CARPATHIAN J. MATH.
23 (2007), No. 1 - 2, 11 - 26

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

A biased discussion of fixed point theory

B. E. RHOADES

ABSTRACT. This paper contains a discussion of the following topics: contractive conditions, 2-metric spaces, D-metric spaces, maps which have no nontrivial periodic points, contractive conditions involving integrals, Mann and Ishikawa iteration, stability, and iteration with errors.

1. INTRODUCTION

My remarks today are biased for two reasons. First I will be talking about aspects of fixed point theory of interest to me. Second, some of my remarks will reflect my personal biases with respect to that particular topic.

Perhaps a subtitle for this talk would be "Some things which work and some which don't".

I shall begin with a little personal history. I became interested in the subject of fixed point theory in the spring of 1973, while on sabbatical leave in Germany and in Israel. Each week I would peruse the contents of the new journals, looking for any papers in summability. In the course of that activity I began to encounter papers dealing with fixed points. At first I ignored these. Then, when they became so frequent, I decided to look at them more closely. I discovered two things. First, the proof technique used in each paper was the same. Second, no one appeared to be reading any other person's work.

The reason there were so many papers on fixed point theory is the appearance of the 1968 paper of Kannan [29]. Prior to then, the main paper on fixed point theory was the 1922 paper of Banach [4]. Banach's contraction principle has been applied to many problems in analysis and fixed point theory. The main drawback to his condition is that it requires the map to be Lipschitz of order $k < 1$. In 1968 Kannan provided an example of a contractive condition that does not require the continuity of the map at every point, although maps satisfying his condition are continuous at fixed points. When I returned to Bloomington I examined many of the journals in our library, and discovered the same two facts. I then decided that it would be worthwhile to prepare a paper which partially ordered some of these many definitions. The end result of this endeavor was my Transactions paper of 1977 [48], which partially ordered 125 contractive definitions for a single map, and another 125 for a pair of maps, and either stated or proved the most general fixed point theorem along each of the strands. Apparently others found

---

Received: 07.11.2006; In revised form: 10.01.2007; Accepted: 19.02.2007
2000 *Mathematics Subject Classification.* 47H10.
Key words and phrases. *Fixed point iteration with errors, fixed point theorems, Ishikawa iteration, Mann iteration.*

this endeavor to be worthwhile. At last count this paper has been cited over 248 times.

No good deed goes unpunished. As soon as this paper appeared I suddenly became the world's expert on contractive definitions, which resulted in my being asked to referee a large number of fixed point papers every year since.

At this time I would like to mention some work related to that paper, which is not as well known.

2. Contractive definitions

The first 25 definitions in [48] involve a single map. I will illustrate by taking one example. X is a complete metric space, and T is a selfmap of X.

$$d(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{24}$$

for all $x, y \in X$, where k is a constant satisfying $0 \leq k < 1$.

The next 25 definitions involve some fixed iterate of T. for example, corresponding to (24) would be

$$d(T^p x, T^p y) \leq k \max\{d(x,y), d(x,T^p x), d(y,T^p y), d(x,T^p y), d(y,T^p x)\}, \tag{49}$$

where p is a fixed positive integer.

The next 25 definitions deal with the p-th iterate of x and the q-th iterate of y. For example, if p and q are fixed positive integers,

$$d(T^p x, T^q y) \leq k \max\{d(x,y), d(x,T^p x), d(y,T^q y), d(x,T^p y), d(y,T^q x)\}. \tag{74}$$

For the next 25 definitions p is a function of x; i.e., $p = p(x)$. Thus, (49) becomes

$$\begin{aligned} d(T^{p(x)} x, T^{p(x)} y) \leq k \max\{&d(x,y), d(x,T^{p(x)} x), d(y,T^{p(x)} y), \\ &d(x,T^{p(x)} y), d(y,T^{p(x)} x)\}. \end{aligned} \tag{99}$$

Finally, for the last 25 definitions p depends on both x and y; i.e., $p = p(x,y)$.

$$\begin{aligned} d(T^{p(x,y)} x, T^{p(x,y)} y) \leq k \max\{&d(x,y), d(x,T^{p(x,y)} x), d(y,T^{p(x,y)} y), \\ &d(x,T^{p(x,y)} y), d(y,T^{p(x,y)} x)\}. \end{aligned} \tag{124}$$

There are a corresponding 125 definitions for two maps. For example, the two-map version of (24) is

$$d(Sx,Ty) \leq k \max\{d(x,y), d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx)\}.$$

In 1990 Kincses and Totik [30] gave an example of a map satisfying (104), but which has no fixed points. They also proved that, if T satifies (84), then T has a unique fixed point. Although [48] partially ordered a large number of definitions, not every two definitions were compared. In 1997 Collaco and Silva [9] filled in the table for the basic 25 definitions.

In 1977 [44] Barada Ray and I proved that maps satisfying (221) possess a unique common fixed point.

Sehie Park also observed that fixed point theorems for many contractive definitions used the same proof technique. In 1980 [40]he proved the following two theorems, where $O(u) := \{u, Tu, T^2u, \ldots\}$.

Theorem 2.1. *Let T be a selfmap of a metric space (X, d). If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ such that $\overline{O}(u)$ is complete and*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{2.1}$$

holds for any $x, y = Tx$ in $O(u)$, then $\{T^n u\}$ converges to some $\xi \in X$, and

$$d(T^i u, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u, Tu) \quad for \quad i \geq 1.$$

Further, if T is orbitally continuous at ξ or if (2.1) holds for any $x, y \in \overline{O}(u)$, then ξ is a fixed point of T.

