \tag{4.12}$$

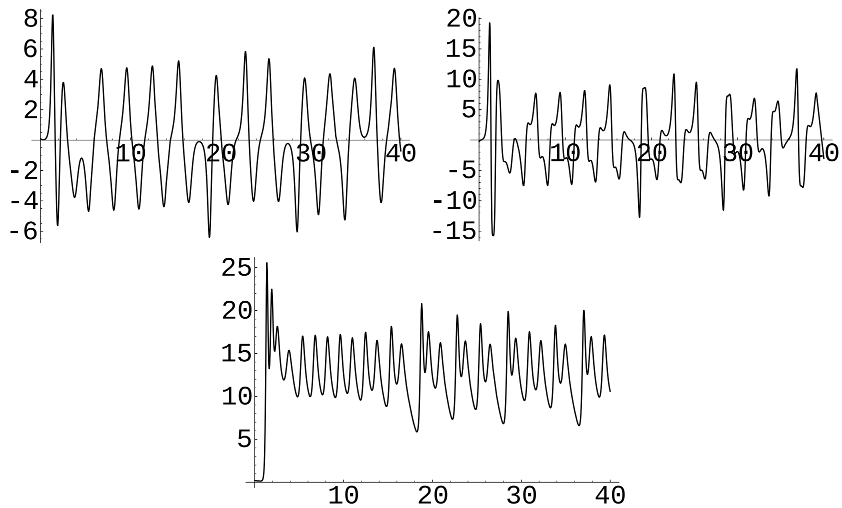

FIGURE 2. The time series $x(t)$ (left), $y(t)$ (right) and $z(t)$ (bellow) of the uncontrolled system (2.1), corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$ and initial values $(0.1, -0.3, 0.2)$.

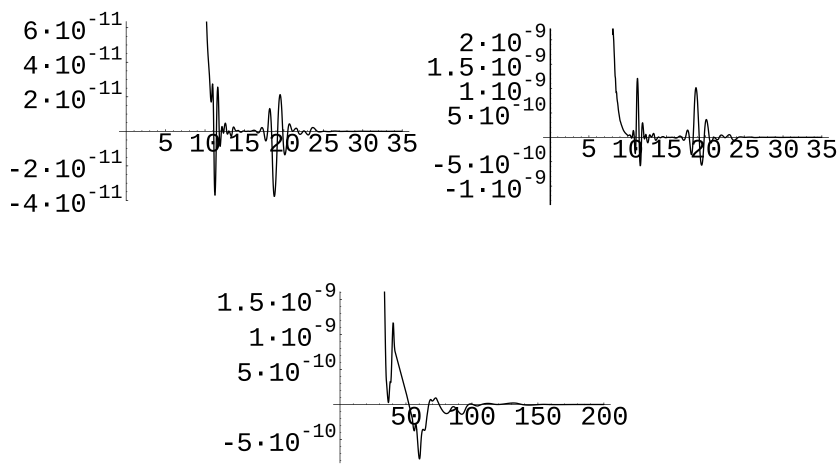

FIGURE 3. The time series $x(t)$ (left), $y(t)$ (right) and $z(t)$ (bellow) of the controlled system (3.3) to the first unstable point $O(0, 0, 0)$, corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$, the initial values $(0.1, -0.3, 0.2)$ and $k_1 = 1, k_2 = a(c - a), k_3 = 0$.

