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Viscosity approximation methods for nonexpansive mapping in Banach spaces

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ABSTRACT. Let C be a closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. The initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (0,1) and $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in [0,1], the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty,\ \alpha_n \to 0$;
(ii) $(1+\beta_n)\gamma_n - 2\beta_n > a,$ for some $a \in [0,1)$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)Tx_n \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T which solves some variational inequality.

1. INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space, C a nonempty closed convex subset of E , and $T : C \to C$ a mapping. Recall that a self mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad x, y \in C.$$

We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f | f : C \to C \text{ a contraction}\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C. Also, recall that T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a real number $t \in (0,1)$ and a contraction $f \in \Pi_C$. We define a mapping $T_t x = tf(x) + (1-t)Tx, \quad x \in C$. It is obvious that T_t is a contraction on C. In fact, for $x, y \in C$,

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we obtain

$$\begin{aligned}\|T_tx - T_ty\| &\le \|t(f(x) - f(y)) + (1-t)(Tx - Ty)\| \\ &\le \alpha t\|x - y\| + (1-t)\|Tx - Ty\| \\ &\le \alpha t\|x - y\| + (1-t)\|x - y\| \\ &= (1 - t(1-\alpha))\|x - y\|.\end{aligned}$$

Let x_t be the unique fixed point of T_t. That is, x_t is the unique solution of the fixed point equation

(1.1) $\quad x_t = tf(x_t) + (1-t)Tx_t.$

A special case has been considered by Browder [1] in a Hilbert space as follows. Fix $u \in C$ and define a contraction S_t On C by

$$S_tx = tu + (1-t)Tx, \;\; x \in C.$$

We use z_t to denote the unique fixed point of S_t, which yields that

$$z_t = tu + (1-t)Tz_t.$$

In 1967, Browder [1] proves the following

Theorem B. *In a Hilbert space, as $t \to 0$, z_t converges strongly to a fixed point of T that is closet to u, that is, the nearest point projection of u onto $F(T)$.*

Also, in 1967, Halpern [3] firstly introduced the iteration scheme

(1.2) $$\begin{cases} x_0 = x \in C, \;\; arbitrarily \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \end{cases}$$

which is the special case of

(1.3) $$\begin{cases} x_0 = x \in C, \;\; arbitrarily \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n. \end{cases}$$

He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that, if the iteration scheme (1.2) converges to a fixed point of T, then these conditions must be satisfied. Ten years later, Lions [5] investigated the general case in Hilbert space under the conditions

$$\lim_{n\to\infty} \alpha_n = 0, \;\; \sum_{n=1}^{\infty} \alpha_n = \infty \;\; \text{and} \;\; \lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n+1})^2}{\alpha_{n+1}} = 0$$

on the parameters. However, Lions'conditions on the parameters were more restrictive and did not include the natural candidate $\{\alpha_n = \frac{1}{n}\}$. In 1980, Reich [6] gave the iteration scheme (1.2) in the case when E is uniformly smooth and $\alpha_n = n^{-\delta}$ with $0 < \delta < 1$.

In 1992, Wittmann [9] studied the iteration scheme (1.2) in the case when E is a Hilbert space and $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \;\; \sum_{n=1}^{\infty} \alpha_n = \infty \;\; and \;\; \sum_{n=1}^{\infty} \|\alpha_{n+1} - \alpha_n\| < \infty.$$

In 1994, Reich [6] obtained a strong convergence of the iterates (1.2) with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping.

On the other hand, for a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$. Xu [8] proposed the following two iterative process

$x_t = tf(x_t) + (1-t)Tx_t,$

$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n.$

He showed that $\{x_t\}$, $\{x_n\}$ converges strongly to a fixed point of T which solves some variational inequality in the Hilbert space and uniformly smooth space, respectively.

In this paper, we use viscosity approximation methods to study strong convergence theorems for nonexpansive mapping.

This paper introduce the composite iteration scheme as follows :

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)Tx_n \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in [0,1]. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, that $\{x_n\}$ defined by (1.4) converges to $Q(f)$, where $Q : \Pi_C \to F(T)$ is defined by (1.8).

