CARPATHIAN J. MATH.
23 (2007), No. 1-2, 32-41

Non-steady Navier-Stokes Equations with Homogeneous Mixed Boundary Conditions and Arbitrarily Large Initial Condition

MICHAL BENEŠ, PETR KUČERA

ABSTRACT. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\partial\Omega \in \mathcal{C}^{0,1}$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ such that Γ_1 and Γ_2 are closed, sufficiently smooth, 1-dimensional measure of $\Gamma_1 \cap \Gamma_2$ is zero and 1-dimensional measure of Γ_1 is positive. Further let $(0,T)$ be a time interval. We prescribe the non-slip boundary conditions on $\Gamma_1 \times (0,T)$ and the boundary condition

$$-\mathcal{P}\mathbf{n} + \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0$$

on $\Gamma_2 \times (0,T)$. Here $\mathbf{u} = (u_1, u_2)$ is velocity, $\mathcal{P}$ represents pressure and $\mathbf{n} = (n_1, n_2)$ is an outer normal vector.

Our aim is to prove the existence and uniqueness of this problem on some time interval $(0, T^*)$ for sufficiently small T^*, $0 < T^* \leq T$.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary, $\partial\Omega \in \mathbb{C}^{0,1}$ and let Γ_1, Γ_2 be open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, $\Gamma_1 \neq \emptyset$ and the 1-dimensional measure of $\partial\Omega - (\Gamma_1 \cup \Gamma_2)$ is zero. The domain Ω represents a channel filled up with a fluid, Γ_1 is a fixed wall and Γ_2 is both the input and the output of the channel.

The authors in [2] and [14] use the Neumann condition on the output of the channel. Some qualitative properties of the Navier–Stokes equations with the mixed boundary conditions are studied in [3], [4], [5], [6], [7], [8], [9], [10].

Let $T \in (0, \infty]$, $Q = \Omega \times (0,T)$. The classical formulation of our problem is as follows:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\mathcal{P} = \boldsymbol{g} \quad \text{in } Q, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } Q, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Gamma_1 \times (0,T), \tag{1.3}$$

$$-\mathcal{P}\boldsymbol{n} + \nu\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = 0 \quad \text{in } \Gamma_2 \times (0,T), \tag{1.4}$$

$$\boldsymbol{u}(0) = \boldsymbol{\gamma} \quad \text{in } \Omega, \tag{1.5}$$

$$\boldsymbol{\gamma} = \boldsymbol{0} \quad \text{on } \Gamma_1. \tag{1.6}$$

Received: date; In revised form: date
2000 *Mathematics Subject Classification.* : 76D05, 76D03.
Key words and phrases. *Navier-Stokes equations, mixed boundary conditions, regularity.*

Functions $\boldsymbol{u}$, $\mathcal{P}$, $\boldsymbol{g}$, $\boldsymbol{\gamma}$ are smooth enough, $\boldsymbol{u} = (u_1, u_2)$ is velocity, $\mathcal{P}$ represents pressure, ν denotes the viscosity, $\boldsymbol{g}$ is a body force and $\boldsymbol{n} = (n_1, n_2)$ is an outer normal vector. The problem (1.1) $-$ (1.6) will be called the nonsteady Navier$-$Stokes problem with the mixed boundary conditions. For simplicity we suppose that $\nu = 1$ throughout this paper.

We solve also the problem, in which (1.2)$-$(1.6) hold and (1.1) is replaced with

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \mathcal{P} = g \quad \text{in } Q. \tag{1.7}$$

The problem (1.2)$-$(1.6) and (1.7) will be called the nonsteady Stokes problem with the mixed boundary conditions.

The Dirichlet boundary condition (1.3) expresses a non-slip behaviour of the fluid on fixed walls of the channel.

Our aim is to prove the existence and uniqueness of this problem on some time interval $(0, T^*)$ for sufficiently small $T^*, 0 < T^* \leq T$.

2. Definition of some function spaces and generalized formulation of the problem

We shall denote by c a generic constant, i.e. a constant whose value may change from line to line. On the other hand, numbered constants will have a fixed value throughout the paper. Constants will always depend only on domain Ω.

Let

$$\mathcal{E}(\Omega) = \left\{ \boldsymbol{u} \in \mathcal{C}^\infty(\overline{\Omega})^2;\ \operatorname{div} u \equiv 0,\ \overline{\operatorname{supp}}\, u \cap \Gamma_2 \equiv \emptyset \right\}.$$

