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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} *anniversary*

The free-boundary flow past an obstacle. Qualitative and numerical results

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ABSTRACT. The investigation of the free boundary flow of an ideal fluid past a smooth obstacle is reduced herein to the study of a system of non-linear integro-differential equations. We study the existence and uniqueness of the solution in case that the obstacle is an arc of circle (symmetrical with respect to Ox - axis which is assumed to have the direction of the fluid flow at infinity upstream) and we calculate it numerically by means of the successive approximations method. We also calculate the drag coefficient and the free lines.

1. INTRODUCTION

Usually, in the papers dealing with the classical theory of the 2d potential flow of an ideal incompressible fluid, d'Alembert's paradox is explained by the neglect of the viscosity. Helmholtz noticed in 1868 (like Kirchhoff in 1869) that d'Alembert's paradox may be avoided (even if ones assumes that the fluid is ideal) if we consider that a wake, a "stagnation zone" (where the velocity vanishes and the pressure is constant) appears behind the obstacle. At the beginning of the XX-th century, Levi-Cività [5] and H. Villat [8] developed the mathematical fundamental of the wake flow. In 1934 J. Leray [4] utilized functional methods (like topological degree teory) for investigating the free-boundary flow past a class of curvilinear obstacles. Later the method was extended by Chaplygin, Riabouchinsky [7], C. Jacob [3], S. Popp [6] to the free-boundary subsonic compressible flow past obstacles. In order to enlarge the class of obstacles for which one may obtaint analytical results, one employed inverse methods. In our paper the problem of the free-boundary flow past a smooth obstacle is reduced to the study of a system of integral equations. We utilize the successive approximations method for studying the flow past the circular obstacle. Then we calculate the drag coefficient and the free-lines. We compare the value of the drag coefficient with values calculated by other authors and we find a very good agreement.

2. HELMHOLTZ'S MODEL

According to this model, behind the fixed obstacle there is a wake where the fluid is at rest and the pressure is constant. According to Bernoulli we have

$$p+\frac{1}{2}\rho V^{2}=p_{0}+\frac{1}{2}\rho V_{0}^{2}=p_{\max} \tag{2.1}$$

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where p is the pressure, ρ is the density, V is the modulus of velocity, p_0 is the pressure at infinity upstream, V_0 is the modulus of velocity at infinity upstream and $p_{\max}$ is the maximum value of the pressure. In fig.1 we present an obstacle with the wake and the two free lines λ_1 and λ_2. Q is an isolated stagnation point and A and B are the detachment points.

Helmholtz model

Helmholtz's model leads to a free-boundary value problem for a domain $\mathcal{D}$ from the Oxy - plane. The unknowns are the coordinates of the velocity $u, v : \mathcal{D} \to \mathbb{R}$ which are assumed to be differentiable, with continuous derivatives. We also utilize the complex potential φ and the stream function ψ which are connected through the relations

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}\,, \qquad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}, \tag{2.2}$$

which arise from the condition of irrotationality and from the equation of continuity (mass conservation):

$$rot\, \mathbf{v} = 0, \qquad div\, \mathbf{v} = 0, \qquad \mathbf{v} = (u, v).$$

We deduce that the functions φ are ψ harmonic in $\mathcal{D}$. The obstacle ϖ and the free lines λ_1 and λ_2 are stream-lines i.e.

$$\mathbf{v} \cdot \mathbf{n} = \frac{\partial \varphi}{\partial n} = \frac{\partial \psi}{\partial s} = 0,$$

(where $\mathbf{n}$ is the normal and $\dfrac{\partial}{\partial n}$ and $\dfrac{\partial}{\partial s}$ are the normal respectively tangential derivative) and we have

$$\psi(x, y) = const. = 0, \qquad \forall (x, y) \in \varpi \cup \lambda_1 \cup \lambda_2. \tag{2.3}$$

Since the free lines are not apriori known, for determining them we have to impose a supplementary condition, arising from the continuity of the pressure $p = p_{0.}$ According to Bernoulli's equation (2.1) it follows

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = u^2 + v^2 = V^2 = V_0^2\ ,\ \forall\, (x, y) \in \lambda_1 \cup \lambda_2. \tag{2.4}$$

At infinity upstream the flow is supposed to be uniform, i.e. denoting by α the angle of the velocity with the Ox - axis, we have:

$$\lim_{\infty} (u, v) = (V_0 \cos \alpha, V_0 \sin \alpha). \tag{2.5}$$

3. Levi - Cività's method

It relies on the conformal mappings method. One introduces the complex variable $z = x + iy$ and the complex potential $f(z) = \varphi(x, y) + i\psi(x, y)$. The complex velocity is

$$w = \frac{df}{dz} = u - iv = V \cos \theta - iV \sin \theta = V \exp(-i\theta) \tag{3.6}$$

where θ is the angle of the velocity with the Ox - axis. Hence V and θ are the polar coordinates in the hodographic plane Ouv.

