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On embedding Fibonacci meshes on Fibonacci cube networks

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ABSTRACT. The Fibonacci cube was presented as a new topology for interconnection networks. Due to his strong recursive structure, the Fibonacci cube posses many attractive properties. In this papers we show how two Fibonacci meshes can be simultaneously embedded in a Fibonacci cube with dilation 1.

1. INTRODUCTION

The Fibonacci cube is a new interconnection network topology introduced by Hsu in [3]. This class of interconnection networks is inspired by the Fibonacci numbers and shows interesting properties due to the important properties of Fibonacci numbers.

A Fibonacci cube of order $n \geq 2$, denoted Γ_n, is an indirected graph with f_n nodes, where f_n is the Fibonacci number of order n, consisting recursively of two disjoint graphs Γ_{n-1} and Γ_{n-2}. Each node in Γ_{n-2} is then connected to a node in Γ_{n-1} and Γ_{n-1} has more nodes than Γ_{n-2}.

On the other side, the Fibonacci cubes are a type of incomplete hypercubes. They are subgraphs of hypercube graphs induced by the nodes that have no two consecutive 1's in their binary representation. The size of the Fibonacci cube does not increase as fast as the hypercube and a Fibonacci cube can be considered as a hypercube with many faulty nodes.

It has been shown that the diameter and node degree of the Fibonacci cube with N nodes are $\mathcal{O}(\log N)$ which are similar to those of the hypercube of $\mathcal{O}(N)$ nodes. The Fibonacci cube contains about $1/5$ fewer edges than the hypercube for the same number of nodes [4].

One of the measures for "goodness" of a network is the embeddability, the possibility of simulating other topologies in the Fibonacci cube. An embedding is defined as a one to one mapping function which maps the vertices of a guest graph to the vertices of a host graph. An edge of the guest graph corresponds to a path in the host graph.

The dilation of an embedding is the maximum distance in the host graph between the images of the adjacent nodes in the guest graph and is a lower bound for communication delay when the host graph simulates the guest graph.

In [3], the embeddings of linear arrays, meshes, Fibonacci trees and hypercubes in the Fibonacci cube are studied. In [2], Cong et al. studied the stimulations of linear arrays, rings and $2D$ meshes on Fibonacci cubes. In [2], the embedding of

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a Fibonacci mesh into its corresponding optimum Fibonacci cube with dilation 1 is presented.

In this paper we show how two disjoint Fibonacci meshes can be embedded into a Fibonacci cube with dilation 1.

2. PRELIMINARIES

The Fibonacci numbers are recursively defined as $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$.

According to Zeckendorf's lemma, any natural number can be uniquely represented as a sum of Fibonacci numbers. If m is an integer, $0 \leq m \leq f_n - 1$, $n \geq 3$, we define the Fibonacci code of m as $(b_{n-1}, b_{n-2}, \ldots, b_2)_F$, where $m = \sum_{j=2}^{n-1} b_j f_j$ and b_j is either 0 or 1 and $b_j \cdot b_{j+1} = 0$ for $2 \leq j < n-1$.

The Fibonacci code for an integer $i \in \{f_{n-1}, f_{n-1}+1, \ldots, f_n - 1\}$ has $n-2$ bits. The Fibonacci codes for the numbers between 0 and $12 = f_7 - 1$ are: $0 = (00000)_F$, $1 = (00001)_F$, $2 = (00010)_F$, $3 = (00300)_F$, $4 = (00101)_F$, $5 = (01000)_F$, $6 = (01001)_F$, $7 = (01010)_F$, $8 = (10000)_F$, $9 = (10001)_F$, $10 = (10010)_F$, $11 = (10100)_F$, $12 = (10101)_F$.

Let N be an integer, $1 \leq N \leq f_n$ for some n and i_F, j_F be the Fibonacci codes of i and j, $0 \leq i, j < N$.

The Fibonacci cube can be defined as follows:

Definition 2.1. [3] The Fibonacci cube of size N is a graph $FC(N) = (V(N), E(N))$ where $V(N) = \{0, 1, \ldots, N-1\}$ and $E(N) = \big\{(i,j) \,|\, H(i_F, j_F) = 1, i, j \in V(N)\big\}$ where $H(i_F, J_F)$ is the Hamming distance between the Fibonacci codes of i and j. Two nodes i, j in the Fibonacci cube are connected if their Fibonacci codes differ in only one position or $|i - j| = f_k$ for some k.