On the other hand, the composite iterations introduced in this paper is a modified Ishikawa iteration. if $\gamma_n = 1$ in (1.4) this can be viewed as a modified Mann iteration

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n. \end{cases} \tag{1.5}$$

Another special case of (1.4) was considered by T. H. Kim and H. K. Xu [4]. They introduced the following iterative process:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.6}$$

where u is an arbitrary (but fixed) element in C and sequences α_n, β_n in (0,1). They proved that the sequence $\{x_n\}$ defined by (1.6) converges strongly to a fixed point of T.

It is our purpose in this paper to introduce this composite iteration scheme for approximating a fixed point of nonexpansive mappings by using viscosity methods in the framework of uniformly smooth Banach spaces. we establish the strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.4). The results improve and extend results of Kim and Xu [4], Wittmann [9], Xu [13], Xu [12] and some others.

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

(1.7) $$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (1.7) is attained uniformly for $(x, y) \in U \times U$.

We need the following definitions and lemmas for the proof of our main results.

Lemma 1.1. *A Banach space E is uniformly smooth if and only if the duality map J is single-valued and $norm$-to-$norm$ uniformly continuous on bounded sets of E.*

Lemma 1.2. *In a Banach space E, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \ x, y \in E$$

where $j(x+y) \in J(x+y)$.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny ([2], [7]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x+t(x-Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction,which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [10, 11]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \ \text{ for all } \ x \in C \ \text{ and } \ y \in D.$$

Lemma 1.3. (Xu [13]) *Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu+(1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : \Pi_C \to F(T)$ by*

(1.8) $$Qf = s - \lim_{t\to 0} x_t, \ f \in \Pi_C.$$

Then $Q(f)$ solves the variational inequality

(1.9) $$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \ \ f \in \Pi_C, p \in F(T).$$

In particular, if $f = u$ is a constant, then (1.9) is reduced to the sunny nonexpansive retract from C onto $F(T)$:

(1.10) $$\langle u - Qu, J(p - Qu)\rangle \leq 0, \ u \in C, \ p \in F(T).$$

Lemma 1.4. (Xu [10], [11]) *Let $\sum_{n=0}^{\infty} \{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum\limits_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ *converges to zero.*

2. Main Results

Theorem 2.1. *Let C be a closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. The initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ in $(0,1)$ and $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$, the following conditions are satisfied*

(i) $\sum\limits_{n=0}^{\infty} \alpha_n = \infty,\ \alpha_n \to 0$;

(ii) $(1+\beta_n)\gamma_n - 2\beta_n > a,$ *for some* $a \in [0,1)$;

(iii) $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum\limits_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n) T x_n \\ y_n = \beta_n x_n + (1-\beta_n) T z_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $Q(f)$, where $Q : \Pi_C \to F(T)$ is defined by (1.8) and $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C, p \in F(T). \tag{2.11}$$

In particular, if $f = u \in C$ is a constant, then (2.11) is reduced to the sunny nonexpansive retraction of Reich from C onto $F(T)$,

$$\langle Q(u) - u, J(Q(u)-p)\rangle \le 0, \ \ u \in C, p \in F(T). \tag{2.12}$$

Proof. First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, if we take a fixed point p of T, we have that

$$\|z_n - p\| \le \gamma_n \|x_n - p\| + (1-\gamma_n)\|Tx_n - p\| \le \|x_n - p\|. \tag{2.13}$$

It follows from (1.3) and (2.13) that

$$\begin{aligned} \|y_n - p\| &\le \beta_n \|x_n - p\| + (1-\beta_n)\|Tz_n - p\| \\ &\le \beta_n \|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\le \|x_n - p\| \end{aligned}$$

we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \max\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_n - p\|\}. \end{aligned}$$

Now, an induction yields

$$\|x_n - p\| \le \max\{\frac{1}{1-\alpha}\|f(p) - p\|, \|x_0 - p\|\}. \ \ n \ge 0. \tag{2.14}$$

This implies that $\{x_n\}$ is bounded, so are $\{Tx_n\}$, $\{f(x_n)\}$ $\{y_n\}$ and $\{z_n\}$. From (1.3) we have

(2.15) $$\|x_{n+1} - y_n\| = \alpha_n\|f(x_n) - y_n\| \to 0, \quad as\ n \to \infty.$$

