Non-steady Navier-Stokes Equations with Homogeneous Mixed Boundary Conditions and Arbitrarily Large Initial Condition

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ABSTRACT. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\partial \Omega \in C^{0,1}$ and $\partial \Omega = \Gamma_1 \cup \Gamma_2$ such that $\Gamma_1$ and $\Gamma_2$ are closed, sufficiently smooth, 1-dimensional measure of $\Gamma_1 \cap \Gamma_2$ is zero and 1-dimensional measure of $\Gamma_1$ is positive. Further let $(0, T)$ be a time interval. We prescribe the non-slip boundary conditions on $\Gamma_1 \times (0, T)$ and the boundary condition 

$$-P n + \frac{\partial u}{\partial n} = 0$$

on $\Gamma_2 \times (0, T)$. Here $u = (u_1, u_2)$ is velocity, $P$ represents pressure and $n = (n_1, n_2)$ is an outer normal vector.

Our aim is to prove the existence and uniqueness of this problem on some time interval $(0, T^* )$ for sufficiently small $T^*, 0 < T^* \leq T$.

1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary, $\partial \Omega \in C^{0,1}$ and let $\Gamma_1, \Gamma_2$ be open disjoint subsets of $\partial \Omega$ such that $\partial \Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}, \Gamma_1 \neq \emptyset$ and the 1-dimensional measure of $\partial \Omega - (\Gamma_1 \cup \Gamma_2)$ is zero. The domain $\Omega$ represents a channel filled up with a fluid, $\Gamma_1$ is a fixed wall and $\Gamma_2$ is both the input and the output of the channel.

The authors in [2] and [14] use the Neumann condition on the output of the channel. Some qualitative properties of the Navier-Stokes equations with the mixed boundary conditions are studied in [3], [4], [5], [6], [7], [8], [9], [10].

Let $T \in (0, \infty], Q = \Omega \times (0, T)$. The classical formulation of our problem is as follows:

1.1

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla P = g \quad \text{in } Q,$$

1.2

$$\text{div } u = 0 \quad \text{in } Q,$$

1.3

$$u = 0 \quad \text{in } \Gamma_1 \times (0, T),$$

1.4

$$-P n + \nu \frac{\partial u}{\partial n} = 0 \quad \text{in } \Gamma_2 \times (0, T),$$

1.5

$$u(0) = \gamma \quad \text{in } \Omega,$$

1.6

$$\gamma = 0 \quad \text{on } \Gamma_1.$$
Functions $u$, $P$, $g$, $\gamma$ are smooth enough, $u = (u_1, u_2)$ is velocity, $P$ represents pressure, $\nu$ denotes the viscosity, $g$ is a body force and $n = (n_1, n_2)$ is an outer normal vector. The problem (1.1) - (1.6) will be called the nonsteady Navier–Stokes problem with the mixed boundary conditions. For simplicity we suppose that $\nu = 1$ throughout this paper.

We solve also the problem, in which (1.2) - (1.6) hold and (1.1) is replaced with

$$
\frac{\partial u}{\partial t} - \Delta u + \nabla P = g \quad \text{in } Q.
$$

The problem (1.2) - (1.6) and (1.7) will be called the nonsteady Stokes problem with the mixed boundary conditions.

The Dirichlet boundary condition (1.3) expresses a non-slip behaviour of the fluid on fixed walls of the channel.

Our aim is to prove the existence and uniqueness of this problem on some time interval $(0, T^*)$ for sufficiently small $T^*$, $0 < T^* \leq T$.

2. Definition of some function spaces and generalized formulation of the problem

We shall denote by $c$ a generic constant, i.e. a constant whose value may change from line to line. On the other hand, numbered constants will have a fixed value throughout the paper. Constants will always depend only on domain $\Omega$.