Definition 2.2. [3] The Fibonacci cube of order n, denoted Γ_n, is a Fibonacci cube with f_n nodes, $\Gamma_0 = (\emptyset, \emptyset)$.

We denote by C_n the set of order n Fibonacci codes, $n \geq 2$. For example $C_2 = \{\lambda\}$, $C_3 = \{0, 1\}$, $C_4 = \{00, 01, 10\}$. Then we can easily see that $C_n = 0 \| C_{n-1} \cup 10 \| C_{n-2}$, $n \geq 4$, where $\|$ denote de concatenation of strings and $|C_n| = f_n$ for $n \geq 2$.

With this observations we can give a recursive definition for Fibonacci cubes:

Definition 2.3. Assume $\Gamma_n = (V(n), E(n))$, $\Gamma_{n-1} = (V(n-1), E(n-1))$ and $\Gamma_{n-2} = (V(n-2), E(n-2))$. The Fibonacci cube of order n is defined recursively by $V(n) = 0 \| V(n-1) \cup 10 \| V(n-2)$ and there is an edge between two nodes in $V(n)$ only if their binary representation differ exactly in one position.

In figure 1, the Fibonacci cubes $\Gamma_1, \Gamma_2, \ldots, \Gamma_6$ are represented.

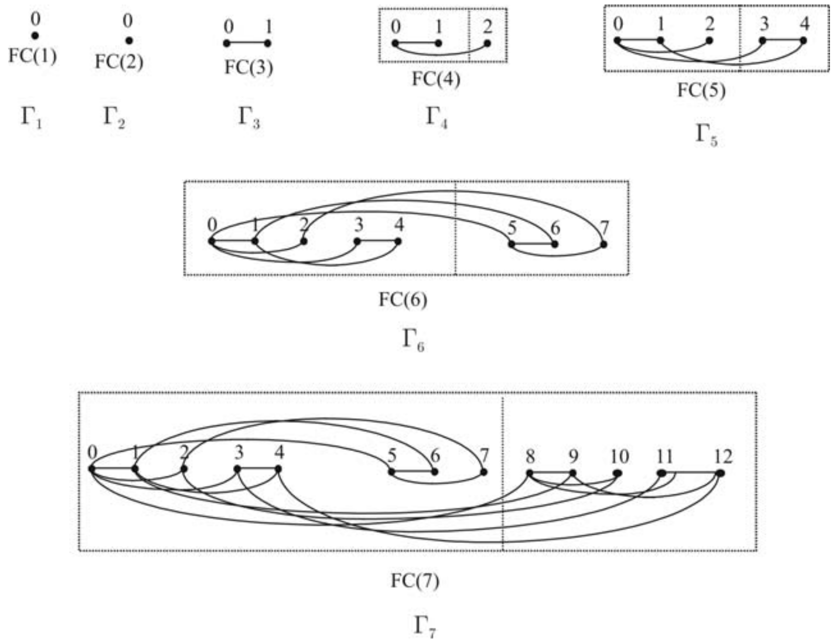


Figure 1

In spite of its asymmetric and relatively sparse interconnections, the Fibonacci cubes possess attractive topological properties in terms of connectivity. This kind of properties were studied in [3] and [4], and are listed bellow:

• the Fibonacci cube of order n has $[2(n-1)f_n - nf_{n-1}]/5$ edges, where f_n denotes the n-th Fibonacci number;

• for $n \geq 2$, $d_n(i) = \begin{cases} d_{n-1}(i)+1, & 0 \leq i < f_{n-2} \\ d_{n-1}(i), & f_{n-2} \leq i < f_{n-1} \\ d_{n-2}(i-f_{n-1})+1, & f_{n-1} \leq i < f_n \end{cases}$, where $d_n(i)$ denotes the degree of node i in Γ_n and $d_0(0)=0$, $d_1(0)=1$, $d_1(1)=1$;

• for $n \geq 3$, $[(n-2)/3] \leq d_n(i) \leq (n-2)$, where $d_n(i)$ denote the degree of node i in Γ_n;

• the Fibonacci cube Γ_n is a connected graph for $n \geq 1$ and has the diameter $D_n = n-2$;

• if $K_e(\Gamma_n)$ and $K_v(\Gamma_n)$ denote de edge, respectively the node connectivity of Γ_n, then $\lfloor n/8 \rfloor \leq K_v(\Gamma_n) \leq \Gamma_e(\Gamma_n) \leq \lfloor (n-2)/3 \rfloor$.