Levi-Cività introduces the complex analytical function $\omega(z)$ by means of the relation

$$w(z) = V_0 \exp(-i\omega(z)). \tag{3.7}$$

From (3.6) and (3.7) we get

$$\omega = \theta + i\tau \ , \ \tau = \ln \frac{V}{V_0}. \tag{3.8}$$

From (2.4) and (3.8) it follows the boundary condition

$$\tau(x, y) = 0 \ , \ \forall (x, y) \in \lambda_1 \cup \lambda_2. \tag{3.9}$$

At infinity upstream we have

$$\lim_{z \to \infty} w(z) = V_0 \exp(-i\alpha) \ , \ \lim_{z \to \infty} \omega(z) = \alpha. \tag{3.10}$$

Taking into account the boundary condition we may conclude that the complex potential maps conformally the domain of motion $\mathcal{D}$ (from the z - plane)onto a domain $\mathcal{G}$ from the plane of the complex variable $f = \varphi + i\psi$, namely the complex plane with a cut along the positive real semi-axis (fig.2).

From the relations (3.6) and (3.7) we deduce

$$V_0(z - z_0) = \int_0^f \exp(i\omega(f))df \tag{3.11}$$

where z_0 is the affix of the stagnation point Q where one considers $f = 0$.

It would be sufficient to know $\omega(f)$ for finding $f(z)$. The function $\omega(f)$ satisfies the following conditions

$$\lim_{f \to \infty} \omega(f) = \alpha \ , \ \tau(\varphi, -0) = 0, \forall \varphi > f_1 \ , \ \tau(\varphi, +0) = 0, \forall \varphi > f_2 \tag{3.12}$$

because (f_1, ∞) is the image of the free - line λ_1 and (f_2, ∞) is the image of the free - line λ_2.

In order to determine $\omega(f)$ it is sufficient to know the restrictions of its real part θ to the segments $(0, f_1)$ respectively $(0, f_2)$:

$$\theta(\varphi, -0) = \theta_1(\varphi), \forall\, \varphi \in (0, f_1) \ , \ \theta(\varphi, +0) = \theta_2(\varphi), \forall\, \varphi \in (0, f_2). \tag{3.13}$$

4. The integral representation of Levi-Cività's function

We determine the solution of the boundary value problem (3.7), (3.12), (3.13) for $\omega(f)$ under the parametric form $\omega = \omega(\zeta)$, $f = f(\zeta)$ where $f(\zeta)$ has an explicit expression and $\omega(\zeta)$ has an integral representation associated to another boundary value problem. We denote by $\mathcal{U}$ the unit half-disk.

$$\mathcal{U} = \{\zeta = \xi + i\eta \in \mathbf{C}; \eta > 0, | \zeta | < 1\} . \tag{4.14}$$

We consider the apriori unknown parameters L (which has the significance of a characteristic length of the obstacle) and $a_0 \in (0, \pi)$. The function $f : \mathcal{U} \to \mathcal{G}$, defined by

$$f(\zeta) = LV_0 \left(\zeta + \frac{1}{\zeta} - u_0 - \frac{1}{u_0} \right)^2, \ \ \zeta \in \mathcal{U}, \tag{4.15}$$

where

$$u_0 = \exp(ia_0)\ ,\ u_0 + \frac{1}{u_0} = 2\cos a_0 \tag{4.16}$$

maps conformally the superior half-disk $\mathcal{U}$ onto the plane with a cut $\mathcal{G}$ (fig. 3).