Theorem 2.2. *Let T be a selfmap of a metric space (X, d). If*

(i) *there exists a point $u \in X$ such that the orbit $O(u)$ has a cluster point $\xi \in X$,*
(ii) *T is orbitally continuous at ξ and $T\xi$, and*
(iii) *T satisfies*
$$d(Tx, Ty) < d(x, y)$$
for each $x, y = Tx \in \overline{O}(u), x \neq y$,

then ξ is a fixed point of T.

These theorems contain as special cases a number of papers involving contractive conditions not covered by my Transactions paper.

As an example of an application of Theorem 2.1, not previously published, is the following [63].

Theorem A. *Let T be an orbitally continuous selfmap of a metric space M that is T-orbitally complete. If T satisfies*

$$\begin{aligned} [d(Tx, Ty)]^3 \leq\ & c_1 d(x, y)[d(y, Ty)]^2 + c_2 d(x, Tx)[d(Tx, Ty)]^2 \\ & + c_3 d(y, T^2x)[d(Tx, T^2x)]^2 + c_4 d(x, Ty)[d(y, Tx)]^2 \end{aligned} \tag{2.2}$$

for all $x, y \in M$ and $c_i \in \mathbb{R}$ such that $(c_1 + c_2)/(1 - c_3) = h \in (0, 1)$, then T has a fixed point, which is unique, when $0 < c_4 < 1$.

Proof. Set $y = Tx$ to get

$$\begin{aligned} [d(Tx, T^2x)]^3 \leq\ & c_1 d(x, Tx)[d(Tx, T^2x)]^2 + c_2 d(x, Tx)[d(Tx, T^2x)]^2 \\ & + c_3 d(Tx, T^2x)[d(Tx, T^2x)]^2 + 0, \end{aligned}$$

which implies that

$$[d(Tx, T^2x)]^3 \leq \frac{c_1 + c_2}{1 - c_3} d(x, Tx)[d(Tx, T^2x)]^2.$$

If x is such that $Tx = T^2x$, then Tx is a fixed point of T. If not, then we have

$$d(Tx, T^2x) \leq h d(x, Tx),$$

and condition (2.1) of Theorem 2.1 is satisfied. Therefore $\{T^n x\}$ converges to a point ξ.

In (2.2) set $x = x_n, y = \xi$ to get

$$\begin{aligned} [d(x_{n+1}, T\xi)]^3 \leq\ & c_1 d(x_n, \xi)[d(\xi, T\xi)]^2 + c_2 d(x, Tx)[d(x_{n+1}, T\xi)]^2 \\ & + c_3 d(\xi, x_{n+2})[d(x_{n+1}, x_{n+2})]^2 + c_4 d(x_n, T\xi)[d(\xi, x_{n+1})]^2. \end{aligned}$$

Taking the limit as $n \to \infty$ yields $\xi = T\xi$. Condition (2.2) implies the uniqueness of ξ when $c_4 < 1$. □

In 1969 Meir and Keeler [34] established a fixed point theorem for a selfmap of a metric space (X, d) satisfying the following condition.

$$\text{Given an } \varepsilon > 0, \text{there exists a } \delta > 0 \text{ such that} \tag{2.3}$$
$$\varepsilon \le d(x,y) < \varepsilon + \delta \quad \text{implies that} \quad d(Tx, Ty) < \varepsilon.$$

In 1980 Sehie Park [41] constructed a table of contractive conditions of Meir-Keeler type, which extended the list in my Transactions paper. A point u in a metric space X is said to be regular if diam$(O(x))$ is finite. The most general fixed point theorem he obtained is the following.

Theorem 2.3. *Let T be a selfmap of a metric space (X, d). Suppose there exists a regular point $u \in X$ such that*

(i) *$O(u)$ has a regular cluster point $p \in X$ and*
(ii) *the following condition holds on $O(u) \cup O(p)$:*
Given an $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any $x, y \in X$,

$$\varepsilon \le m(x,y) < \varepsilon + \delta_0 \quad \text{implies that} \quad d(Tx, Ty) \le \varepsilon_0,$$

where

$$m(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Then T has a unique fixed point $p \in \overline{O}(u)$ and $T^n u \to p$.

In 1988 [50] I showed that, for the then known contractive conditions, the map is continuous at the fixed point.

Some papers have been published where the contractive condition is a special case of one already in print. One such example is the following result of Fisher [13].

Theorem 2.4. *Let T be a selfmap of a complete metric space X satisfying*

$$[d(Tx,Ty)]^2 \le c\{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)\} \tag{2.4}$$

for all $x, y \in X$, $0 \le c < 1/2$. Then T has a unique fixed point.

Using the fact that, for any real nonnegative numbers $a, b, ab \le \max\{a^2, b^2\}$, (2.4) implies that

$$\begin{aligned}[d(Tx,Ty)]^2 &\le 2c\max\{[d(x,Tx)]^2, [d(x,Ty)]^2, [d(y,Ty)]^2, [d(y,Tx)]^2\} \\ &= 2c(\max\{d(x,Tx), d(x,Ty), d(y,Ty), d(y,Tx)\})^2\end{aligned}$$

which implies that

$$d(Tx,Ty) \le \sqrt{2c}\max\{d(x,Tx), d(x,Ty), d(y,Ty), d(y,Tx)\}.$$

Since $c < 1/2$, the above inequality is a special case of (24), which was defined in 1971 by Ciric [8].

3. 2-METRIC SPACES

In 1976 Diminie and White [11] defined the concept of a 2-metric space.

A 2-metric space is a space X in which, for each triple of points a, b, c, there exists a real-valued nonnegative function ρ satisfying:

(1a) for each pair of points $a, b, a \neq b$, there exists a point $c \in X$ such that $\rho(a, b, c) \neq 0$,

(1b) $\rho(a, b, c) = 0$ when at least two of the points are equal,

(2) $\rho(a, b, c) = \rho(a, c, b) = \rho(b, c, a)$, and

(3) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$.