Let $V^{k,p}$ be a closure of $\mathcal{E}(\overline{\Omega})$ in the norm of $W^{k,p}(\Omega)^2$, $k \geq 0$ (k need not be an integer) and $1 \leq p < \infty$. Then $V^{k,p}$ is a Banach space with the norm of the space $W^{k,p}(\Omega)^2$. For simplicity, we denote $V^{1,2}$ and $V^{0,2}$, respectively, as V and H. Note, that V and H, respectively, are Hilbert spaces with scalar products $\left(\!\left(. \,, \, . \right)\!\right)_V$ and $\left(\!\left(. \,, \, . \right)\!\right)_H$,

$$\left(\!\left(. \,, \, . \right)\!\right)_V = \left(\!\left(\boldsymbol{\Phi}, \boldsymbol{\Psi} \right)\!\right)_V = \int_\Omega \nabla \boldsymbol{\Phi} \cdot \nabla \boldsymbol{\Psi} \, d(\Omega) = \int_\Omega \frac{\partial \Phi_i}{\partial x_j} \frac{\partial \Psi_i}{\partial x_j} \, d(\Omega)$$

and

$$\left(\!\left(. \,, \, . \right)\!\right)_H = \left(\!\left(\boldsymbol{\Phi}, \boldsymbol{\Psi} \right)\!\right)_H = \int_\Omega \boldsymbol{\Phi} \cdot \boldsymbol{\Psi} \, d(\Omega) = \int_\Omega \Phi_i \Psi_i \, d(\Omega)$$

and they are closed subspaces of spaces $W^{1,2}(\Omega)^2$ and $L^2(\Omega)^2$.

Let

(2.8)

$$D = \{ \boldsymbol{w} \in V;\ \text{there exists } \boldsymbol{f} \in H \text{ such that } \left(\!\left(\boldsymbol{w}, \boldsymbol{v} \right)\!\right)_V = \left(\!\left(\boldsymbol{f}, \boldsymbol{v} \right)\!\right)_H \text{ for every } \boldsymbol{v} \in V \}$$

and

$$\|\boldsymbol{w}\|_D = \|\boldsymbol{f}\|_H,$$

where $\boldsymbol{w}$, $\boldsymbol{f}$ are corresponding functions via (2.8). Let $\boldsymbol{w}_i$ and $\boldsymbol{f}_i$ are corresponding functions via (2.8). Note that D is the Hilbert space with the scalar product $(\!(.,.)\!)_D$ such that

$$(\!(\boldsymbol{w}_1, \boldsymbol{w}_2)\!)_D = \left(\!\left(\boldsymbol{f}_1, \boldsymbol{f}_2 \right)\!\right)_H.$$

Similarly it can be shown as in ([16], Chapter I.,2.6) that there exist functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \dots, \boldsymbol{\phi}_k, \dots \in V \subset H$ and real positive numbers $\lambda_1, \lambda_2, \dots, \lambda_k, \dots \to \infty$ for $k \to \infty$, such that

$$\big(\!\big(\boldsymbol{\phi}_k, \boldsymbol{v}\big)\!\big)_V = \lambda_k \big(\!\big(\boldsymbol{\phi}_k, \boldsymbol{v}\big)\!\big)_H$$

for every $\boldsymbol{v} \in V$. $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \dots$ is a system that is complete in both H and V, orthonormal in H and orthogonal in V. Note that

$$H = \Big\{\boldsymbol{v};\ \boldsymbol{v} = \sum_{k=1}^{\infty} a_k \boldsymbol{\phi}_k,\ a_k \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} a_k^2 < \infty\Big\}, \tag{2.9}$$

$$V = \Big\{\boldsymbol{v};\ \boldsymbol{v} = \sum_{k=1}^{\infty} a_k \boldsymbol{\phi}_k,\ a_k \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} \lambda_k a_k^2 < \infty\Big\} \tag{2.10}$$

and

$$D = \Big\{\boldsymbol{v};\ \boldsymbol{v} = \sum_{k=1}^{\infty} a_k \boldsymbol{\phi}_k,\ a_k \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} \lambda_k^2 a_k^2 < \infty\Big\}. \tag{2.11}$$

Let $\mathcal{X}$ be an arbitrary Banach space and $p \in [1, \infty)$. As usual $L^p(\alpha, \beta;\ \mathcal{X})$ and $L^\infty(\alpha, \beta;\ \mathcal{X})$ (for $-\infty \le \alpha < \beta \le \infty$) denote the Banach spaces

$$\Big\{\phi;\ \phi(t) \in \mathcal{X} \text{ for almost every } t \in (\alpha, \beta),\ \int_\alpha^\beta \|\phi(t)\|_{\mathcal{X}}^p\, dt < \infty\Big\}$$

and

$$\Big\{\phi;\ \ \phi(t) \in \mathcal{X} \text{ for almost every } t \in (\alpha, \beta), \text{ ess} \sup_{t \in (\alpha, \beta)} \|\phi(t)\|_{\mathcal{X}} < \infty\Big\}$$

with the norms

$$\|\phi\|_{L^p(\alpha, \beta;\, \mathcal{X})} = \Big(\int_\alpha^\beta \|\phi(t)\|_{\mathcal{X}}^p\, dt\Big)^{1/p}$$

and

$$\|\phi\|_{L^\infty(\alpha, \beta;\, \mathcal{X})} = \text{ess} \sup_{t \in (\alpha, \beta)} \|\phi(t)\|_{\mathcal{X}}.$$

Note that ([11]) and ([1]) yield existence of γ, $1 < \gamma < 2$ such that

$$D \hookrightarrow\hookrightarrow W^{\gamma, 2}(\Omega)^2 \hookrightarrow W^{1, 2+2\delta}(\Omega)^2, \tag{2.12}$$

where $\delta = \frac{\gamma - 1}{2 - \gamma}$.