Let

$$
\mathcal{E}(\Omega) = \{ u \in C^{\infty}(\overline{\Omega})^2; \text{div} u \equiv 0, \text{supp} u \cap \Gamma_2 \equiv \emptyset \}.
$$

Let $V^{k,p}$ be a closure of $\mathcal{E}(\Omega)$ in the norm of $W^{k,p}(\Omega)^2$, $k \geq 0$ ($k$ need not be an integer) and $1 \leq p < \infty$. Then $V^{k,p}$ is a Banach space with the norm of the space $W^{k,p}(\Omega)^2$. For simplicity, we denote $V^{1,2}$ and $V^{0,2}$, respectively, as $V$ and $H$. Note, that $V$ and $H$, respectively, are Hilbert spaces with scalar products $((., .))_V$ and $((., .))_H$,

$$
((., .))_V = \left( ((\Phi, \Psi))_V \right) = \int_{\Omega} \nabla \Phi \cdot \nabla \Psi \ d(\Omega) = \int_{\Omega} \frac{\partial \Phi}{\partial x_j} \frac{\partial \Psi}{\partial x_j} \ d(\Omega)
$$

and

$$
((., .))_H = \left( ((\Phi, \Psi))_H \right) = \int_{\Omega} \Phi \cdot \Psi \ d(\Omega) = \int_{\Omega} \Phi \Psi_i \ d(\Omega)
$$

and they are closed subspaces of spaces $W^{1,2}(\Omega)^2$ and $L^2(\Omega)^2$.

Let

$$
D = \{ w \in V; \text{there exists } f \in H \text{ such that } ((w, v))_V = ((f, v))_H \text{ for every } v \in V \}
$$

and

$$
\|w\|_D = \|f\|_H,
$$

where $w$, $f$ are corresponding functions via (2.8). Let $w_i$ and $f_i$ be corresponding functions via (2.8). Note that $D$ is the Hilbert space with the scalar product $((., .))_D$ such that

$$
((w_1, w_2))_D = ((f_1, f_2))_H.
$$
Similarly it can be shown as in ([16], Chapter I, 2.6) that there exist functions
φ₁, φ₂, . . . , φₖ, . . . ∈ \( V \subset H \) and real positive numbers λ₁, λ₂, . . . , λₖ, . . . → ∞ for
\( k \to \infty \), such that
\[
((φₖ, v))_V = λₖ((φₖ, v))_H
\]
for every \( v ∈ V \). φ₁, φ₂, . . . is a system that is complete in both \( H \) and \( V \), or-
thonormal in \( H \) and orthogonal in \( V \). Note that
(2.9) \( H = \{ v; v = \sum_{k=1}^{∞} a_k φ_k, a_k ∈ \mathbb{R} \text{ and } \sum_{k=1}^{∞} a_k^2 < \infty \} \),
(2.10) \( V = \{ v; v = \sum_{k=1}^{∞} a_k φ_k, a_k ∈ \mathbb{R} \text{ and } \sum_{k=1}^{∞} λ_k a_k^2 < \infty \} \)
and
(2.11) \( D = \{ v; v = \sum_{k=1}^{∞} a_k φ_k, a_k ∈ \mathbb{R} \text{ and } \sum_{k=1}^{∞} λ_k^2 a_k^2 < \infty \} \).

Let \( X \) be an arbitrary Banach space and \( p ∈ [1, \infty) \). As usual \( L^p(α, β; X) \) and \( L^∞(α, β; X) \)
(for \( -∞ ≤ α < β ≤ ∞ \)) denote the Banach spaces
\[
\{ φ; φ(t) ∈ X \text{ for almost every } t ∈ (α, β), \int_α^β \|φ(t)\|^p_X \, dt < \infty \}
\]
and
\[
\{ φ; φ(t) ∈ X \text{ for almost every } t ∈ (α, β), \text{ ess sup }_{t ∈ (α, β)} \|φ(t)\|_X < \infty \}
\]
with the norms
\[
\|φ\|_{L^p(α, β; X)} = \left( \int_α^β \|φ(t)\|^p_X \, dt \right)^{1/p}
\]
and
\[
\|φ\|_{L^∞(α, β; X)} = \text{ ess sup }_{t ∈ (α, β)} \|φ(t)\|_X.
\]
Note that ([111]) and ([11]) yield existence of \( γ, 1 < γ < 2 \) such that
(2.12) \( D ↪ \hookrightarrow W^{γ, 2}(Ω) \hookrightarrow W^{1,2+2δ}(Ω) \),
where \( δ = \frac{γ - 1}{2γ} \).