A sequence $\{x_n\}$ in X is said to be Cauchy if $\lim \rho(x_n, x_m, a) = 0$ for all $a \in X$. A sequence $\{x_n\}$ in X is convergent and $x \in X$ is the limit, if $\lim \rho(x_n, x, a) = 0$ for each $a \in X$.

Following the appearance of this paper, a number of fixed point theorems were proved in 2-metric spaces, including the following one by me [49].

Theorem 3.5. *Let X be a complete 2-metric space, T a selfmap of X satisfying: there exists an $h, 0 \leq h < 1$ such that, for each $x, y \in X$,*

$$\rho(Tx, Ty, a) \leq h \max\{\rho(x, y, a), \rho(x, Tx, a), \rho(y, Ty, a), \rho(x, Ty, a), \rho(y, Tx, a)\}.$$

Then T possesses a unique fixed point z and $\lim T^n x_0 = z$ for each $x_0 \in X$.

In 1986 Hsiao [18] made the following observation. A selfmap of a 2-metric space X is said to have property (H) if

$$d(T^i x, T^j x, T^k x) = 0 \quad \text{for all} \quad i, j, k \in \mathbb{N} \cup \{0\}.$$

He then proved that all of the fixed point theorems in the literature, including Theorem 3.5, have Property (H). Consequently every orbit lies in a straight line.

4. D-METRIC SPACES

In 1984 Dhage [10] defined the concept of a D-metric space.

A nonempty set X, together with a function $D : X \times X \times X \to [0, \infty)$ is called a D-metric space if D satisfies

(i) $D(x, y, z) = 0$ if and only if $x = y = z$,

(ii) $D(x, y, z) = D(p(x, y, z))$, where p is a permutation of x, y, z, and

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, , a, z \in X$.

He claimed that D-metric convergence defines a Hausdorff topology and that the D-metric is (sequentially) continuous in all three variables. Assuming the correctness of his dissertation he and several other authors wrote papers dealing with fixed points for maps defined on D-metric spaces. In 2004, two dissertations [43] and [36], contained examples which show that : (1) in a D-metric space D-metric convergence need not always define a topology; (2) even when the D-metric convergence defines a topology it need not be Hausdorff; and (3) even when the D-metric convergence defines a metrizable topology the D-metric need not be continuous even in a single variable.

5. Probabilistic metric spaces

Beginning in the 70's a number of papers have been written proving fixed point theorems in probabilistic metric spaces.

Let H denote the distribution functon defined by $H(x) = 0$ if $x \leq 0$, and $H(x) = 1$ for $x > 0$.

Let X be a set, $\mathcal{F}$ a function on $X \times X$ such that $\mathcal{F}(x, y) = F_{x,y}$ is a distribution function. Then the pair $(X, \mathcal{F})$ is called a probabilistic metric space (PM-space) if the following conditions are satisfied:

I. $F_{x,y}(0) = 0$ for all $x, y \in X$,
II. $F_{x,y} = H$ if and only if $x = y$,
III. $F_{x,y} = F_{y,x}$, and,
IV. if $F_{x,y}(\varepsilon) = 1$ and $F_{y,z}(\delta) = 1$, then $F_{x,z}(\varepsilon + \delta) = 1$.

If, in addition,

V $F_{x,z}(\varepsilon + \delta) \geq T(F_{x,y}(\varepsilon), F_{yz}(\delta))$,

where T is a 2-place function on the unit square satisfying

(1) $T(0, 0) = 0$ and $T(a, 1) = a$,
(2) $T(a, b) = T(b, a)$, and
(3) $T(T(a, b), c) = T(a, T(b, c))$,

then $(X, \mathcal{F}, T)$ is called a Menger space.

Extending some of the work of Schweizer and Sklar [61], Hicks [17] has shown that, for many probabilistic inequalities defined on a complete Menger space whose T-norm satisfies properties (1) -(3), there is an equivalent metric d on the space which converts the probabilistic inequality to a contractive type condition, thereby essentially eliminating the need to consider probabilistic metric spaces.

6. Generalizations of commutativity

Many fixed point theorems have also being proved for two maps S and T, where the iteration process is defined by intertwining the two maps; i.e., let $x_0 \in X$ and define $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$.

In 1976 Jungck [25] proved the following.

Theorem 6.6. *Let I be a continuous selfmap of a complete metric space (X, d). Then I has a fixed point in X iff there exists an $\alpha \in (0, 1)$ and a mapping $T : X \to X$ which commutes with I and which satisfies*

$$I(X) \subset T(X) \quad and \quad d(Tx, Ty) \leq \alpha d(Ix, Iy) \quad for\,all \quad x, y \in X.$$

Moreover, the common fixed point is unique.

A large number of fixed point papers then appeared, which replaced x and y on the right hand side of a contractive condition with continuous functions I and J. For example, condition (21) in this context would take the following form,

$$d(Sx, Ty) \leq k \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), [d(Ix, Ty) + d(Jy, Sx)]/2\}$$

Recognizing that commutativity was too restrictive a condition, Sessa, in 1982 defined weakly commuting. Two maps S and I are said to be weakly commuting if

$$d(SIx, ISx) \leq d(S, Ix) \quad \text{for each} \quad x \in X.$$

This definition was soon generalized to R-weakly commuting [38]. Two maps S and I are said to be R-weakly commuting if there exists a positive number R such that, for each $x \in X$,

$$d(SIx, ISx) \leq Rd(Sx, Ix).$$

In 1986 Jungck [26] defined the concept of compatibility. Two maps S and I are said to be compatible if $\lim d(SIx_n, ISx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Ix_n = t$ for some $t \in X$. Jungck [26], in 1986, extended the concept of compatibility to weakly compatible. Two maps S and I are said to be weakly compatible if they commute at coincidence points. A further extension was made by Jungck [27] in 1996 to weakly biased maps. Two maps S and I are said to be weakly I biased if $Sx = Ix$ implies $d(IAx, Ix) \leq d(SIx, Sx)$.