Definition 2.1. *Let $0 < T^* \le T$, $\boldsymbol{f} \in L^2(0, T^*;\ H)$, $\boldsymbol{u}_0 \in V$. Then $\boldsymbol{u}$ is called a generalized solution of the Stokes problem with the mixed boundary conditions and with data $\boldsymbol{f}$ and $\boldsymbol{u}_0$ on $(0, T^*)$ (problem (1.2)–(1.7) for $T = T^*$) if $\boldsymbol{u} \in L^2(0, T^*;\ D) \cap L^\infty(0, T^*;\ V)$, $\boldsymbol{u}' \in L^2(0, T^*;\ H)$ and*

$$\big(\!\big(\boldsymbol{u}'(t), \boldsymbol{v}\big)\!\big)_H + \big(\!\big(\boldsymbol{u}(t), \boldsymbol{v}\big)\!\big)_V = \big(\!\big(\boldsymbol{f}(t), \boldsymbol{v}\big)\!\big)_H \tag{2.13}$$

holds for every $\boldsymbol{v} \in V$ and for almost every $t \in (0, T^*)$.

Theorem 2.1. *Let $\boldsymbol{f} \in L^2(0,T;\, H)$, $\boldsymbol{u}_0 \in V$. There exists a unique generalized solution $\boldsymbol{u}$ of the Stokes problem with the mixed boundary conditions and with data $\boldsymbol{f}$ and $\boldsymbol{u}_0$ on $(0,T)$. Moreover $\boldsymbol{u} \in \mathcal{C}([0,T];\, V)$ and*

$$\|\boldsymbol{u}\|_{L^2(0,T;\, D)} + \|\boldsymbol{u}\|_{L^\infty(0,T;\, V)} + \|\boldsymbol{u}'\|_{L^2(0,T;\, H)} \le c_1(\|\boldsymbol{f}\|_{L^2(0,T;\, H)} + \|\boldsymbol{u}_0\|_V). \tag{2.14}$$

Proof: Since $\boldsymbol{f} \in L^2(0,T;\, H)$ and $\boldsymbol{u}_0 \in V$, we have

$$\boldsymbol{f} = \sum_{k=1}^{\infty} \mu_k(t)\boldsymbol{\phi}_k, \qquad \boldsymbol{u}_0 = \sum_{k=1}^{\infty} a_k \boldsymbol{\phi}_k, \tag{2.15}$$

where

$$\sum_{k=1}^{\infty} \int_0^T \mu_k^2(t)\, dt + \sum_{k=1}^{\infty} a_k^2 < \infty. \tag{2.16}$$

Let ϑ_k be a solution of the ordinary differential equation

$$\vartheta_k'(t) + \lambda_k \vartheta_k(t) = \mu_k(t) \tag{2.17}$$

with the initial condition

$$\vartheta_k(0) = a_k \tag{2.18}$$

for $k = 1, 2, \dots$ Then

$$\vartheta_k(t) = \int_0^t e^{\lambda_k(s-t)} \mu_k(s)\, ds + a_k e^{-\lambda_k t}$$

for almost every $t \in (0,T)$. Hence $\vartheta_k \in W^{1,2}((0,t))$.

Multiplying (2.17) by $2\vartheta_k'$ and integrating over $(0,t)$ we get

$$2\int_0^t {\vartheta_k'}^2(s)\, ds + \lambda_k \vartheta_k^2(t) = \lambda_k {\vartheta_k}^2(0) + 2\int_0^t \mu_k(s){\vartheta_k}'(s)\, ds \le$$
$$\le \vartheta_k^2(0) + \int_0^t {\vartheta_k}'^2(s)\, ds + \int_0^t \mu_k^2(s)\, ds$$

for $k = 1, 2, \dots$ and for almost every $t \in (0,T)$ and therefore

$$\int_0^t {\vartheta_k'}^2(s)\, ds + \lambda_k \vartheta_k^2(t) \le \vartheta_k^2(0) + \int_0^t \mu_k^2(s)\, ds. \tag{2.19}$$