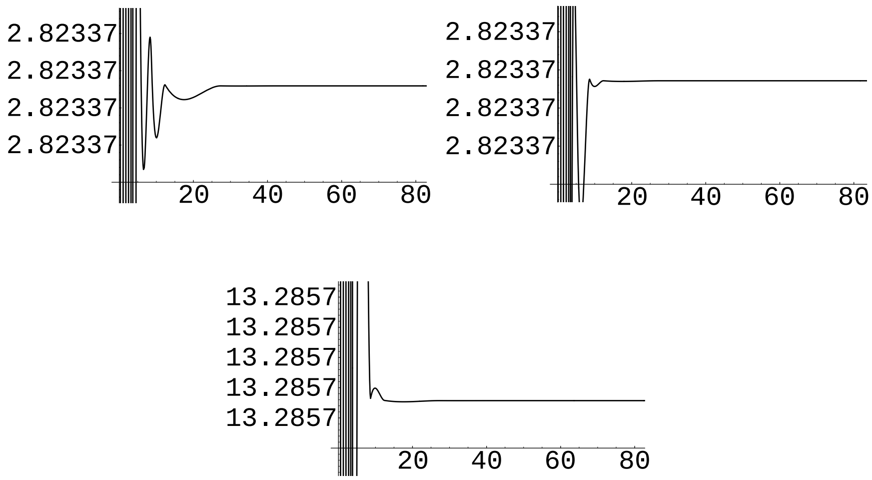

FIGURE 4. The time series $x(t)$ (left), $y(t)$ (right) and $z(t)$ (bellow) of the controlled system (3.3) to the second unstable point E_1, corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$, the initial values $(0.1, -0.3, 0.2)$ and $k_1 = \frac{1}{4}(c-a)$, $k_2 = 4a$, $k_3 = \frac{1}{16}b\frac{(-3a+c)^2}{a^2}$.

with the eigenvalues: $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{-3a^2 - 4ak + 4ac}$, $\lambda_3 = -b$.

Then, if $c - a < k \le c - \frac{3}{4}a$ the three eigenvalues are negative real numbers and if $k > c - \frac{3}{4}a$ the eigenvalues $\lambda_{1,2}$ are complex but with negative real parts, such that, the steady state $O(0,0,0)$ is asymptotically stable whenever $c - a < k$. But in this case the system does not have another fixed point, so the system is completely controlled. Numerical illustrations can be seen in Figs. 5, 6.

5. CONCLUSIONS

In this paper we present methods to suppress chaos in a dynamical system. First, applying Lyapunov function we guide the chaotic trajectories to the unstable fixed points. Simple controls are used. Such controls are desirable in practical applications. Then, using a linear control, the system is controlled to a stable state. Analytical results are accompanied by numerical illustrations.

6. ACKNOWLEDGEMENTS

This work was supported through a research fellowship offered by the Romanian Government by CNBSS and partially through a European Community

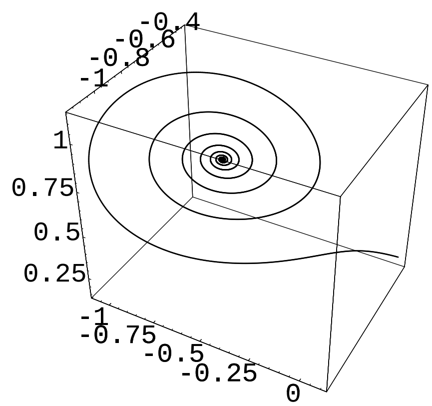

FIGURE 5. Controlling the chaotic trajectory of the system (2.1) by the controller $u_1(t) = -kx$, with $k = c$ corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$ and the initial values $(0.1, -0.3, 0.2)$.

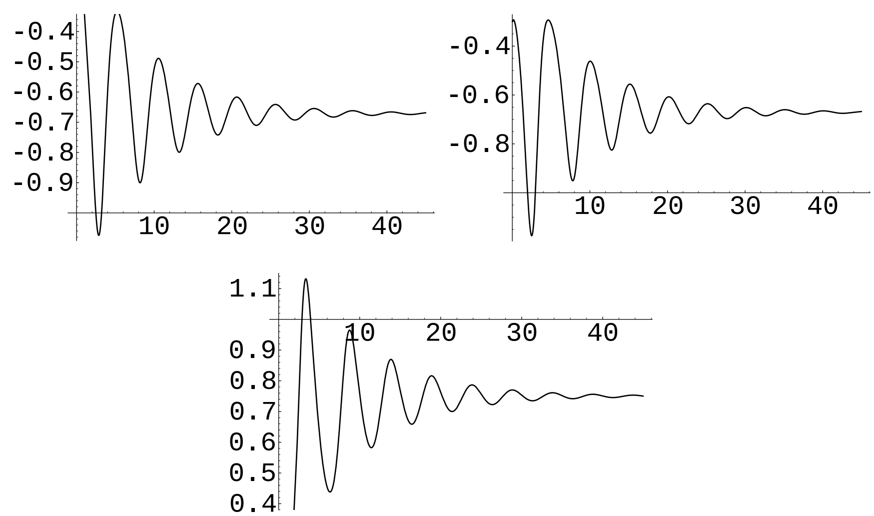

FIGURE 6. The time series $x(t)$ (left), $y(t)$ (right) and $z(t)$ (bellow) of the controlled system (4.11), corresponding to the parameters $(a, b, c) = (2.1, 0.6, 30)$, the initial values $(0.1, -0.3, 0.2)$ and $k = c$.

Marie Curie Fellowship, in the framework of the CTS, contract number HPMT-CT-2001-00278.

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