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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

A Schurer-Stancu type quadrature formula

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ABSTRACT. Starting from the Schurer-Stancu approximation formula (1.4) we construct the quadrature formula (2.5). The coefficients of (2.5) are expressed at (2.6). We establish the case when (2.5) has the degree of exactness 1 and in this case we give the form of the remainder term. Also an optimal quadrature of Schurer-Stancu type is established. As particular cases, the Stancu, Schurer and respectively Bernstein quadrature formulas are obtained.

1. PRELIMINARIES

Let p be a given non-negative integer and let α, β be real parameters satisfying conditions $0 \le \alpha \le \beta$.

The Schurer-Stancu operators [2] $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$ are defined for any $f \in C([0,1+p])$, any $x \in [0,1+p]$ and any positive integer m by

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right) \tag{1.1}$$

where

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} \tag{1.2}$$

are the fundamental Schurer's polynomials [6].

Note that the operators (1.1) belong to a class of more general multiparameter linear operators, first considered by Professor D. D. Stancu in 1997 [12]. Many approximation properties of operators (1.1) were investigated in our recent monograph [4].

Let us to recall some of these properties, which will be essentially used in the present paper.

Theorem 1.1. ([2], [4]) *Let $e_j(x) = x^j$ (j - non-negative integer) be the test monomials. The following identities:*

(i) $\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} e_0\right)(x) = 1$;

(ii) $\left(\widetilde{S}_{m,p}^{(\alpha,\beta)} e_1\right)(x) = \dfrac{m+p}{m+\beta} x + \frac{\alpha}{m+\beta}$;

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(iii) $\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}e_2\right)(x)=\dfrac{1}{(m+\beta)^2}\{(m+p)^2x^2+(m+p)x(1-x)+2\alpha(m+p)x+\alpha^2\}$,
hold, for any $x \in [0, 1+p]$ and any non-negative integer m.

Theorem 1.2. ([2], [4]) *Let $\varphi_x : [0, 1+p] \to \mathbb{R}$ be defined by*

$$\varphi_x(t) = |t - x|.$$

For any $x \in [0, 1+p]$ the following identity

$$\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}\varphi_x^2\right)(x) = \frac{1}{(m+\beta)^2}\left\{((p-\beta)x+\alpha)^2 + (m+p)x(1-x)\right\} \tag{1.3}$$

holds.

In [3] and [4] it was considered the Schurer-Stancu approximation formula

$$f = \widetilde{S}_{m,p}^{(\alpha,\beta)}f + \widetilde{R}_{m,p}^{(\alpha,\beta)}f \tag{1.4}$$

and were proved some results regarding its remainder term $\widetilde{R}_{m,p}^{(\alpha,\beta)}f$.

2. Main results

Let $f \in C([0, 1+p])$ be given and let

$$\int_0^1 f(x)dx = \sum_{k=0}^{m+p} A_{m+p,k}^{(\alpha,\beta)} f\left(\frac{k+\alpha}{m+\beta}\right) + r_{m,p}^{(\alpha,\beta)}(f) \tag{2.5}$$

be the Schurer-Stancu type quadrature formula.

For $p = 0$, (2.5) reduces to the Stancu's quadrature formula ([13], [14]), while for $\alpha = \beta = 0$ and $p \neq 0$ (2.5) is the Schurer's quadrature formula. For $\alpha = \beta = p = 0$, (2.5) is the Bernstein's quadrature formula.

Lemma 2.1. *The coefficients of quadrature formula* (2.5) *are expressed by*

$$A_{m+p,k}^{(\alpha,\beta)} = \frac{1}{m+p+1} \tag{2.6}$$

for any $k = \overline{0, m+p}$.

Proof. From (1.1) and (2.5) follows

$$\begin{aligned} A_{m+p,k}^{(\alpha,\beta)} &= \int_0^1 \widetilde{p}_{m,k}(x)dx = \binom{m+p}{k}\int_0^1 x^k(1-x)^{m+p-k} \\ &= \binom{m+p}{k} B(k+1, m+p-k+1) \end{aligned}$$

where $B(k+1, m+p-k+1)$ denotes the Euler's function of first kind (the Beta-function). Taking into account the well-known properties of this function one obtains (2.6). □

Next, we are dealing with the degree of exactness of (2.5). We need

Lemma 2.2. *Let $e_j(x) = x^j$ (j - non-negative integer) be the test monomials. The following identities*

$$r_{m,p}^{(\alpha,\beta)}(e_0) = 0; \tag{2.7}$$

$$r_{m,p}^{(\alpha,\beta)}(e_1) = \frac{\beta - 2\alpha - p}{2(m+\beta)}; \tag{2.8}$$

$$r_{m,p}^{(\alpha,\beta)}(e_2) = \frac{2(\beta-p)(2m+\beta+p) - (m+p)(6\alpha+1) - 6\alpha^2}{6(m+\beta)^2}, \tag{2.9}$$

hold.

Proof. From (1.4) and (2.5) we get

$$r_{m,p}^{(\alpha,\beta)}(f) = \int_0^1 \left\{ f(x) - \left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) \right\} dx. \tag{2.10}$$

Next one applies Theorem 1.1. □

Remark 2.1.

(i) In general $r_{m,p}^{(\alpha,\beta)}(e_1) \neq 0$, i.e., the degree of exactness of (2.5) is 0.