Thus (2.19) yields

$$\sum_{k=1}^{\infty} \int_0^t {\vartheta_k'}^2(s)\, ds + \sum_{k=1}^{\infty} \lambda_k \vartheta_k^2(t) \le \sum_{k=1}^{\infty} \int_0^T {\vartheta_k'}^2(s)\, ds + \sum_{k=1}^{\infty} \lambda_k \vartheta_k^2(t) \le \tag{2.20}$$
$$2\sum_{k=1}^{\infty} \vartheta_k^2(0) + 2\sum_{k=1}^{\infty} \int_0^T \mu_k^2(s)\, ds$$

for almost every $t \in (0,T)$ (remind that k doesn't depend on t) and therefore we get

$$\boldsymbol{u} = \sum_{k=1}^{\infty} \vartheta_k(t)\boldsymbol{\phi}_k \in L^\infty(0,T;\, V), \quad \boldsymbol{u}' \in L^2(0,T;\, H) \tag{2.21}$$

and

(2.22) $$\|\boldsymbol{u}\|_{L^\infty(0,T;\,V)} + \|\boldsymbol{u}'\|_{L^2(0,T;\,H)} \le 2\|\boldsymbol{f}\|_{L^2(0,T;\,H)} + 2\|\boldsymbol{u}_0\|_V.$$

(2.17) yields also inequalities

$$\lambda_k^2 \vartheta_k^2(t) \le 2\mu_k^2(t) + 2{\vartheta'}_k^2(t)$$

for every $k = 1, 2, \dots$ and for almost every $t \in (0, T)$. Therefore we get

$$\sum_{k=1}^{\infty} \lambda_k^2 \int_0^t \vartheta_k^2(s)\, ds \le \sum_{k=1}^{\infty} \lambda_k^2 \int_0^T \vartheta_k^2(s)\, ds \le 2\sum_{k=1}^{\infty} \int_0^T \mu_k^2(s)\, ds + 2\sum_{k=1}^{\infty} \int_0^T {\vartheta'}_k^2(s)\, ds.$$

The last inequality and (2.20) yield

(2.23) $$\sum_{k=1}^{\infty} \lambda_k^2 \int_0^t \vartheta_k^2(s)\, ds \le \sum_{k=1}^{\infty} \lambda_k^2 \int_0^T \vartheta_k^2(s)\, ds \le 6\sum_{k=1}^{\infty} \int_0^T \mu_k^2(s)\, ds + 4\sum_{k=1}^{\infty} \vartheta_k^2(0)$$

for almost every $t \in (0, T)$. Therefore we get

(2.24) $$\boldsymbol{u} \in L^2(0, T;\, D)$$

and

(2.25) $$\|\boldsymbol{u}\|_{L^2(0,T;\,D)} \le c(\|\boldsymbol{f}\|_{L^2(0,T;\,H)} + \|\boldsymbol{u}_0\|_V).$$

The last inequality and (2.22) imply (2.14). It is easy to see that

$$\big((\boldsymbol{u}'(t), \boldsymbol{v})\big)_H + \big((\boldsymbol{u}(t), \boldsymbol{v})\big)_V = \langle \boldsymbol{f}(t), \boldsymbol{v})\big)_H$$

for every $\boldsymbol{v} \in V$ and for almost every $t \in (0, T)$ and that

$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Since $\frac{d}{dt}(\|\boldsymbol{u}(s)\|_V^2) = \sum_{k=1}^{\infty} 2\lambda_k \vartheta_k(t){\vartheta'}_k(t)$ for almost every $t \in (0, T)$ and

$$\sum_{k=1}^{\infty} 2\int_0^T |\lambda_k \vartheta_k(s){\vartheta'}_k(s)|\, ds \le$$
$$\le \lambda_k^2 \int_0^T {\vartheta_k}^2(s)\, ds + \int_0^T {\vartheta'}_k^2(s)\, ds < \infty$$

we get

(2.26) $$\|\boldsymbol{u}(.)\|_V \in \mathcal{C}([0, T]).$$

(2.21), (2.26) and the fact that V is a Hilbert space imply $\boldsymbol{u} \in \mathcal{C}([0, T];\, V)$.

Suppose that $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are two solutions of our problem. Denote $\boldsymbol{u} = \boldsymbol{u}_2 - \boldsymbol{u}_1$. Then

$$\big((\boldsymbol{u}'(t), \boldsymbol{v})\big)_H + \big((\boldsymbol{u}(t), \boldsymbol{v})\big)_V = 0$$

for every $\boldsymbol{v} \in V$ and for almost every $t \in (0, T)$ and

$$\boldsymbol{u}(0) = \boldsymbol{0}_V.$$

Then

$$\big((\boldsymbol{u}'(t), \boldsymbol{u}(t))\big)_H + \big((\boldsymbol{u}(t), \boldsymbol{u}(t))\big)_V = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_H^2 + \|\boldsymbol{u}(t)\|_V^2 = 0$$

for almost every $t \in (0,T)$. Therefore $\|\boldsymbol{u}\|^2_{L^2(0,T;\,V)} = 0$. The theorem is proved.