When the parameters L and a_0 are determined by the relations

$$f(1) = 4LV_0(1-\cos a_0)^2 = f_1\ ,\ f(-1) = 4LV_0(1+\cos a_0)^2 = f_2 \tag{4.17}$$

to the radii $(0,1)$ and $(-1,0)$ of the unit half-disk it correspond the half-lines (f_1,∞) respectively (f_2,∞), which are images of the free-lines λ_1 and λ_2. To the arcs of the unit half-circle it corresponds the segments $(0,f_1)$ respectively $(0,f_2)$ which are images of the arcs AQ and QB according to the relation

$$f(\exp(i\sigma)) = 4LV_0(\cos\sigma - \cos a_0)^2. \tag{4.18}$$

Derivating and integrating we have

$$\frac{df}{d\zeta} = 2LV_0\left(\frac{1}{\zeta} - \frac{1}{u_0}\right)(1-\zeta u_0)\left(1-\frac{1}{\zeta^2}\right), \tag{4.19}$$

$$\frac{dz}{d\zeta} = \frac{1}{w}\frac{df}{d\zeta} = 2L\exp(i\omega(\zeta))\left(\frac{1}{\zeta} - \frac{1}{u_0}\right)(1-\zeta u_0)\left(1-\frac{1}{\zeta^2}\right), \tag{4.20}$$

$$z = z(\zeta) = z_A + \int_1^\zeta \frac{dz}{d\zeta}d\zeta. \tag{4.21}$$

The formulae (4.15), (4.20) and (4.21) are the parametric representations of the complex potential $f(z)$. In this manner the initial problem was reduced to the determination of the function $\omega\,(\zeta)$. The imaginary part of this function vanishes on $(-1,1)$

$$\tau\,(\xi,0) = 0,\ \forall\xi\in(-1,1). \tag{4.22}$$

For determining $\omega\,(\zeta)$ it suffices to know the real part on the half-circle AQB :

$$\theta(\cos\sigma,\sin\sigma) = \theta_r(\sigma),\ \sigma\in(0,\pi). \tag{4.23}$$

We prolong the functions $f\,(\zeta)$ and $\omega\,(\zeta)$ in the inferior unit half-disk by means of the relations

$$f\left(\overline{\zeta}\right) = \overline{f(\zeta)},\ \omega\left(\overline{\zeta}\right) = \overline{\omega\,(\zeta)},\ \mid\zeta\mid\le 1. \tag{4.24}$$

Hence we have

$$\theta_r(-\sigma) = \theta_r(\sigma),\ \sigma\in(0,\pi).$$

According to Schwarz-Villat ([3]) formula we have the integral representation

$$\omega\,(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta_r(\sigma)\left(\frac{2\zeta}{\exp(i\sigma)-\zeta}+1\right)d\sigma,\ \mid\zeta\mid< 1. \tag{4.25}$$

Since $\theta_r(\sigma)$ is an odd function, from (4.25) we deduce

$$\omega\,(\zeta) = \frac{1}{\pi}\int_0^{\pi}\theta_r(\sigma)\left(\frac{\zeta}{\exp(i\sigma)-\zeta}+\frac{\zeta}{\exp(-i\sigma)-\zeta}+1\right)d\sigma,\ \mid\zeta\mid< 1. \tag{4.26}$$

and from (4.26), (3.12) and (4.15)

$$\omega\left(0\right)=\frac{1}{\pi}\int_{0}^{\pi}\theta_r(\sigma)d\sigma=\alpha. \tag{4.27}$$

We consider the complex function $N(\cdot,\cdot)$:

$$N\left(\zeta,a\right)=\ln\frac{1-\zeta\exp\left(ia\right)}{1-\zeta\exp\left(-ia\right)}=-2i\sum_{m\geq1}\frac{1}{m}\zeta^{m}\sin ma, \mid\zeta\mid<1, a\in(0,\pi) \tag{4.28}$$

with $N\left(\zeta,0\right)=N\left(\zeta,\pi\right)=0$.

We also introduce the real şi simmetric kernel $K(\cdot,\cdot)$:

$$K(t,a)=K(\pi-t,\pi-a)=ReN(\exp(it),a)=\ln\frac{\sin\frac{t+a}{2}}{\mid\sin\frac{t-a}{2}\mid},\ a,t\in(0,\pi) \tag{4.29}$$

$$K(t,a)=2\sum_{m\geq1}\frac{1}{m}\sin ma\sin mt. \tag{4.30}$$

Employing the new notations we may write:

$$\omega\left(\zeta\right)=\alpha+\frac{i}{\pi}\int_{0}^{\pi}\theta_r\left(\sigma\right)\frac{\partial}{\partial\sigma}N\left(\zeta,\sigma\right)d\sigma. \tag{4.31}$$