(ii) For $\beta = 2\alpha + p$, (2.8) and (2.9) yield $r_{m,p}^{(\alpha,2\alpha+p)}(e_1) = 0$, $r_{m,p}^{(\alpha,2\alpha+p)}(e_2) \neq 0$, i.e., (2.5) has the degree of exactness 1.

In what follows, we are dealing with the Schurer-Stancu quadrature formula having the degree of exactness 1, i.e., with the following quadrature formula

$$\int_0^1 f(x)dx = \frac{1}{m+p+1} \sum_{k=0}^{m+p} f\left(\frac{k+\alpha}{m+p+2\alpha}\right) + r_{m,p}^{(\alpha,2\alpha+p)}(f). \tag{2.11}$$

We shall investigate the remainder term of (2.11). First, let us prove the following

Lemma 2.3. *For $\beta = 2\alpha + p$ and $m + p > 4\alpha^2$ the following identity*

$$\max_{x\in[0,1]} \left(\widetilde{S}_{m,p}^{(\alpha,2\alpha+p)} \varphi_x^2\right)(x) = \frac{m+p}{4(m+p+2\alpha)^2} \tag{2.12}$$

holds.

Proof. Applying Theorem 1.2 for $\beta = 2\alpha + p$ one obtains

$$\left(\widetilde{S}_{m,p}^{(\alpha,2\alpha+p)} \varphi_x^2\right) = (x)\frac{1}{(m+p+2\alpha)^2} \left\{\alpha^2(1-2x)^2 + (m+p)x(1-x)\right\}.$$

For $m + p > 4\alpha^2$, the function $f : [0,1] \to \mathbb{R}$

$$f(x) = \alpha^2(1-2x)^2 + (m+p)x(1-x)$$

attains its maximum value $\frac{m+p}{4}$ (at the point $x = \frac{1}{2}$). □

Remark 2.2. For $p = 0$, we get a result due to D. D. Stancu and A. Vernescu [14], regarding the Stancu's operators $S_m^{(\alpha,\beta)}$.

Theorem 2.3. *If*
(i) $f \in C([0,1+p]) \cap C^2([0,1])$;
(ii) $\beta = 2\alpha + p$ *and* $m + p > 4\alpha^2$,
the remainder term of Schurer-Stancu quadrature formula (2.11) *can be represented under the form*

$$r_{m,p}^{(\alpha,2\alpha+p)}(f) = \frac{(2\alpha-1)m + 2\alpha^2 + (2\alpha-1)p}{12(m+p+2\alpha)^2} f''(\xi) \tag{2.13}$$

where $0 < \xi < 1$.

Proof. Taking into account the hypotheses, we can apply the Peano's theorem [13] and we get

$$r_{m,p}^{(\alpha,2\alpha+p)}(f) = \frac{1}{2} f''(\xi) K_m(\alpha, p)$$

where $0 < \xi < 1$ and $K_m(\alpha, p)$ is the Peano's kernel, i.e.,

$$K_m(\alpha, p) = r_{m,p}^{(\alpha,2\alpha+p)}(e_2) = \frac{(2\alpha-1)m + 2\alpha^2 + (2\alpha-1)p}{6(m+p+2\alpha)^2}.$$

□

Remark 2.3. The minimum value of the remainder term of the quadrature formula (2.11) is obtained for $\alpha = \frac{1}{2}$, i.e.,

$$\min_{0 \le \alpha < \frac{1}{2}\sqrt{m+p}} r_{m,p}^{(\alpha,2\alpha+p)} = \frac{1}{24(m+p+1)^2} f''(\xi). \tag{2.14}$$

Theorem 2.4. *If*
(i) $f \in C([0,1+p]) \cap C^2([0,1])$;
(ii) $m + p > 4\alpha^2$,
the optimal quadrature formula of Schurer-Stancu type is the following

$$\int_0^1 f(x)dx = \frac{1}{m+p+1} \sum_{k=0}^{m+p} f\left(\frac{2k+1}{2m+2p+2}\right) + \frac{1}{24(m+p+1)^2} f''(\xi). \tag{2.15}$$

Proof. The quadrature formula (2.15) has the degree of exactness 1 and its remainder term has the minimum value possible (in the set of quadrature formulas of Schurer-Stancu type). □

Remark 2.4.

(i) For $p = 0$, from (2.15) we get the Stancu's optimal quadrature formula [4], i.e.,

$$\int_0^1 f(x)dx = \frac{1}{m+1} \sum_{k=0}^{m} f\left(\frac{2k+1}{2m+2}\right) + \frac{1}{24(m+1)^2} f''(\xi). \tag{2.16}$$

(ii) For $\alpha = \beta = 0$ and $p \neq 0$, from (2.5) we get the Schurer quadrature formula

$$\int_0^1 f(x)dx = \frac{1}{m+p+1} \sum_{k=0}^{m+p} f\left(\frac{k}{m}\right) + r_{m,p}(f). \tag{2.17}$$

It is immediately that the optimal Schurer quadrature formula is the Bernstein quadrature formula, i.e.

$$\int_0^1 f(x)dx = \frac{1}{m+1}\sum_{k=0}^{m} f\left(\frac{k}{m}\right) - \frac{1}{12m}\, f''(\xi). \tag{2.18}$$

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