**Definition 2.1.** Let \( 0 < T^* ≤ T, f ∈ L^2(0, T^*; H), u_0 ∈ V \). Then \( u \) is called a
generalized solution of the Stokes problem with the mixed boundary conditions and with
data \( f \) and \( u_0 \) on \( (0, T^*) \) (problem (1.2)–(1.7) for \( T = T^* \)) if \( u ∈ L^2(0, T^*; D) \cap L^∞(0, T^*; V), u' ∈ L^2(0, T^*; H) \) and
(2.13) \( ((u'(t), v))_H + ((u(t), v))_V = ((f(t), v))_H \)
holds for every \( v ∈ V \) and for almost every \( t ∈ (0, T^*) \).
Theorem 2.1. Let \( \mathbf{f} \in L^2(0, T; H) \), \( \mathbf{u}_0 \in V \). There exists a unique generalized solution \( \mathbf{u} \) of the Stokes problem with the mixed boundary conditions and with data \( \mathbf{f} \) and \( \mathbf{u}_0 \) on \( (0, T) \). Moreover \( \mathbf{u} \in C([0, T]; V) \) and

\[
\|\mathbf{u}\|_{L^2(0, T; D)} + \|\mathbf{u}\|_{L^\infty(0, T; V)} + \|\mathbf{u}'\|_{L^2(0, T; H)} \leq c_1 (\|\mathbf{f}\|_{L^2(0, T; H)} + \|\mathbf{u}_0\|_V).
\]

Proof: Since \( \mathbf{f} \in L^2(0, T; H) \) and \( \mathbf{u}_0 \in V \), we have

\[
\mathbf{f} = \sum_{k=1}^{\infty} \mu_k(t) \phi_k, \quad \mathbf{u}_0 = \sum_{k=1}^{\infty} a_k \phi_k,
\]

where

\[
\sum_{k=1}^{\infty} \int_0^T \mu_k^2(t) \, dt + \sum_{k=1}^{\infty} a_k^2 < \infty.
\]

Let \( \vartheta_k \) be a solution of the ordinary differential equation

\[
\vartheta_k'(t) + \lambda_k \vartheta_k(t) = \mu_k(t)
\]

with the initial condition

\[
\vartheta_k(0) = a_k
\]

for \( k = 1, 2, \ldots \). Then

\[
\vartheta_k(t) = \int_0^t e^{\lambda_k(s-t)} \mu_k(s) \, ds + a_k e^{-\lambda_k t}
\]

for almost every \( t \in (0, T) \). Hence \( \vartheta_k \in W^{1,2}((0, t)) \).

Multiplying (2.17) by \( 2 \vartheta_k' \) and integrating over \( (0, t) \) we get

\[
2 \int_0^t \vartheta_k'^2(s) \, ds + \lambda_k \vartheta_k^2(t) = \lambda_k \vartheta_k^2(0) + 2 \int_0^t \mu_k(s) \vartheta_k'(s) \, ds \leq \\
\leq \vartheta_k^2(0) + \int_0^t \vartheta_k'^2(s) \, ds + \int_0^t \mu_k^2(s) \, ds
\]

for \( k = 1, 2, \ldots \) and for almost every \( t \in (0, T) \) and therefore

\[
\int_0^t \vartheta_k'^2(s) \, ds + \lambda_k \vartheta_k^2(t) \leq \vartheta_k^2(0) + \int_0^t \mu_k^2(s) \, ds.
\]

Thus (2.19) yields

\[
\sum_{k=1}^{\infty} \int_0^t \vartheta_k'^2(s) \, ds + \sum_{k=1}^{\infty} \lambda_k \vartheta_k^2(t) \leq \sum_{k=1}^{\infty} \int_0^T \vartheta_k'^2(s) \, ds + \sum_{k=1}^{\infty} \lambda_k \vartheta_k^2(t) \leq \\
2 \sum_{k=1}^{\infty} \vartheta_k^2(0) + 2 \sum_{k=1}^{\infty} \int_0^T \mu_k^2(s) \, ds
\]

for almost every \( t \in (0, T) \) (remind that \( k \) doesn’t depend on \( t \)) and therefore we get

\[
\mathbf{u} = \sum_{k=1}^{\infty} \vartheta_k(t) \phi_k \in L^\infty(0, T; V), \quad \mathbf{u}' \in L^2(0, T; H)
\]
and
\[ \|u\|_{L^\infty(0,T;V)} + \|u'\|_{L^2(0,T;H)} \leq 2\|f\|_{L^2(0,T;H)} + 2\|u_0\|_V. \]  