5. The second integral representation

Let S be the curvilinear coordinate on the obstacle and $g(S)$ (which is supposed to be derivable with a continuous derivative), the angle between the $Ox-$axis and the tangent to the obstacle AQB (fig.1). In the stagnation point Q the streamline bifurcates. The angle θ_r of the velocity is equal to the angle $g(S)$ on the arc QB and it is equal to the angle $g(S)-\pi$ on the arc AQ :

$$\theta_r=\alpha_0+g(S),\ \alpha_0=\begin{cases}-\pi,\ \sigma\in(0,a_0)\\ 0,\ \sigma\in(a_0,\pi).\end{cases} \tag{5.32}$$

Hence we have the representation

$$\omega=\omega_0+\omega_g \tag{5.33}$$

with

$$\omega_0=\alpha-iN\left(\zeta,a_0\right)=\alpha+i\ln\frac{1-\frac{\zeta}{u_0}}{1-\zeta u_0}, \tag{5.34}$$

$$\omega_g=\frac{i}{\pi}\int_{0}^{\pi}g(S(\sigma))\frac{\partial}{\partial\sigma}N\left(\zeta,\sigma\right)d\sigma. \tag{5.35}$$

From (4.27) and (5.32) we get the condition $a_0=\frac{1}{\pi}\int_0^\pi g(S(\sigma))d\sigma-\alpha$.

From the formulas (3.7), (4.20) and (5.34) we get:

$$w_0\left(\zeta\right)=V_0\exp\left(-i\omega_0\left(\zeta\right)\right)=V_0\exp\left(-i\alpha\right)\frac{1-\frac{\zeta}{u_0}}{1-\zeta u_0}, \tag{5.36}$$

$$w\left(\zeta\right)=w_0\left(\zeta\right)\exp\left(-i\omega_g\left(\zeta\right)\right), \tag{5.37}$$

$$\frac{dz_0}{d\zeta} = 2L\exp(i\omega_0)\left(\frac{1}{\zeta}-\frac{1}{u_0}\right)(1-\zeta u_0)\left(1-\frac{1}{\zeta^2}\right) \tag{5.38}$$
$$= 2L\exp(i\alpha)(1-\zeta u_0)^2\left(\zeta-\frac{1}{\zeta}\right)\frac{1}{\zeta^2},$$

$$\frac{dz}{d\zeta} = \frac{dz_0}{d\zeta}\exp\left(i\omega_g\left(\zeta\right)\right) \tag{5.39}$$
$$= 2L\exp(i\alpha)u_0\left(\zeta u_0+\frac{1}{\zeta u_0}-2\right)\left(\zeta-\frac{1}{\zeta}\right)\frac{1}{\zeta}\exp\left(i\omega_g\left(\zeta\right)\right).$$

The parametric representation of the arc AQB with respect to the parameter σ follows from the relations

$$\frac{dz_0}{d\sigma} = \frac{dz_0}{d\zeta}\frac{d\zeta}{d\sigma} = i\zeta\frac{dz_0}{d\zeta} = 8L\exp\left(i\left(\alpha+a_0\right)\right)\left[1-\cos\left(\sigma+a_0\right)\right]\sin\sigma, \tag{5.40}$$

$$\frac{dS}{d\sigma} = \left|\frac{dz}{d\sigma}\right| = 8L\mid\exp\left[i\omega_g\left(\exp\left(i\sigma\right)\right)\right]\mid\left[1-\cos\left(\sigma+a_0\right)\right]\sin\sigma. \tag{5.41}$$

Integrating we obtain

$$S_B-S_A = \int_0^\pi\frac{dS}{d\sigma}d\sigma = 8L\int_0^\pi\mid\exp\left[i\omega_g\left(\exp\left(i\sigma\right)\right)\right]\mid\left[1-\cos\left(\sigma+a_0\right)\right]\sin\sigma d\sigma. \tag{5.42}$$

In case that the detachment points are known, this relation allows to find the parameter L.