If $\boldsymbol{\theta},\ \boldsymbol{\psi},\ \boldsymbol{v} \in V$, then $\boldsymbol{b}(\boldsymbol{\theta},\boldsymbol{\psi},\boldsymbol{v})$ denotes the trilinear form

$$\boldsymbol{b}(\boldsymbol{\theta},\boldsymbol{\psi},\boldsymbol{v}) = \int_\Omega \theta_j \frac{\partial \psi_i}{\partial x_j} v_i \, d(\Omega). \tag{2.27}$$

Remark 2.1. *Let* $\boldsymbol{\theta},\ \boldsymbol{\psi} \in D$. *Then (2.12) yields that* $\boldsymbol{b}(\boldsymbol{\theta},\boldsymbol{\psi},.) \in H$. *If* $\boldsymbol{u},\boldsymbol{w} \in L^2(0,T^*;\, D) \cap L^\infty(0,T^*;\, V)$ *then* $\boldsymbol{b}(\boldsymbol{u},\boldsymbol{w},.) = \boldsymbol{b}(\boldsymbol{u}(t),\boldsymbol{w}(t),.) \in L^2(0,T^*;\, H)$.

Now we define a generalized formulation of the Navier-Stokes problem.

Definition 2.2. *Let* $0 < T^* \le T$, $\boldsymbol{f} \in L^2(0,T;\, H)$, $\boldsymbol{u}_0 \in V$. *Then* $\boldsymbol{u}$ *is called a generalized solution of the problem (1.1) – (1.6) on* $(0,T^*)$ *(a generalized solution of the Navier-Stokes problem with the mixed boundary conditions) with data* $\boldsymbol{f}$ *and* $\boldsymbol{u}_0$ *if* $\boldsymbol{u} \in L^2(0,T^*;\, D) \cap L^\infty(0,T^*;\, V)$, $\boldsymbol{u}' \in L^2(0,T^*;\, H)$,

$$\big(\!\big(\boldsymbol{u}'(t),\boldsymbol{v}\big)\!\big)_H + \big(\!\big(\boldsymbol{u}(t),\boldsymbol{v}\big)\!\big)_V + \boldsymbol{b}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \big(\!\big(\boldsymbol{f}(t),\boldsymbol{v}\big)\!\big)_H \tag{2.28}$$

holds for every $\boldsymbol{v} \in V$ *and for almost every* $t \in (0,T^*)$, *and*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{2.29}$$

3. THE MAIN RESULT

Our aim is to prove the following result:

Theorem 3.2. *Let* $\boldsymbol{u}_0 \in D$, $\boldsymbol{f} \in L^2(0,T;\, H)$. *Then there exists* T^* *and* $\boldsymbol{u} \in L^2(0,T^*;\, D) \cap L^\infty(0,T^*;\, V)$, $\boldsymbol{u}' \in L^2(0,T^*;\, H)$ *such that* $\boldsymbol{u}$ *is a generalized solution of the Navier-Stokes problem on* $(0,T^*)$.

Let $0 < T^* \le T$. We make use the following reflexive Banach spaces.

$$X_{1,T^*} = \big\{\boldsymbol{\varphi};\ \boldsymbol{\varphi} \in L^2(0,T^*, W^{1+2\delta,2}(\Omega)^2) \cap L^9\big(0,T^*;\, V\big)\big\}$$

and

$$X_{2,T^*} = \big\{\boldsymbol{\varphi};\ \boldsymbol{\varphi} \in L^2(0,T^*;\, D), \boldsymbol{\varphi}' \in L^2(0,T^*;\, H)\big\},$$

respectively, with norms

$$\|\boldsymbol{\varphi}\|_{X_{1,T^*}} = \|\boldsymbol{\varphi}\|_{L^2(0,T^*,W^{1+2\delta,2}(\Omega)^2)} + \|\boldsymbol{\varphi}\|_{L^9(0,T^*;V)}$$

and

$$\|\boldsymbol{\varphi}\|_{X_{2,T^*}} = \|\boldsymbol{\varphi}\|_{L^2(0,T^*;\,D)} + \|\boldsymbol{\varphi}'\|_{L^2(0,T^*;\,H)}.$$

Since

$$X_{2,T^*} \hookrightarrow\hookrightarrow L^2(0,T^*, W^{1+2\delta,2}(\Omega)^2)$$

and

$$X_{2,T^*} \hookrightarrow\hookrightarrow L^9\big(0,T^*;\, V\big)$$

the embedding

$$X_{2,T^*} \hookrightarrow\hookrightarrow X_{1,T^*} \tag{3.30}$$

holds.

Let $\boldsymbol{u}$ be a generalized solution of problem (1.1)−(1.6) with a right hand side $\boldsymbol{f} \in L^2(0,T;\, H)$ and initial condition

$$\boldsymbol{u}_0 \in D. \tag{3.31}$$

Let

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}. \tag{3.32}$$

Then $\boldsymbol{w} \in L^2(0,T;\, D) \cap L^\infty(0,T;\, V)$, $\boldsymbol{w}' \in L^2(0,T;\, H)$, the form