In 1997 Jeong and I [22] established a version of Park's result for four maps. Specifically, we proved the following.

Theorem 6.7. *Let S, T, I, and J be selfmaps of a metric space (X, d) such that*

$$I(X) \subset T(X), J(X) \subset S(X).$$

Assume that $\{y_n\}$ is complete, where $\{x_n\}$ is defined by $Tx_{2n+1} = Ix_{2n}, Sx_{2n+2} = Jx_{2n+1}$, and $\{y_n\}$ is defined by $y_{2n} = Sx_{2n}, y_{2n+1} = Tx_{2n+1}$. Suppose that there exists a $\lambda \in [0, 1)$ such that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \quad forall \quad y_n \neq y_{n+1}.$$

Then either

(a) *I and S have a coincidence point,*
(b) *J and T have a coincidence point,*
(c) *I, S, and T have a common coincidence point,*
(d) *J, S, and T have a coincidence point, or*
(e) *$\{y_n\}$ converges to a point $z \in X$, and*

$$d(y_i, z) \leq \frac{\lambda^i}{1-\lambda} d(y_0, y_1) \quad \text{for each} \quad i > 0.$$

We then obtained results of Ahmad and Imdad [3], Ahmad, Imdad, and Khan [2], Imdad and Ahmad [19], Divicarrio and Sessa [12], Murthy and Sharma [35], Chourasia [7], and Gajic and Stojakovic [15] as special cases.

Two maps S and I are said to be pointwise R-weakly commuting if, given an $x \in X$, there exists an $R > 0$ such that

$$d(SIx, ISx) \leq Rd(Sx, Ix).$$

Pant [39] proved the following.

Theorem 6.8. *Let (S, I) and (T, J) be pointwise R-weakly commuting pairs of selfmappings of a complete metric space (X, d) satisfying*

(i) $SX \subset JX, TX \subset IX$,
(ii) $d(Sx, Ty) \leq hM(x, y)$, *for* $0 \leq h < 1, x, y, \in X$.

where

$$M(x, y) := \max\{d(Ix, Jy), d(Sx, Ix), d(Jy, Ty), [d(Sx, Jy) + d(Ty, Is)]/2\}.$$

Suppose that (S, I) or (T, J) is weakly compatible. Then S, T, I, and J have a unique common fixed point.

Pant [39] provides the following example. $X = [2, 20]$, d is the usual metric on X.

$$S2 = 2, Sx = 3 \quad \text{if} \quad x > 2,$$

$$I2 = 2, Ix = 6 \quad \text{if} \quad x > 2,$$

$$Tx = 2 \quad \text{for} \quad x = 2, x > 5,\ Tx = 6 \quad \text{if} \quad 2 < x \leq 5,$$

$$J2 = 2, Jx = 12 \quad \text{if} \quad 2 < x \leq 5,\ Jx = x - 3 \quad \text{if} \quad x > 5.$$

These four maps satisfy the hypotheses of his theorem, but they are not continuous at the common fixed point.

Earlier this year [28] Jungck and I defined the concept of occasionally weakly commuting. Two maps S and I are said to be occasionally weakly commuting (owc) if there is a point $x \in X$ at which S and I commute.

Proofs of fixed point theorems for four compatible or weakly compatible maps all have the same pattern. Step one is to show that there exists a common coincidence point for one pair of maps. The second step is to show that step one gives rise to a common coincidence point for the second pair of maps. In step three it is shown that these pairwise coincidence points are equal. In step four it is shown that this common coincidence point is a common fixed point. Uniqueness is established in step five.

Here is one of our results.

Theorem 6.9. *Let X be a set with a symmetric r. Suppose that I, J, S, T are selfmaps of X and that the pairs $\{I, S\}$ and $\{J, T\}$ are each owc. If*

$$r(Sx, Ty) < M(x, y) \tag{6.5}$$

for each $x, y \in X$ for which $Sx \neq Ty$, where

$$M(x, y) := \max\{r(Ix, Jy), r(Ix, Sx), r(Jy, Ty), r(Ix, Ty), r(Jy, Sx)\}.$$

Then there is a unique point $w \in X$ such that $Sw = Tw = w$ and a unique point $z \in X$ such that $Tz = Jz = z$. Moreover, $z = w$, so that there is a unique common fixed point of I, J, S, and T.

As a consequence of our work, to prove a fixed point theorem involving four maps, it is sufficient to prove that there exists a common coincidence point for one pair of maps. The existence of a common fixed point then follows from our theorems.

In 2002 Aamri and Moutawakil [1] defined property (E, A). Two maps S and I satisfy property (E, A) if there exists a sequence $\{x_n\}$ such that $\lim Sx_n = \lim Ix_n = t$ for some $t \in X$. In other words property (E, A) hypothesizes the existence of a common coincidence point.

Our theorem then contains the results of [1] as a special case.

7. Maps for which $F(T) = F(T^n)$

Let T be a selfmap which has a fixed point z. Then z is a fixed point of T^n for every positive integer n. Is the converse true; i.e., if z is a fixed point of T^n for some positive integer $n > 1$, is z a fixed point of T?

The answer is no in general as the following example shows. Let $X = [0, 1], T$ defined by $Tx = 1 - x$. Then T has one fixed point at $x = 1/2$. However, T^2 is the identity map, so every point is a fixed point.

The conjecture is, if T satisfies a contractive condition that does not include nonexpansive mappings, then $F(T^n) = F(T)$ for every positive integer n. Jeong and I have written a long paper [28], which has just appeared, verifying this conjecture for every contractive condition, involving a single map, in the literature. We have submitted a second paper [24], which verifies the conjecture for pairs of maps for which either some form of commutativity is not required, or for which the maps commute.