3. EMBEDDINGS IN FIBONACCI CUBES

A Hamiltonian path can be embedded in a Fibonacci cube. The construction of such a path is based on the construction of a Gray code sequence in Γ_n. The Gray codes for Γ_3 and Γ_4 are $G_3 = \{0,1\}$ and $G_4 = \{01,00,10\}$. If G_n and G_{n+1} denote the Gray codes for Γ_n and Γ_{n+1} and G'_n and G'_{n+1} denote the reversed sequence of G_n respectively G_{n+1}, then we can define the Gray code sequence for Γ_{n+2} as $G_{n+2} = \left\{0 \| G'_{n+1}, 10 \| G'_n\right\}$, for $n > 2$ [2]. So, it follows that

Theorem 3.1. *A linear array L_n with f_n nodes can be embedded with dilation* 1 *in the Fibonacci cube* Γ_n.

For example, the linear array embedded in Γ_6 is shown in figure 2.

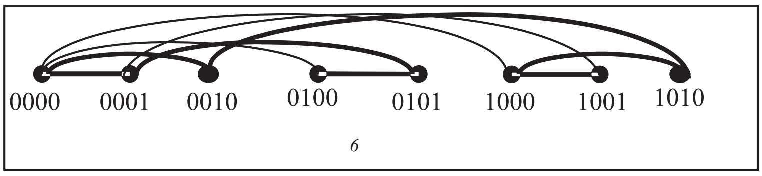


Figure 2

In [3], is proved that the Fibonacci cube Γ_n can be embedded with dilation 1 in the hypercube of order $n-2$, $H(n-2)$. Because there are no cycles of odd length in a hypercube, there is no hamiltonian cycle in Γ_n if $|f_n|$ is an odd number.

In [2], Cong proved that a ring with f_n nodes can be embedded with dilation 2 in Γ_n.

The problem of embedding Fibonacci meshes was studied in [3] and [2]. A Fibonacci mesh $M_{m,k}$ is defined as a $2D$ mesh, $f_m \times f_k$ mesh.

Hsu showed that two disjoint Fibonacci meshes $M_{n,n}$ and $M_{n+1,n+1}$ can simultaneously be embedded in Γ_{2n+1} with dilation 1, respectively $M_{n,n+1}$ and $M_{n,n-1}$ in Γ_{2n} for all $n \geq 1$.

In [2] the construction of a Fibonacci mesh $M_{n,n+1}$ embedded in its corresponding optimum Fibonacci cube Γ_{2n} is given.

The basic idea of this construction is to label the lines and columns of Fibonacci mesh, according to the Gray codes for Γ_n and Γ_{n+1}. The columns are labelled according to the Gray code G_{n+1} and the rows according to the Gray code G_n.
The label for the node in row i and column j is obtained by concatenation between the label for the j-th column a 0 bit and then the label for the i-th row. The 0 is placed to avoid two consecutive 1's. The label for a node has $[(n+1)-2]+1+(n-2) = 2n-2$ bits and no consecutive 1's. The dilation is clearly 1, so we have a dilation 1 embedding of $M_{n,n+1}$ Fibonacci mesh into Γ_{2n}. Cong also showed that Γ_{2n} is the smallest Fibonacci cube with at least $f_n \cdot f_{n+1}$ nodes in which $M_{n,n+1}$ can be embedded with dilation 1.

For example, the construction of $M_{4,5}$ to embed in Γ_8 is represented in figure 3.

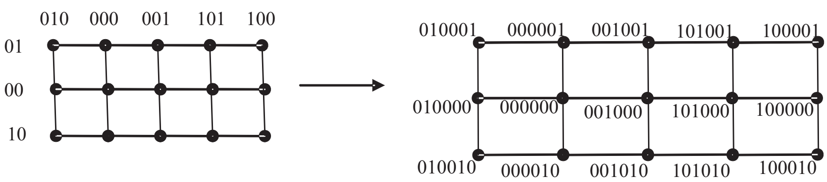


Figure 3

We can use this idea and construct two disjoint Fibonacci meshes $M_{n,n+1}$ and $M_{n-1,n}$ which can be embedded in Γ_{2n} with dilation 1 and all the nodes of Γ_{2n} are covered by these meshes.

Theorem 3.2. *Two disjoint Fibonacci meshes $M_{n,n+1}$ and $M_{n-1,n}$ can be simultaneously embedded in a Fibonacci cube Γ_{2n} with dilation 1 and covering all the nodes of the cube.*

Proof. The $2D$ Fibonacci mesh $M_{n,n+1}$ can be considered as a matrix with f_n rows and f_{n+1} columns and the $2D$ Fibonacci mesh $M_{n-1,n}$ can be considered as a matrix with f_{n-1} rows and f_n columns.