$$\begin{aligned}(3.33) \qquad & \big(\!\big(\boldsymbol{w}', \boldsymbol{v}\big)\!\big)_H + \big(\!\big(\boldsymbol{w}, \boldsymbol{v}\big)\!\big)_V = \\ &= \big(\!\big(\boldsymbol{f}, \boldsymbol{v}\big)\!\big)_H - \big(\!\big(\boldsymbol{u}_0, \boldsymbol{v}\big)\!\big)_V - \boldsymbol{b}(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{v}) - \boldsymbol{b}(\boldsymbol{w}, \boldsymbol{u}_0, \boldsymbol{v}) - \boldsymbol{b}(\boldsymbol{u}_0, \boldsymbol{w}, \boldsymbol{v}) - \boldsymbol{b}(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v})\end{aligned}$$

holds for every $\boldsymbol{v} \in V$ and for almost every $t \in (0,T)$ and

$$\boldsymbol{w}(0) = 0. \tag{3.34}$$

Let $\boldsymbol{F} : X_{1,T^*} \to L^2(0,T;\, H)$ be an operator such that

$$\begin{aligned}\big(\!\big(\boldsymbol{F}(\boldsymbol{\phi}), \boldsymbol{v}\big)\!\big)_H = \big(\!\big(\boldsymbol{F}(\boldsymbol{\phi})(t), \boldsymbol{v}\big)\!\big)_H &= \big(\!\big(\boldsymbol{f}(t), \boldsymbol{v}\big)\!\big)_H - \big(\!\big(\boldsymbol{u}_0, \boldsymbol{v}\big)\!\big)_V - \boldsymbol{b}(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{v}) - \\ &- \boldsymbol{b}(\boldsymbol{\phi}(t), \boldsymbol{u}_0, \boldsymbol{v}) - \boldsymbol{b}(\boldsymbol{u}_0, \boldsymbol{\phi}(t), \boldsymbol{v}) - \boldsymbol{b}(\boldsymbol{\phi}(t), \boldsymbol{\phi}(t), \boldsymbol{v}).\end{aligned}$$

Remark 3.2. *Note that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0$ is a generalized solution of the problem (1.1)−(1.6) if and only if the equality*

$$\big(\!\big(\boldsymbol{w}', \boldsymbol{v}\big)\!\big)_H + \big(\!\big(\boldsymbol{w}, \boldsymbol{v}\big)\!\big)_V = \big(\!\big(\boldsymbol{F}(\boldsymbol{w}), \boldsymbol{v}\big)\!\big)_H \tag{3.35}$$

holds for every $\boldsymbol{v} \in V$ and for almost every $t \in (0,T)$ and (3.34) holds.

We prove the following lemma:

Lemma 3.1. *$\boldsymbol{F}$ is a continuous operator from X_{1,T^*} into $L^2(0,T^*;\, H)$. Moreover there exists $K > 0$ such that the inequality*

$$\|\boldsymbol{F}(\boldsymbol{\varphi})\|_{L^2(0,T^*;\, H)} \le c_2 (T^*)^{1/12} \|\boldsymbol{\varphi}\|^2_{X_{1,T^*}} + c_3 (T^*)^{1/6} \|\boldsymbol{\varphi}\|_{X_{1,T^*}} + K \tag{3.36}$$

holds for every $\boldsymbol{\varphi} \in X_{1,T^*}$.

Proof of Lemma 3.1: It is easy to see that there exists $K > 0$ such that

$$\| \big(\!\big(\boldsymbol{f}(t), .\big)\!\big)_H - \big(\!\big(\boldsymbol{u}_0, \boldsymbol{v}\big)\!\big)_V - \boldsymbol{b}(\boldsymbol{u}_0, \boldsymbol{u}_0, \boldsymbol{v}) \|_{L^2(0,T^*;\, H)} \le K. \tag{3.37}$$

Further the inequality

$$\begin{aligned}(3.38) \qquad \|\boldsymbol{b}(\boldsymbol{\varphi}, \boldsymbol{u}_0, .)\|_{L^2(0,T^*;\, H)} &\le \|\boldsymbol{u}_0\|_{W^{1,2+2\delta}(\Omega)^2} \Big(\int_0^{T^*} \|\boldsymbol{\varphi}\|^2_{L^p(\Omega)^2} \Big)^{1/2} \le \\ &\le \|\boldsymbol{u}_0\|_{W^{1,2+2\delta}(\Omega)^2} \Big(\int_0^{T^*} \|\boldsymbol{\varphi}\|^3_{L^p(\Omega)^2} \Big)^{1/3} (T^*)^{1/6} \le \\ &\le c \|\boldsymbol{u}_0\|_{W^{1,2+2\delta}(\Omega)^2} \|\boldsymbol{\varphi}\|_{X_{1,T^*}} (T^*)^{1/6}\end{aligned}$$

holds for sufficiently large p. Similarly we obtain the inequality

$$\begin{aligned}(3.39)\qquad &\|\boldsymbol{b}(\boldsymbol{u}_0,\boldsymbol{\varphi},.)\|_{L^2(0,T^*;\,H)} \le \|\boldsymbol{u}_0\|_{L^\infty(\Omega)^2}\Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_V^2\Big)^{1/2} \le\\ &\le \|\boldsymbol{u}_0\|_{W^{1,2+2\delta}(\Omega)^2}\Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_V^9\Big)^{1/9}(T^*)^{7/9} \le\\ &\le c\|\boldsymbol{u}_0\|_{W^{1,2+2\delta}(\Omega)^2}\|\boldsymbol{\varphi}\|_{X_{1,T^*}}(T^*)^{7/9}.\end{aligned}$$