We shall say that a pair of maps S and I have property Q if $F(S) \cap F(I) = F(S^n) \cap F(I^n)$ for each positive integer n. The following is a simple example.

Theorem 7.10. *Let S and I be two commuting selfmaps of a complete metric space (X, d) satisfying*

$$d(Sx, Sy) \leq kd(Ix, Iy)$$

for each $x, y \in X$, where $0 \leq k < 1$. Then S and I satisfy property Q.

Proof. Let $u \in F(S^n) \cap F(I^n)$. Then

$$\begin{aligned} d(u, Su) &= d(S^n u, S^{n+1} u) \leq kd(IS^{n-1} u, IS^n u) \\ &= kd(S^{n-1} Iu, S^n Iu) \leq k^2 d(IS^{n-2} Iu, IS^{n-1} Iu) \\ &\leq \cdots \leq k^n d(I^n u, SI^n u) = k^n d(u, Su), \end{aligned}$$

which implies that $u = Su$.

In a similar manner

$$\begin{aligned} d(u, Iu) &= d(S^n u, S^n Iu) \leq kd(IS^{n-1} u, IS^{n-1} Iu) \\ &\leq \cdots \leq k^n d(I^n u, I^{n=1} u) = k^n d(u, Iu), \end{aligned}$$

and $u = Iu$. □

For contractive conditions involving maps for which compatibility, or some similar condition, is required, the question is still open.

8. Integral conditions

In 2002 Branciari [5] proved the following theorem.

Theorem 8.11. *Let (X, d) be a compete metric space, $c \in [0, 1), T$ a selfmap of X such that, for each $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \varphi(t) dt \leq c \int_0^{d(x,y)} \varphi(t) dt,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ os a Lebesgue integrable mapping which is summable, nonnegative, and such that, for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(t)dt > 0.$$

Then T as a unique fixed point $z \in X$, and, for each $x \in X$, $\lim T^n x = z$.

There are obvious generalizations of this theorem, and several papers have appeared containing such generalizations. Clearly these integral conditions contain the corresponding definitions by setting $\varphi(t) \equiv 1$. However, in some cases one has equivalence. The following result appeared in a paper I was recently sent to referee.

Theorem 8.12. *Let (X, d) be a compact metric space, S, T selfmaps of X satisfying, for each $x, y \in X$,*

$$\begin{aligned}(8.6)\qquad \psi\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) < a\psi\Big(\int_0^{d(fx,gy)} \varphi(t)dt\Big) + b\psi\Big(\int_0^{d(fx,Sx)} \varphi(t)dt\Big) \\ + c\psi\Big(\int_0^{d(gy,Ty)} \varphi(t)dt\Big) + \frac{e}{2}\Big[\psi\Big(\int_0^{d(fx,Ty)} \varphi(t)dt\Big) \\ + \psi\Big(\int_0^{d(gy,Sx)} \varphi(t)dt\Big)\Big]\end{aligned}$$

whenever the right hand side of (8.6) is positive, where $0 \le a + b + c + e/2 < 1$, ψ is a nonnegative increasing function such that $\psi(t) < t$ for each $t > 0$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable, nonnegative, and such that, for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varphi(t)dt > 0.$$

Then S and T have a unique common fixed point in X.

First of all (8.6) implies that

$$\begin{aligned}\psi\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) < (a+b+c+e)\max\Big\{\psi\Big(\int_0^{d(fx,gy)} \varphi(t)dt\Big), \\ \psi\Big(\int_0^{d(fx,Sx)} \varphi(t)dt\Big), \psi\Big(\int_0^{d(gy,Ty)} \varphi(t)dt\Big), \\ \Big[\psi\Big(\int_0^{d(fx,Ty)} \varphi(t)dt\Big) + \psi\Big(\int_0^{d(gy,Sx)} \varphi(t)dt\Big)\Big]/2\Big\}.\end{aligned}$$

But, for any nonnegative real numbers a and b, since ψ is increasing,

$$\max\{\psi(a), \psi(b)\} = \psi(\max\{a, b\}).$$

Therefore the above inequality implies that

$$\begin{aligned}(8.7)\qquad \psi\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big) < (a+b+c+e)\psi\Big(\int_0^{m(x,y)} \varphi(t)dt\Big) \\ < \psi\Big(\int_0^{m(x,y)} \varphi(t)dt\Big),\end{aligned}$$

where

$$m(x,y) := \max\{d(fx,gy), d(fx,Sx), d(gy,Ty), [d(fx,Ty)+d(gy,Sx)]/2\}.$$

Again using the fact that ψ is increasing, (8.7) implies that

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le \int_0^{m(x,y)} \varphi(t)dt. \tag{8.8}$$

But equality in (8.8) contradicts (8.7). Therefore we must have

$$\int_0^{d(Sx,Ty)} \varphi(t)dt < \int_0^{m(x,y)} \varphi(t)dt. \tag{8.9}$$

But, since

$$h(s) := \int_0^s \varphi(t)dt$$

implies that h is positive and nondecreasing, (8.9) implies that

$$d(Sx,Ty) \le m(x,y).$$

But equality contradicts (8.7). Therefore we have

$$d(Sx,Ty) < m(x,y). \tag{8.10}$$

Suppose that (8.10) is true for all x and y in X. Define

$$j(x,y) = \frac{d(Sx,Ty)}{m(x,y)}. \tag{8.11}$$

Then j is continuous on the compact set $X \times X$ and so attains its maximum value. Call it c. Inequality (8.10) implies that $c < 1$. Then (8.11) implies that

$$d(Sx,Ty) \le cm(x,y),$$

and the existence of a unique common fixed point follows from a result of Singh and Mishra [62].