(2.22) yields also inequalities
\[ \lambda_k^2 \vartheta_k^2(t) \leq 2\mu_k^2(t) + 2\vartheta_k^2(t) \]
for every \( k = 1, 2, \ldots \) and for almost every \( t \in (0, T) \). Therefore we get
\[ \sum_{k=1}^\infty \lambda_k^2 \int_0^t \vartheta_k^2(s) \, ds \leq \frac{2}{\lambda_k^2} \int_0^T \vartheta_k^2(s) \, ds \leq 2 \sum_{k=1}^\infty \mu_k^2(s) \, ds + 2 \int_0^T \vartheta_k^2(s) \, ds. \]

The last inequality and (2.20) yield
\[ \sum_{k=1}^\infty \lambda_k^2 \int_0^t \vartheta_k^2(s) \, ds \leq 6 \sum_{k=1}^\infty \mu_k^2(s) \, ds + 4 \sum_{k=1}^\infty \vartheta_k^2(t) \]
for almost every \( t \in (0, T) \). Therefore we get
\[ \|u\|_{L^2(0,T;D)} \leq c(\|f\|_{L^2(0,T;H)} + \|u_0\|_V). \]

The last inequality and (2.22) imply (2.14). It is easy to see that
\[ ((u'(t), v))_H + ((u(t), v))_V = \langle f(t), v \rangle_H \]
for every \( v \in V \) and for almost every \( t \in (0, T) \) and that
\[ u(0) = u_0. \]

Since \( \frac{d}{dt}(\|u(s)\|_V^2) = \sum_{k=1}^\infty 2\lambda_k \vartheta_k(t) \vartheta_k(t) \) for almost every \( t \in (0, T) \) and
\[ \sum_{k=1}^\infty 2 \int_0^T |\lambda_k \vartheta_k(s) \vartheta_k(s)| \, ds \leq \lambda_k^2 \int_0^T \vartheta_k^2(s) \, ds + \int_0^T \vartheta_k^2(s) \, ds < \infty \]
we get
\[ \|u(.)\|_V \in C([0,T]). \]

(2.21), (2.26) and the fact that \( V \) is a Hilbert space imply \( u \in C([0,T]; V) \).

Suppose that \( u_1, u_2 \) are two solutions of our problem. Denote \( u = u_2 - u_1 \).

Then
\[ ((u'(t), v))_H + ((u(t), v))_V = 0 \]
for every \( v \in V \) and for almost every \( t \in (0, T) \) and
\[ u(0) = 0_V. \]

Then
\[ ((u'(t), u(t)))_H + ((u(t), u(t)))_V = \frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 + \|u(t)\|_V^2 = 0 \]
for almost every $t \in (0, T)$. Therefore $\|u\|_{L^2(0,T; V)}^2 = 0$. The theorem is proved.

If $\theta, \psi, v \in V$, then $b(\theta, \psi, v)$ denotes the trilinear form
\begin{equation}
(2.27) \quad b(\theta, \psi, v) = \int_{\Omega} \theta_j \frac{\partial \psi_i}{\partial x_j} v_i \, d(\Omega).
\end{equation}

Remark 2.1. Let $\theta, \psi \in D$. Then (2.12) yields that $b(\theta, \psi, ,) \in H$. If $u, w \in L^2(0, T^*; D) \cap L^\infty(0, T^*; V)$ then $b(u, w, ,) = b(u(t), w(t), ,) \in L^2(0, T^*; H)$.

Now we define a generalized formulation of the Navier-Stokes problem.

Definition 2.2. Let $0 < T^* \leq T$, $f \in L^2(0, T; H)$, $u_0 \in V$. Then $u$ is called a generalized solution of the problem (1.1) – (1.6) on $(0, T^*)$ (a generalized solution of the Navier-Stokes problem with the mixed boundary conditions) with data $f$ and $u_0$ if $u \in L^2(0, T^*; D) \cap L^\infty(0, T^*; V)$, $u' \in L^2(0, T^*; H)$,
\begin{equation}
(2.28) \quad ((u'(t), v))_H + ((u(t), v))_V + b(u, u, v) = ((f(t), v))_H
\end{equation}
holds for every $v \in V$ and for almost every $t \in (0, T^*)$, and
\begin{equation}
(2.29) \quad u(0) = u_0.
\end{equation}

3. The main result

Our aim is to prove the following result:

Theorem 3.2. Let $u_0 \in D$, $f \in L^2(0, T; H)$. Then there exists $T^*$ and $u \in L^2(0, T^*; D) \cap L^\infty(0, T^*; V)$, $u' \in L^2(0, T^*; H)$ such that $u$ is a generalized solution of the Navier-Stokes problem on $(0, T^*)$.