Next, we claim that

(2.16) $$\|x_{n+1} - x_n\| \to 0.$$

In order to prove (2.16) from

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n \\ x_n = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})y_{n-1} \end{cases}$$

we have

$$x_{n+1} - x_n = (1-\alpha_n)(y_n - y_{n-1}) + (\alpha_{n-1} - \alpha_n)(y_{n-1} - f(x_{n-1})) + \alpha_n(f(x_n) - f(x_{n-1})).$$

It follows that

(2.17) $$\begin{aligned} \|x_{n+1} - x_n\| \le (1-\alpha_n)\|y_n - y_{n-1}\| + |\alpha_{n-1} - \alpha_n|\|y_{n-1} - f(x_{n-1})\| \\ + \alpha\alpha_n\|x_n - x_{n-1}\|. \end{aligned}$$

Similarly, from

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tz_n \\ y_{n-1} = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})Tz_{n-1} \end{cases}$$

we have

$$\begin{aligned} y_n - y_{n-1} = (1-\beta_n)(Tz_n - Tz_{n-1}) + \beta_n(x_n - x_{n-1}) \\ + (Tz_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n). \end{aligned}$$

It follow that

(2.18) $$\begin{aligned} \|y_n - y_{n-1}\| &\le (1-\beta_n)\|Tz_n - Tz_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ &\quad + \|Tz_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n| \\ &\le (1-\beta_n)\|z_n - z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\ &\quad + \|Tz_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|. \end{aligned}$$

On the other hand, from

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)Tx_n \\ z_{n-1} = \gamma_{n-1}x_{n-1} + (1-\gamma_{n-1})Tz_{n-1} \end{cases}$$

we obtain

$$\begin{aligned} z_n - z_{n-1} = (1-\gamma_n)(Tx_n - Tx_{n-1}) + \gamma_n(x_n - x_{n-1}) \\ + (\gamma_{n-1} - \gamma_n)(Tx_{n-1} - x_{n-1}). \end{aligned}$$

It follows that

(2.19) $$\|z_n - z_{n-1}\| \le \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|Tx_{n-1} - x_{n-1}\|.$$

Substituting (2.19) into (2.18), we get

(2.20) $$\begin{aligned} \|y_n - y_{n-1}\| &\le (1-\beta_n)(\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|Tx_{n-1} - x_{n-1}\|) \\ &\quad + \beta_n\|x_n - x_{n-1}\| + \|Tz_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n| \end{aligned}$$

that is,

$$\|y_n - y_{n-1}\| \le \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|Tx_{n-1} - x_{n-1}\| + \|Tz_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|. \tag{2.21}$$

Similarly, substituting (2.21) into (2.17) yields that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1-\alpha_n)(\|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n|\|Tx_{n-1} - x_{n-1}\| \\ &\quad + \|Tz_{n-1} - x_{n-1}\||\beta_{n-1} - \beta_n|) + |\alpha_{n-1} - \alpha_n|\|y_{n-1} - f(x_{n-1})\| \\ &\quad + \alpha\alpha_n\|x_n - x_{n-1}\| \\ &\le (1-(1-\alpha)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + M(|\alpha_{n-1} - \alpha_n| + |\beta_{n-1} - \beta_n| + |\gamma_{n-1} - \gamma_n|) \end{aligned} \tag{2.22}$$

where M is a constant such that

$$M \ge \max\{\|y_{n-1} - f(x_{n-1})\|, \|x_{n-1} - Tx_{n-1}\|, \|x_{n-1} - Tz_{n-1}\|\}$$

for all n. By assumptions (i)-(iii), we have that

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty}(1-\alpha)\alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty}(|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|) < \infty.$$

Hence, Lemma 1.4 is applicable to (2.22) and we obtain

$$\|x_{n+1} - x_n\| \to 0 \qquad as\ n \to \infty \tag{2.23}$$

Again, it follows from (2.23) that

$$\begin{aligned} \|Tx_n - x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - Tz_n\| + \|Tz_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - Tz_n\| + \|z_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - Tx_n\| \\ &\quad + \beta_n\|Tz_n - Tx_n\| + \|z_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - Tx_n\| \\ &\quad + \beta_n\|x_n - z_n\| + \|z_n - x_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - Tx_n\| \\ &\quad + (1+\beta_n)\|x_n - z_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - Tx_n\| \\ &\quad + (1+\beta_n)(1-\gamma_n)\|x_n - Tx_n\|. \end{aligned} \tag{2.24}$$

That is,

$$(\gamma_n - 2\beta_n + \gamma_n\beta_n)\|Tx_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

From condition (ii) and (2.13), (2.23) we know

$$\|Tx_n - x_n\| \to 0. \tag{2.25}$$

Next, we claim that

(2.26) $$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0,$$

where $q = Qf = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$ (Lemma 1.4).