6. The system of integro-differential equations

The parametric equations of the arc AQB follow from

$$\frac{dx}{dS} = \cos g(S),\quad \frac{dy}{dS} = \sin g(S),\quad S\in(S_A,S_B). \tag{6.43}$$

The curvature $-h(S)$ of the arc AQB is:

$$-h(S) = \frac{dg}{dS}. \tag{6.44}$$

Passing to the limit on the unit circle we get

$$\omega_g\left(\exp\left(it\right)\right) = G(t)+iT(t) = \frac{i}{\pi}\int_0^\pi N\left(\exp\left(it\right),\sigma\right)h(S(\sigma))\frac{dS}{d\sigma}d\sigma. \tag{6.45}$$

Separating the imaginary parts and taking into account (4.29), we get the system of nonlinear integro-differential equations for the functions $T(\cdot)$ and $S(\cdot)$:

$$T(t) = \frac{1}{\pi}\int_0^\pi\ln\frac{\sin\frac{t+\sigma}{2}}{\mid\sin\frac{t-\sigma}{2}\mid}K(t,\sigma)h(S(\sigma))S'(\sigma)d\sigma, \tag{6.46}$$

$$S'(\sigma) = 8L\left[1-\cos\left(\sigma+a_0\right)\right]\sin\sigma\exp\left(-T\left(\sigma\right)\right). \tag{6.47}$$

The function $-h(\cdot)$ is the known curvature of the obstacle. Obviously we have

$$S(\sigma) = S_A+8L\int_0^\sigma\exp\left(-T\left(t\right)\right)\left[1-\cos\left(t+a_0\right)\right]\sin t dt. \tag{6.48}$$

7. The symmetric obstacle having the shape of an arc of circle

In this case the curvature is constant and we have:

$$\alpha = 0,\ \ a_0 = \frac{\pi}{2},\ \ \lambda = 8Lh(S) = const. \geq 0. \tag{7.49}$$

Substituting S' from (6.47) to (6.46) we obtain the nonlinear integral equation

$$T(t) = F(T(t)) = \frac{\lambda}{\pi}\int_0^{\pi} \exp(-T(\sigma)) \ln \frac{\sin\dfrac{t+\sigma}{2}}{|\sin\dfrac{t-\sigma}{2}|}(1+\sin\sigma)\sin\sigma d\sigma. \tag{7.50}$$

The operator F is defined on $C\left([0,\pi]\right)$, the space of continuous functions on $[0,\pi]$, endowed with the norm

$$\|T\| = \sup\left\{|T(t)|\,;\ \ 0 \leq t \leq \pi\right\}. \tag{7.51}$$

The kernel of this operator has a positive values and it also has a logarithmic singularity. Denoting $F_0(t) = F(0)(t)$ we get

$$0 \leq F_0(t) \leq \frac{2\lambda}{\pi}\int_0^{\pi} K(t,\sigma)\sin\sigma d\sigma = 2\lambda\sin t \leq 2\lambda. \tag{7.52}$$

The images of the functions are minorized by 0 and the images of the positive functions are majorized as follows

$$T \geq 0 \Longrightarrow 0 \leq F(T) \leq F(0) \leq 2\lambda\sin(\cdot). \tag{7.53}$$

Since

$$T, Y \geq 0 \Longrightarrow |\exp(-T) - \exp(-Y)| \leq |T - Y| \leq \|T - Y\| \tag{7.54}$$

we get

$$T, Y \geq 0 \Longrightarrow |F(T) - F(Y)| \leq \|T - Y\|\, F(0) \leq 2\lambda\,\|T - Y\|\sin(\cdot). \tag{7.55}$$

We deduce that for $\lambda \in [0, \frac{1}{2})$ the operator F is a contraction, therefore the integral equation (7.50) has a unique solution which may be calculated with the successive approximations method.

The case of this obstacle is also studied in [3], page 584-587. We notice that $t, \sigma \in (0,\pi)$; we have therefore $\ln \dfrac{\sin\dfrac{t+\sigma}{2}}{|\sin\dfrac{t-\sigma}{2}|} \geq 0$, whence it follows

$$\widetilde{T}\,(t) \geq \widehat{T}\,(t)\,, \forall t \in (0,\pi) \Rightarrow F(\widetilde{T}\,(t)) \leq F(\widehat{T}\,(t)), \forall t \in (0,\pi)\,.$$

Starting with the initial function $T_0(t) \equiv 0$, denoting $T_n = F\left(T_{n-1}\right)$ and taking into account that $T_0\left(t\right) \leq T_2\left(t\right) \leq T_1\left(t\right)$, we get

$$T_0\left(t\right) \leq T_2\left(t\right) \leq ... \leq T_{2n}\left(t\right) \leq ... \leq T_{2n+1}\left(t\right) \leq ... \leq T_3\left(t\right) \leq T_1\left(t\right).$$