The columns in $M_{n,n+1}$ are labelled according to the Gray code G_{n+1} defined for Fibonacci cube and the rows are labelled according to the Gray code G_n defined for Fibonacci cube. The node located on line i and column j has the label $\beta_j 0\alpha_i$, where $\beta_j \in G_{n+1}$ and $\alpha_i \in G_n$. This label has $[(n+1)-2]+1+(n-2) = 2n-2$ bits and no consecutive 1's, so is a Fibonacci code in Γ_{2n}.

The columns in $M_{n-1,n}$ are labelled according to the Gray code C_n and the rows according to the Gray cod G_{n-1}. The node located on line i, column j, $i \in \{1,\ldots,f_{n-1}\}$, $j \in \{1,\ldots,f_n\}$ has the label $\gamma_j 010\delta_i$ where $\gamma_j \in G_n$, $\delta_i \in G_{n-1}$. This label has $(n-2)+3+(n-1)-2 = 2n-2$ bits and no consecutive 1's so this label is a Fibonacci code in Γ_{2n}.

The label of a node in $M_{n,n+1}$ has a 0 on its $n-1$ position and a node in $M_{n-1,n}$ has a 1 on its $n-1$ position. This means there are two disjoint meshes. The total number of nodes in those two meshes are $f_{n+1}\cdot f_n + f_n \cdot f_{n-1} = f_{2n}$, therefore all the nodes in Γ_{2n} are covered.

The dilation in this embeddings is obviously 1. Each node in $M_{n,n-1}$ has a link to a node in $M_{n+1,n}$, their labels differ in $(n-1)$-th position.

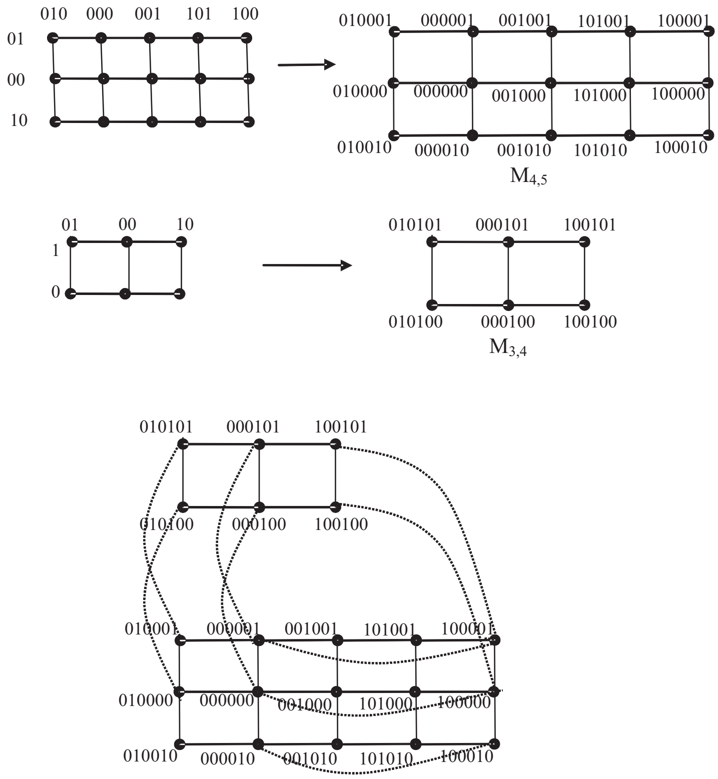
$\square$

For the Fibonacci cube Γ_8 the construction and the embedding of the two meshes $M_{4,5}$ and $M_{3,4}$ are represented in figure 4.

Using a similar construction, for dilation 1 embedding of two square meshes into Fibonacci cubes, we can state that

Theorem 3.3. *Two disjoint square Fibonacci meshes $M_{n,n}$ and $M_{n+1,n+1}$ can be simultaneously embedded in a Fibonacci cube Γ_{2n+1} with dilation 1 and covering all the nodes of the cube.*

The embeddings presented in this paper and in [2], [3] show that the Fibonacci cube allow efficient simulation of other topologies, almost as efficient as the hypercube. The embeddings of the Fibonacci meshes shows that in presence of faulty links, a Fibonacci cube can be reconfigured using meshes.



$M_{4,5}$ and $M_{3,4}$ in Γ_8

Figure 4

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