If there exists a point (x,y) for which $m(x,y) = 0$, then we have $fx = gy = Sx = Ty$. Since (f,S) are weakly compatible,

$$Sfx = S^2x = fSx.$$

If $S^2x \ne Sx$, then, from (8.10),

$$d(S^2x,Ty) < \max\{d(fSx,gy), d(fSx,S^2x), d(gy,Ty), [d(fSx,Ty)+d(gy,S^2x)]/2\};$$

i.e.,

$$\begin{aligned} d(S^2x,Sx) &< \max\{d(S^2x,Sx), 0, 0, [d(S^2x,Sx)+d(Sx,S^2x)]/2\} \\ &= d(S^2x,Sx), \end{aligned}$$

a contradiction. Therefore Sx is a common fixed point of f and S.

Similarly, the assumption that $TSx \ne Sx$ leads to a contradiction. Therefore Sx is also a common fixed point of g and T. The uniqueness of the fixed point follows from (2.3).

9. Fixed point iterations

I now wish to talk about fixed point iteration processes.

Consider the function T defined by $Tx = 1 - x$ over $X = [0, 1]$. Suppose we choose $x = a, 0 \leq a \leq 1, a \neq 1/2$. Then repeated function iteration of a yields the sequence $\{a, 1-a, a, 1-a, \ldots\}$ which does not converge. However, T has a fixed point at $1/2$. Therefore some other iteration process is needed in order to obtain the fixed point. One of the first ones, and one of the most popular ones is that of Mann [33]. Let $x_0 \in X$, and define

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n,$$

where $c_n = 1/(n+1)$.

In 1955 Krasnoselskij [31] defined the special case of Mann iteration with each $c_n = 1/2$. In 1957 Schaefer [60] defined an iteration method, which he named the Krasnoselskij method, by choosing each $c_n = \lambda$, where $0 < \lambda < 1$.

Mann proved that, if T is a continuous selfmap of a finite interval $[a, b]$ with exactly one fixed point, then his iteration scheme converges to that fixed point. Reinerman [45], in 1969, extended this result to $\{c_n\}$ satisfying $c_0 = 1, 0 < c_n < 1$ for $n \geq 1$, and $\sum c_n = \infty$. In 1971 Franks and Marzec [14] extended the result of Mann to include maps with more than one fixed point. In 1974 [46] I extended the Franks and Marzec result to the $\{c_n\}$ of Reinermann.

In 1974 Ishikawa [20] defined the following iteration process.

$$x_0 = y_0 \in X, y_n = (1 - \beta_n)x_n T x_n, x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

where $0 \leq \alpha_n \leq \beta_n \leq 1, \lim \beta_n = 0, \sum \alpha_n \beta_n = \infty$.

Let H be a Hilbert space, E a convex compact subset of H, T a Lipschitzian pseudo-contractive selfmap of E. He proved that this process converges to a fixed point of T. In 2001 Chidume and Mutangadura [6] gave an example of a Lipschitz continuous pseudocontractive map for which no Mann iteration converges.

The Ishikawa process does not contain Mann iteration as a special case, since, setting each $\beta_n = 0$ eliminates all of the α_n. I observed [47], in 1976, that if one replaces the condition

$$0 \leq \alpha_n \leq \beta_n \leq 1$$

with

$$0 \leq \alpha_n, \beta_n \leq 1$$

then this modified Ishikawa iteration would include Mann iteration. Many mathematicians spent the next 30 years extending results for Mann iteration to this modified Ishikawa iteration, and then recovering the original theorem for Mann iteration by setting each $\beta_n = 0$.

In 1992 Wittmann [64] defined the following iteration process for Hilbert spaces. Let $x \in H$,

$$x_0 = x, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n.$$

He then proved that, if T is a nonexpansive map with $fF(T) \neq \emptyset$, $\lim \alpha_n = 0, \sum \alpha_n = \infty, and \sum |\alpha_{n+1} - \alpha_n| < \infty$, then, for any $x \in H$, $\{x_n\}$ converges strongly to a fixed point of T.

It is known that the Mann iteration process converges to a fixed point of any continuous selfmap of a closed an bounded interval on the real line. Linda Saliga and I [53] have shown that the Wittmann process for certain continuous nonincreasing selfmaps need not converge.

Beginning in 2003 Stefan Soltuz and I wrote a series of papers demonstrating that Mann and Ishikawa iterations are equivalent for many different classes of mappings.([54], [55], [56], [57], [58], and [59])

Unfortunately any iteration process is apt to introduce computational errors. A serious practical question is whether these errors will destroy the convergence to a fixed point. In an attempt to address this question, Troy Hicks and his student Alberta Harder [16], based on a theorem of Ostrowski ([21], page 101), made the following definition of stability.

Let X be a metric space, T a selfmap of X, $x_0 \in X$. Suppose that $x_{n+1} = f(T, x_n)$ is some iteration procedure, involving T, which yields a sequence $\{x_n\}$ of points of X. Further, suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X, and define $\varepsilon_n = d(y_{n+1}, f(T, y_n))$ for $n = 0, 1, \ldots$. If $\lim \varepsilon_n = 0$ implies that $\lim y_n = p$, then the iteration process $x_{n+1} = f(T, x_n)$ is said to be T-stable, or stable with respect to T. They then showed that a number of contractive conditions are T-stable with respect to function iteration.

In 1990 [51] I showed stability for a wider class of maps, and for some fixed point iteration procedures. The most general results in this direction were established by Osilike. In 1995 [37] He proved the following.