Since

$$\|\boldsymbol{\varphi}\|_{W^{1,2+\delta}(\Omega)^2} \le \|\boldsymbol{\varphi}\|_V^{1/2}\|\boldsymbol{\varphi}\|_{W^{1,2+2\delta}(\Omega)^2}^{1/2}$$

we get the inequality

$$\begin{aligned}(3.40)\quad &\|\boldsymbol{b}(\boldsymbol{\varphi},\boldsymbol{\varphi},.)\|_{L^2(0,T^*;\,H)} \le \Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_V^3\|\boldsymbol{\varphi}\|_{W^{1,2+2\delta}(\Omega)^2}\Big)^{1/2} \le\\ &\Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_{W^{1,2+2\delta}(\Omega)^2}^2\Big)^{1/4}\Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_V^9\Big)^{1/6}(T^*)^{1/6} \le \|\boldsymbol{\varphi}\|_{X_{1,T^*}}^2(T^*)^{1/12}.\end{aligned}$$

The inequalities (3.37)$-$(3.40) yield $\boldsymbol{F}(\boldsymbol{\varphi}) \in L^2(0,T;\,H)$ and the inequality (3.36).

Let $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in X_{1,T^*}$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_1$. Then

$$\boldsymbol{F}(\boldsymbol{\varphi}_2) - \boldsymbol{F}(\boldsymbol{\varphi}_1) = \boldsymbol{b}(\boldsymbol{\varphi},\boldsymbol{u}_0,.) + \boldsymbol{b}(\boldsymbol{u}_0,\boldsymbol{\varphi},.) + \boldsymbol{b}(\boldsymbol{\varphi}_2,\boldsymbol{\varphi},.) + \boldsymbol{b}(\boldsymbol{\varphi},\boldsymbol{\varphi}_1,.)$$

and

(3.41)

$$\begin{aligned}&\|\boldsymbol{b}(\boldsymbol{\varphi}_2,\boldsymbol{\varphi},.)\|_{L^2(0,T^*;\,H)} \le \Big(\int_0^{T^*}\|\boldsymbol{\varphi}_2\|_V^2\|\boldsymbol{\varphi}\|_{W^{1,2+\delta}(\Omega)^2}^2\Big)^{1/2} \le\\ &\le \Big(\int_0^{T^*}\|\boldsymbol{\varphi}_2\|_V^2\|\boldsymbol{\varphi}\|_{W^{1,2+2\delta}(\Omega)^2}\|\boldsymbol{\varphi}\|_V\Big)^{1/2} \le\\ &\le \Big(\int_0^{T^*}\|\boldsymbol{\varphi}_2\|_V^9\Big)^{1/9}\Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_V^9\Big)^{1/18}\Big(\int_0^{T^*}\|\boldsymbol{\varphi}\|_{W^{1,2+\delta}(\Omega)^2}^2\Big)^{1/4}(T^*)^{1/6} \le\\ &\le \|\boldsymbol{\varphi}\|_{X_{1,T^*}}\|\boldsymbol{\varphi}_2\|_{X_{1,T^*}}(T^*)^{1/6}.\end{aligned}$$

Similarly

$$(3.42)\qquad \|\boldsymbol{b}(\boldsymbol{\varphi},\boldsymbol{\varphi}_1,.)\|_{L^2(0,T^*;\,H)} \le \|\boldsymbol{\varphi}\|_{X_{1,T^*}}\|\boldsymbol{\varphi}_1\|_{X_{1,T^*}}.$$

Inequalities (3.38), (3.39), (3.41) and (3.42) yield that $\boldsymbol{F}$ is a continuous operator from X_{1,T^*} into $L^2(0,T^*;\,H)$. The proof is complete.

Definition 3.3. *Let $\mathcal{T} : X_{1,T^*} \to X_{2,T^*}$ be an operator such that $\mathcal{T}(\boldsymbol{\varphi}) = \boldsymbol{w}$ if and only if*

$$(3.43)\qquad \big((\boldsymbol{w}',\boldsymbol{v})\big)_H + \big((\boldsymbol{w},\boldsymbol{v})\big)_V = \big((\boldsymbol{F}(\boldsymbol{\varphi}),\boldsymbol{v})\big)_H,$$

holds for every $\boldsymbol{v} \in V$ and $\boldsymbol{w}(0) = 0$.

Lemma 3.2. *The operator $\mathcal{T}$ is a continuous operator from X_{1,T^*} into X_{2,T^*}. Moreover,*

$$c_1\|\mathcal{T}(\boldsymbol{\varphi})\|_{X_{1,T^*}} \leq \|\mathcal{T}(\boldsymbol{\varphi})\|_{X_{2,T^*}} \leq c_2\|\boldsymbol{F}(\boldsymbol{\varphi})\|_{L^2(0,T^*;\,H)}. \tag{3.44}$$

Proof of Lemma 3.2: Inequality (2.14) and Lemma 3.1 imply that $\mathcal{T}$ is a continuous operator from X_{1,T^*} into X_{2,T^*}.