Let $0 < T^* \leq T$. We make use the following reflexive Banach spaces.

\[ X_{1, T^*} = \{ \varphi; \varphi \in L^2(0, T^*, W^{1+2\delta^2}(\Omega))^2 \cap L^9(0, T^*; V) \} \]

and
\[ X_{2, T^*} = \{ \varphi; \varphi \in L^2(0, T^*, D), \varphi' \in L^2(0, T^*; H) \} \],

respectively, with norms
\[ \| \varphi \|_{X_{1, T^*}} = \| \varphi \|_{L^2(0, T^*, W^{1+2\delta^2}(\Omega))^2} + \| \varphi \|_{L^9(0, T^*; V)} \]

and
\[ \| \varphi \|_{X_{2, T^*}} = \| \varphi \|_{L^2(0, T^*, D)} + \| \varphi' \|_{L^2(0, T^*, H)}. \]

Since
\[ X_{2, T^*} \hookrightarrow L^2(0, T^*, W^{1+2\delta^2}(\Omega))^2 \]
and
\[ X_{2, T^*} \hookrightarrow L^9(0, T^*; V) \]
the embedding
\begin{equation}
(3.30) \quad X_{2, T^*} \hookrightarrow X_{1, T^*}
\end{equation}
holds.

Let \( u \) be a generalized solution of problem (1.1)–(1.6) with a right hand side \( f \in L^2(0, T; H) \) and initial condition
\[
(3.31) \quad u_0 \in D.
\]

Let
\[
(3.32) \quad u = u_0 + w.
\]

Then \( w \in L^2(0, T; D) \cap L^\infty(0, T; V) \), \( w' \in L^2(0, T; H) \), the form
\[
(3.33) \quad (\langle w', v \rangle \rangle_H + (\langle w, v \rangle \rangle_V =
\]
holds for every \( v \in V \) and for almost every \( t \in (0, T) \) and
\[
(3.34) \quad w(0) = 0.
\]

Let \( F : X_{1,T^*} \to L^2(0, T; H) \) be an operator such that
\[
(\langle F(\phi), v \rangle H = (\langle F(\phi)(t), v \rangle H =
\]
holds for every \( v \in V \) and for almost every \( t \in (0, T) \) and (3.34) holds.

We prove the following lemma:

**Lemma 3.1.** \( F \) is a continuous operator from \( X_{1,T^*} \) into \( L^2(0, T^* ; H) \). Moreover there exists \( K > 0 \) such that the inequality
\[
(3.35) \quad \|F(\phi)\|_{L^2(0, T^* ; H)} \leq c_2(T^*)^{1/2}\|\phi\|_{X_{1,T^*}}^2 + c_3(T^*)^{1/6}\|\phi\|_{X_{1,T^*}} + K
\]
holds for every \( \phi \in X_{1,T^*} \).

**Proof of Lemma 3.1:** It is easy to see that there exists \( K > 0 \) such that
\[
(3.36) \quad \|\langle f(t), v \rangle \rangle_H - (\langle u_0, v \rangle \rangle_V - b(u_0, u_0, v)\|_{L^2(0, T^* ; H)} \leq K.
\]

Further the inequality
\[
(3.37) \quad b(\phi, u_0, )\|_{L^2(0, T^* ; H)} \leq \|u_0\|_{W^{1,2+2k}(\Omega)^2} \left( \int_0^{T^*} \|\phi\|_{L^p(\Omega)^2}^2 \right)^{1/2} \leq
\]
\[
\leq \|u_0\|_{W^{1,2+2k}(\Omega)^2} \left( \int_0^{T^*} \|\phi\|_{L^p(\Omega)^2}^3 \right)^{1/3} (T^*)^{1/6} \leq
\]
\[
\leq c\|u_0\|_{W^{1,2+2k}(\Omega)^2} \|\phi\|_{X_{1,T^*}} (T^*)^{1/6}
\]
Let \( (3.39) \) holds for sufficiently large \( p \). Similarly we obtain the inequality
\[
(3.39) \quad \|b(u_0, \varphi, \cdot)\|_{L^2(0, T^*; H)} \leq \|u_0\|_{L^\infty(\Omega)^2} \left( \int_0^{T^*} \|\varphi\|^2_V \right)^{1/2} \leq \|u_0\|_{W^{1,2+\delta}(\Omega)^2} \left( \int_0^{T^*} \|\varphi\|^2_V \right)^{1/9} (T^*)^{7/9} \leq c\|u_0\|_{W^{1,2+\delta}(\Omega)^2} \|\varphi\|_{X_{1,T^*}} (T^*)^{7/9}.
\]