Theorem 9.13. *Let X be a Banach space, T a selfmap of X satisfying, for each $x, y \in X$,*

$$\|Tx - Ty\} \leq a\|x - y\| + L\|x - Tx\|,$$

where $0 \leq a < 1, L \geq 0$. Suppose that T has a fixed point p. Let $x_0 \in X, x_{n+1} = Tx_n$. Let $\{y_n\} \subset X$, and define $\varepsilon_n = d(y_{n+1}, Ty_n)$. Then

$$\|y_{n+1} - p\| \leq \|x_{n+1} - p\| + L\sum_{j=0}^{n} a^{n-j}\|x_j - Tx_j\| + a^{n+1}\|x_0 - y_0\| + \sum_{j=0}^{n} a^{n-j}\varepsilon_j$$

and $\lim y_n = p$ iff $\lim \varepsilon_n = 0$.

He proved a similar result for Mann and Ishikawa iterations.

An iteration scheme is said to be weakly T stable if $\{\varepsilon\}$ also satisfies $\sum \varepsilon_n < \infty$. It is a reasonable conjecture that every iteration scheme is weakly T stable.

In 1995 Liu [32] defined Mann and ishikawa processes with errors. The Mann process with errors is defined as follows.

Let K be a nonempty subset of a normed linear space E, $T : K \to E$. Let $x_0 \in K$, and define

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n + u_n,$$

where $\{\alpha_n\} \subset [0, 1)$ satisfies appropriate conditions and u_n is a vector in K such that $\sum \|u_n\| < \infty$.

Xu objected to the process of Liu on the grounds that the condition $\sum \|u_n\| < \infty$ is not compatible with the randomness of the occurrence of errors. In 1998 Xu [65] made the following definition.

Let $X_0 \in K$,

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n,$$

where $\{u_n\}$ is a bounded sequence and $\{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$ with

$$a_n + b_n + c_n = 1$$

for each n.

Define $\alpha_n = b_n + c_n$. Then we have

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T x_n + c_n u_n \\ &= (1 - \alpha_n) x_n + \alpha_n x_n + c_n (u_n - T x_n), \end{aligned}$$

so that, if $R(T)$ is bounded, then $v_n := u_n - T x_n$ is a bounded sequence, and Xu's definition reduces to that of Liu, since, in using the definition of Xu, it is always assumed that $\sum c_n < \infty$.

As I pointed out in 2004, [52], the construction of Xu cannot be carried out. For, in order to determine the values of a_n, b_n, and c_n, it is necessary to know the value of u_n for each n. But, if u_n is an arbitrary bounded sequence, its values are not known.

Acknowledgement. The author takes this opportunity to thank the referee for the careful reading of the manuscript.

References

[1] Aamri, M. and El Moutawakil, D., *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270**(2002), 181-188

[2] Ahmad, A., Imdad, M. and Khan, A. R., *Four mappings satisfying a rational inequality*, Bull. Calcutta Math. Soc. **87**(1995), 323-328

[3] Ahmad, A. and Imdad, M., *A common fixed point theorem for four mappings satisfying a rational inequality*, Publ. Math. Debrecen **41**(1992), 181-187; **41**(1992), 323-328

[4] Banach, S., *Sur les operations dans les ensembles absraites et leurs applications*, Fund. Math. **3**(1922), 133-181

[5] Branciari, A., *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sc. **29**(2002), 531-536

[6] Chidume, C. E. and Mutangadura, S. A., An example of the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc. **129** (2001), 2359-2363

[7] Chourasia, G., *Common fixed point theorems for three mappings in complete metric space*, Math. Student **58**(1990), 31-34

[8] Ćirić, Lb. J., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45**(1974), 267-273

[9] Paula Collaċo and Carvalho Silva, J., *A complete comparison of 25 contractive conditions*, Nonlinear Analysis **30**(1997), 471-476

[10] Dhage, B. C., *A study of some fixed point theorems*, Ph. D. Thesis, Marathwada University, Aurangabad, India (1984)

[11] Diminie, C. and White, A. G. Jr., *Strictly convex linear* 2*-normed spaces*, Math. Nachr. **59**(1976), 319-324; **10**(1982), 623-631

[12] Divicarro, M. L. and Sessa, S., *Some remarks on common fixed points of four mappings*, Jñānābha **15**(1985), 139-149

[13] Fisher, B., *Some theorems on fixed points*, Studia Sci. Math. Hung.**12**(1977), 159-160

[14] Franks, R. L. and Marzec, R. P., *A theorem on mean value iterations*, Proc. Amer. Math. Soc. **30**(1971), 324-326

[15] Gajic, Lj. and Stojakovic, M., *On compatible mappings in fixed point theory*, Zb. Rad. Prirod-Mat. Fak. **24**(1994), 39-51