Proof of Theorem 3.2: Let

$$B_R = \{\boldsymbol{\varphi} \in X_{1,T^*};\ \|\boldsymbol{\varphi}\|_{X_{1,T^*}} \leq R\}.$$

Lemma 3.1 and Lemma 3.2 imply that for a sufficiently small T^* and for a sufficiently large R, $\mathcal{T}$ maps B_R into itself. By Lemma 3.2 and (3.30), $\mathcal{T}$ is totally continuous operator from X_{1,T^*} into X_{1,T^*}. Moreover, the Banach space X_{1,T^*} is reflexive. Therefore there exists $\boldsymbol{w} \in B_R$ such that $\mathcal{T}(\boldsymbol{w}) = \boldsymbol{w}$. Set $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0$. By Remark 3.2, $\boldsymbol{u}$ is a generalized solution of the problem (1.1)$-$(1.6). The theorem is proved.

Acknowledgement. The research was supported by the grant CTU0600111 of the Czech technical university (author one), by the Grant Agency of the Czech Academy of Sciences (grant No. IAA100190612) (author two) and by the research plan of the Ministry of Education of the Czech Republic No. MSM 6840770010 (author two).

References

[1] Beneš M.: The qualitative properties of the Stokes and Navier-Stokes system for the mixed boundary problem in a nonsmooth domain. Accepted in journal Mathematics and Computers in Simulation.

[2] Glowinski R.: Numerical methods for nonlinear variational problems. Springer Verlag, Berlin-Heidelberg-Tokio-New York, 1984.

[3] Kračmar S., Neustupa J.: Global existence of weak solutions of a nonsteady variational inequalities of the Navier-Stokes type with mixed boundary conditions. Proc. of the conference ISNA'92, August-September 1992, Part III, Publ. Centre of the Charles Univ., Prague, 156-157.

[4] Kračmar S., Neustupa J.: Modelling of flows of a viscous incompressible fluid through a channel by means of variational inequalities. ZAMM 74, 1994, 6, 637-639.

[5] Kračmar S., Neustupa J.: Some Initial Boundary Value Problems of the Navier-Stokes Type with Mixed Boundary Conditions. Proc. of the seminar Numerical Mathematics in Theory and Practise, Pilsen, 1993.

[6] Kračmar S., Neustupa J.: Simulation of Steady Flows through Channels by Variational Inequalities. Proc. of the conference Numerical Modelling in Continuum Mechanics. Prague 1994, publ.1995.

[7] Kračmar S., Neustupa J.: A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions. Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000). Nonlinear Anal. 47 (2001), no. 6, 4169–4180.

[8] Kučera P., Skalák Z.: Solutions to the Navier-Stokes Equations with Mixed Boundary Conditions. Acta Applicandae Mathematicae, 275-288, Kluver Academic Publishers, 1998, vol. 54, no. 3.

[9] Kučera P.: Solution of the Stationary Navier-Stokes Equations with Mixed Boundary Conditions in a Bounded Domain. Conference on Analysis, Numerics and Applications of Differential and Integral Equations, University of Stuttgart, october 1996, (1998), Pitman Research Notes in Mathematics Series n.379, Longman, 127-131

[10] KUČERA P.: A structure of the set of critical points to the Navier-Stokes equations with mixed boundary conditions. Proceedings of the conference Navier-Stokes equations: theory and numerical methods (Varenna, 1997), Pitman Res. Notes Math. Ser., 388, Longman, Harlow, (1998), 201–205. Editor: R.Salvi.

[11] KUFNER A., JOHN O., FUČÍK S.: Function spaces. Academia, Prague, 1977.

[12] LIONS J. L., MAGENES E.: Non-Homogeneous Boundary Value Problems and Applications. Vols. I and II., Springer-Verlag, Berlin 1972.

[13] NEČAS J., HLAVÁČEK I.: Mathematical theory of elastic and elastico-plastic bodies: An introduction. Elsevier 1981, Amsterdam, Oxford, New York.

[14] RANNACHER R.: Numerical analysis of the Navier-Stokes equations. Proceedings of conference ISNA'92, part I., 1992, 361-380.

[15] TAYLOR A. E.: Introduction to functional analysis. Wiley, 1958.

[16] TEMAM R.: Navier-Stokes Equations, theory and numerical analysis, North-Holland Publishinged edition, Company, Amsterodam, New York, Oxford. Revis (1979).

CZECH TECHNICAL UNIVERSITY IN PRAGUE
DEPARTMENT OF MATHEMATICS
THÁKUROVA 7, 166 29 PRAGUE 6, CZECH REPUBLIC
E-mail address: benes@mat.fsv.cvut.cz, kucera@mat.fsv.cvut.cz