Since the sequences $\left\{T_{2n}\left(t\right)\right\}_{n\geq 0}$ and $\left\{T_{2n+1}\left(t\right)\right\}_{n\geq 0}$ are monotone and bounded, we get

$$T^{(0)}\left(t\right) = \lim_{n\to\infty} T_{2n}\left(t\right),\ T^{(1)}\left(t\right) = \lim_{n\to\infty} T_{2n+1}\left(t\right),\ T^{(0)}\left(t\right) \leq T^{(1)}\left(t\right).$$

We obviously have

$$F\left(T^{(0)}\right)=T^{(1)}, F\left(T^{(1)}\right)=T^{(2)}.$$

If we obtain by means of some numerical experiments that $T^{(0)}(t)=T^{(1)}(t)$, we conclude that $F(T)=T$ where

$$T^{(0)}(t)=T^{(1)}(t)=T(t)=\lim_{n\to\infty}T_n(t),\ T_n=F^n(T_0). \tag{7.56}$$

$$F^n(T_0(t))=F(F^{n-1}(T_0(t))),\ n\geq 2. \tag{7.57}$$

The function $F(T_0(t))$ is calculated approximatively as follows: one considers an equidistant grid $\{t_1=0,t_2,...,t_N,t_{N+1}=\pi\}$ of the interval $[0,\pi]$ and using a certain quadrature formula, one calculates $\{T_1(t_i); i=1,...,N+1\}$ and repeating the procedure,$\{T_n(t_i); i=1,...,N+1\}$. We stop the iterative process when for a certain $0<\epsilon<<1$,

$$|T_{n+1}(t_i)-T_n(t_i)|\leq\epsilon,\ i=1,...,N+1. \tag{7.58}$$

Since $T(t)=Re\left[-i\omega_g(\exp(it))\right]$ and $T(-t)=-T(t)$, we have according to Schwarz-Villat formula

$$\omega_g(\zeta)=\frac{2\zeta}{\pi}\int_0^\pi T(t)\frac{\sin t}{1-2\zeta\cos t+\zeta^2}dt \tag{7.59}$$

whence, derivating we get

$$\omega_g'(\zeta)=\frac{2}{\pi}\int_0^\pi T(t)\frac{\sin t}{1-2\zeta\cos t+\zeta^2}dt-\frac{4\zeta}{\pi}\int_0^\pi T(t)\frac{\zeta-\cos t}{(1-2\zeta\cos t+\zeta^2)^2}dt \tag{7.60}$$

$$\omega_g'(0)=\frac{2}{\pi}\int_0^\pi T(t)dt. \tag{7.61}$$

After calculating $\{T(t_i); i=1,...,N+1\}$, one may calculate numerically $\omega_g'(0)$.

In our paper we considered $N+1=41$ equidistant nodes on the interval $[0,\pi]$ and we utilized Simpson's formula for the approximate calculus of the integrals. Finally we compute

$$\omega'(0)=\omega_0'(0)+\omega_g'(0)=-2+\omega_g'(0). \tag{7.62}$$

We performed some numerical calculations for $\lambda=1.1372$. From (5.42) it follows that for this parameter the length of the arc of circle with radius1, symmetric with respect to the Ox-axis este 1.9222, corresponding to the detachment angle $55.04°$.

The free-boundary flow past this obstacle was studied through various methods by J. Hureau et al. [2], G. Birkhoff and E. H. Zarantonello [1], Brodetski, Schmieden (for the last two authors some references may be found in [3], page 593). In their papers the following drag coefficient was calculated

$$C_D=\frac{\pi\lambda(\omega'(0))^2}{8}. \tag{7.63}$$

In the table below we present the values of the drag coefficient calculated by various authors

	Hureau	Birkhoff	Brodet ski
C_D	0.4986	0.499	0.493

Utilizing the method presented herein we found $C_D = 0.4986$. Employing the relations (4.20) (where $u_0 = i$), (4.21), (5.34), (7.50) and (7.59) we can calculate various points $z(\zeta), \zeta \in [-1,1]$ on the free lines. In fig.4 we present the free lines by continuous lines and the obstacle by (+).

8. Conclusions

The 2d free-boundary incompressible flow past a smooth convex obstacle was reduced to the study of a system of two non-linear integro-differential equations. In case that the obstacle is an arc of circle one has to solve only a non-linear integral equation. We proved that the integral equation has a unique solution which may be found by means of the successive approximations method. We have calculated the drag coefficient for the obstacle consisting in an arc of circle and we have compared the result with results obtained in other papers. The position of the free lines was also calculated.

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