Similarly we get the inequality
\[
(3.40) \quad \|b(\varphi, \varphi, \cdot)\|_{L^2(0, T^*; H)} \leq \left( \int_0^{T^*} \|\varphi\|^2_V \|\varphi\|_{W^{1,2+\delta}(\Omega)^2} \right)^{1/2} \leq \left( \int_0^{T^*} \|\varphi\|^2_V \right)^{1/4} \left( \int_0^{T^*} \|\varphi\|^2_V \right)^{1/6} (T^*)^{1/6} \leq \|\varphi\|_{X_{1,T^*}} (T^*)^{1/12}.
\]

The inequalities (3.37)–(3.40) yield \( F(\varphi) \in L^2(0, T^*; H) \) and the inequality (3.36).

Let \( \varphi_1, \varphi_2 \in X_{1,T^*} \) and \( \varphi = \varphi_2 - \varphi_1 \). Then
\[
F(\varphi_2) - F(\varphi_1) = b(\varphi, u_0, \cdot) + b(u_0, \varphi, \cdot) + b(\varphi_2, \varphi, \cdot) + b(\varphi, \varphi_1, \cdot)
\]
and
\[
(3.41) \quad \|b(\varphi_2, \varphi, \cdot)\|_{L^2(0, T^*; H)} \leq \left( \int_0^{T^*} \|\varphi_2\|^2_V \|\varphi\|_{W^{1,2+\delta}(\Omega)^2} \|\varphi\|_{V} \right)^{1/2} \leq \left( \int_0^{T^*} \|\varphi_2\|^2_V \right)^{1/9} \left( \int_0^{T^*} \|\varphi_2\|^2_V \right)^{1/18} \left( \int_0^{T^*} \|\varphi\|_{W^{1,2+\delta}(\Omega)^2} \right)^{1/4} (T^*)^{1/6} \leq \|\varphi\|_{X_{1,T^*}} \|\varphi_2\|_{X_{1,T^*}} (T^*)^{1/6}.
\]

Similarly
\[
(3.42) \quad \|b(\varphi, \varphi_2, \cdot)\|_{L^2(0, T^*; H)} \leq \|\varphi\|_{X_{1,T^*}} \|\varphi_2\|_{X_{1,T^*}}.
\]

Inequalities (3.38), (3.39), (3.41) and (3.42) yield that \( F \) is a continuous operator from \( X_{1,T^*} \) into \( L^2(0, T^*; H) \). The proof is complete.

**Definition 3.3.** Let \( T : X_{1,T^*} \rightarrow X_{2,T^*} \) be an operator such that \( T(\varphi) = w \) if and only if
\[
((w', v), \varphi)_H + ((w, v), \varphi)_V = ((F(\varphi), v), \varphi)_H,
\]
holds for every \( v \in V \) and \( w(0) = 0 \).
Lemma 3.2. The operator $T$ is a continuous operator from $X_{1,T^*}$ into $X_{2,T^*}$. Moreover,
\begin{equation}
    c_1 \|T(\varphi)\|_{X_{1,T^*}} \leq \|T(\varphi)\|_{X_{2,T^*}} \leq c_2 \|F(\varphi)\|_{L^2(0,T^*; H)}.
\end{equation}

Proof of Lemma 3.2: Inequality (2.14) and Lemma 3.1 imply that $T$ is a continuous operator from $X_{1,T^*}$ into $X_{2,T^*}$.

Proof of Theorem 3.2: Let
\[ B_R = \{ \varphi \in X_{1,T^*}; \|\varphi\|_{X_{1,T^*}} \leq R \}. \]

Lemma 3.1 and Lemma 3.2 imply that for a sufficiently small $R$, $T$ maps $B_R$ into itself. By Lemma 3.2 and (3.30), $T$ is totally continuous operator from $X_{1,T^*}$ into $X_{1,T^*}$. Moreover, the Banach space $X_{1,T^*}$ is reflexive. Therefore there exists $w \in B_R$ such that $T(w) = w$. Set $u = w + u_0$. By Remark 3.2, $u$ is a generalized solution of the problem (1.1)-(1.6). The theorem is proved.

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References


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