From x_t solves the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Thus we have

$$\|x_t - x_n\| = \|(1-t)(Tx_t - x_n) + t(f(x_t) - x_n)\|.$$

It follows from Lemma 1.2 that

(2.27) $$\begin{aligned}\|x_t - x_n\|^2 &\le (1-t)^2\|Tx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle\\ &\le (1-2t+t^2)\|x_t - x_n\|^2 + f_n(t)\\ &\quad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2\end{aligned}$$

where

(2.28) $$f_n(t) = (2\|x_t - x_n\| + \|x_n - Tx_n\|)\|x_n - Tx_n\| \to 0, \text{ as } n \to 0.$$

It follows that

(2.29) $$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t).$$

Letting $n \to \infty$ in (2.29) and noting (2.28) yields

(2.30) $$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_1,$$

where $M_1 > 0$ is a constant such that $M_1 \ge \|x_t - x_n\|^2$ for all $t \in (0,1)$ and $n \ge 1$.

Letting $t \to 0$ from (2.30) we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0.$$

So, for any $\varepsilon > 0$, there exists a positive number δ_1 such that, for $t \in (0, \delta_1)$ we get

(2.31) $$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{\varepsilon}{2}.$$

On the other hand, since $x_t \to q$ as $t \to 0$, from Lemma 1.1, there exists $\delta_2 > 0$ such that, for $t \in (0, \delta_2)$ we have

$$\begin{aligned}&|\langle f(q)-q, J(x_n-q)\rangle - \langle x_t - f(x_t), J(x_t - x_n)\rangle|\\ \le\ &|\langle f(q)-q, J(x_n-q)\rangle - \langle f(q)-q, J(x_n-x_t)\rangle|\\ &+|\langle f(q)-q, J(x_n-x_t)\rangle - \langle x_t - f(x_t), J(x_t-x_n)\rangle|\\ \le\ &|\langle f(q)-q, J(x_n-q) - J(x_n-x_t)\rangle| + |\langle f(q) - f(x_t) - q + x_t, J(x_n-q)\rangle|\\ \le\ &\|f(q)-q\|\,\|J(x_n-q) - J(x_n-x_t)\| + \|f(q)-f(x_t)-q+x_t\|\,\|x_n-q\| < \frac{\varepsilon}{2}.\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta)$, we have

$$\langle f(q)-q, J(x_n-q)\rangle \le \langle x_t - f(x_t), J(x_t-x_n)\rangle + \frac{\varepsilon}{2}$$

that is,

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le \limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t-x_n)\rangle + \frac{\varepsilon}{2}\,.$$

It follows from (2.31) that

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le \varepsilon.$$

Since ε is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0 \tag{2.32}$$

Finally, we show that $x_n \to q$ strongly and this concludes the proof. Indeed, using Lemma 1.2 again we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(y_n-q)+\alpha_n(f(x_n)-q)\|^2\\
&\le (1-\alpha_n)^2\|y_n-q\|^2 + 2\alpha_n\langle f(x_n)-q, J(x_{n+1}-q)\rangle\\
&\le (1-\alpha_n)^2\|x_n-q\|^2\\
&\quad + 2\alpha_n\langle f(x_n)-f(q), J(x_{n+1}-q)\rangle + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\alpha\|x_n-q\|\|x_{n+1}-q\|\\
&\quad + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + \alpha_n\alpha(\|x_n-q\|^2+\|x_{n+1}-q\|^2)\\
&\quad + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}\|x_n-q\|^2 - \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q, J(x_{n+1}-q)\rangle\\
&\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-q\|^2 - \frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(q)-q, J(x_{n+1}-q)\rangle + M_2\alpha_n^2\\
&= (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n})\|x_n-q\|^2\\
&\quad + \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}(\frac{M_2(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(q)-q), J(x_n+1-q)\rangle.
\end{aligned}$$

Now we apply Lemma 1.4 and use (2.32) to see that $\|x_n-q\|\to 0$. This completes the proof. □

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