[16] Harder, Alberta M. and Hicks, T. L., *Stablity results for fixed point iteration procedures*, Math. Japonica **33**(1988), 693-706
[17] Hicks, T. L., *Fixed point theory in probabilistic metric spaces, II*, Math. Japonica **44**(1996), 487-493
[18] Hsiao, C-R., *A property of contractive type mappings in 2-metric spaces*, Jñānābha **16**(1986), 223-229
[19] Imdad, M. and Ahmad, A., *Four mappings with a common fixed point*, Rev. Research, Novi Sad **24**(1984), 23-30
[20] Ishikawa, S., *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44**(1974), 147-150
[21] Istrăţescu, V. I., Fixed Point Theory, Math. and Its Appl. **7**, D. Reidel Publishing Co. (1981)
[22] Jeong, G. S. and Rhoades, B. E., *Some remarks for improving fixed point theorems for more than two maps*, Indian J. Pure Appl. Math. **28**(1997), 1177-1196
[23] Jeong, G. and Rhoades, B. E., *Maps for which* $F(T) = F(T^n)$, Fixed Point Theory, **7**(2006)
[24] Jeong, G. and Rhoades, B. E., *More maps for which* $F(T) = F(T^n)$, submitted
[25] Jungck, G., *Commuting mappings and fixed points*, Amer. Math. Monthly **83**(1976), 261-263
[26] Jungck, G., *Compatible mappings and common fixed points*, Int. J. Math. & Math. Sci. **9**(1986), 771-779
[27] Jungck, G., *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. **4**(1996), 199-215
[28] Jungck, G. and Rhoades, B. E., *Fixed point theorems for occasionally weakly compatible mappings*, to appear in Fixed Point Theory
[29] Kannan, R., *Some results on fixed points*, Bull. Calcutta Math. Soc. **69**(1968), 71-76
[30] Kincses, J. and Totik, V., *Theorems and counterexamples on contractive mappings*, Math. Balkanica **4**(1990), 68-90
[31] Krasnoselskij, M. A., *Two remarks on the method of successive approximations* (Russian), Uspehi Mat. Nauk. **10**(1955), 123-127
[32] Liu, L., *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194**(1995), 114-125
[33] Mann, W. R., *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4**(1953), 506-510
[34] Meir, A. and Keeler, E., *A theorem on contraction mappings*, J. Math. Anal. Appl. **28**(1969), 326-329
[35] Murthy, P. P. and Sharma, A. K., *Extension of Naimpally and Singh's theorems under weak commutativity*, Bull. Calcutta Math. Soc. **83**(1991), 371-372
[36] Mustafa, Z. Y., *A mew approach to generalization of metric spaces*, Ph. D. Thesis, University of Newcastle, Australia (2004)
[37] Osilike, M. O., *Stability results for fixed point iteration procedures*, J. Nigerian Math. Soc. **14/14**(1995/96), 17-29
[38] Pant, R. P., *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188**(1994), 436-440
[39] Pant, R. P., *Common fixed points of four mappings*, Bull. Calcutta Math. Soc. **90**(1998), 281-286
[40] Park, S., *A unified approach to fixed points of contractive maps*, J. Korean Math. Soc. **16**(1980), 95-105
[41] Park, S., *On general contractive type conditions*, J. Korean Math. Soc. **17**(1980), 131-140
[42] Park, S. and Rhoades, B. E., *Meir-Keeler type contractive conditions*, Math. Japonica **26**(1981), 13-20
[43] Srinasa R. N., *A study of the topological properties of D-metric spaces and fixed point theorems in metric and D-metric spaces*, Ph. D. Thesis Acharya Nagarjuna University, (2004)
[44] Ray, B. K. and Rhoades, B. E., *Fixed point theorems for mappings with a contractive iterate*, Pacific J. Math. **71**(1977), 517-520
[45] Reinermann, J., *Über Toeplitzsche Iterationsverfahren und einige ihre Anwendungen in der konstruktiven Fixpunktheorie*, Arch. Math. (Basel) **20**(1969), 59-64
[46] Rhoades, B. E., *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc. **196**(1974), 161-176
[47] Rhoades, B. E., *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. **56**(1976), 741-750
[48] Rhoades, B. E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226**(1977), 257-290
[49] Rhoades, B. E., *Contraction type mappings on a 2-metric space*, Math. Nachr. **91**(1979), 151-155
[50] Rhoades, B. E., *Contractive conditions and continuity*, Contemporary Math. **72**(1988), 233-245
[51] Rhoades, B. E., *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math. **21**(1990), 1-9
[52] Rhoades, B. E., *Comments on some iteration processes with errors*, Fixed Point Theory and Appl. **5**(2004), 125-126

[53] Rhoades, B. E. and Saliga, L., *Some fixed point iteration procedures, II*, Nonlinear Analysis Forum **6**(2001), 193-217

[54] Rhoades, B. E. and Şoltuz, Ş. M., *The equivalence of Mann and Ishikawa iteration methods*, Int. J. Math. Math. Sci. **2003**(2003), 2451-459

[55] Rhoades, B. E. and Şoltuz, Ş. M., *The equivalence of Mann iteration and Ishikawa iteration for non-LIpschitzian operators*, Int. J. Math. Math. Sci. **2003**(2003), 2645-2652

[56] Rhoades, B. E. and Şoltuz, Ş. M., *The eqivalence between the convergences of Ishikawa and Mann iterations for asymptotically pseudocontractive maps*, J. Math. Anal. Appl. **283**(2003), 681-688

[57] Rhoades, B. E. and Şoltuz, Ş. M., *The eqivalence of Mann and Ishikawa iteration for Lipschitzian ψ-uniformly psuedocontractive and ψ-uniformly accretive maps*, Tamkang J. Math. **35**(2004), 235-245

[58] Rhoades, B. E. and Şoltuz, Ş. M., *The equivalence between the convergences of Ishikawa and Mann iterations for asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive maps*, J. Math. Anal. Appl. **289**(2004), 2766-278

[59] Rhoades, B. E. and Şoltuz, Ş. M., *The eqivalence of Mann and Ishikawa iteration for ψ-uniformly pseudocontractive or ψ-uniformly accretive maps*, Int. J. Math. Math. Sci. **2004**(2004), 2443-2452

[60] Schaefer, H., *Über die Methode sukzessiver Approximationen*, Jahresber. Deutsch. Math. Verein **59**(1957), 131-140

[61] Schweizer, B. and Sklar, A., *Probabilistic Metric Spaces*, Elsevier, North-Holland (1983)

[62] Singh, S. L. and Mishra, S. N., *Remarks on Jachymski's fixed point theorems for compatible maps*, Indian J. Pure Appl. Math. **28**(1997), 611-615

[63] Srivastava, R., *Fixed point theorems in a T-orbitally complete metric space*, Math. Student **64**(1995), 15-20

[64] Wittmann, R., *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58**(1992), 486-491

[65] Xu, Y., *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224**(1998), 91-101

DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405-7106, U.S.A.
E-mail address: rhoades@indiana.edu