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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

The fixed points method for the stability of some functional equations

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ABSTRACT. We use the fixed point method to obtain stability theorems of Ulam-Hyers type for some functional equations. The method uses the fixed point alternative as a meaningful device on the road to a better understanding of the stability property, plainly related to some fixed point of a concrete operator.

1. INTRODUCTION

The study of stability problems for functional equations originated from a question of S. M. Ulam concerning the stability of group homomorphisms:

Let G be a metric group with a metric d. Given $\varepsilon > 0$, does there exist a $k > 0$ such that ***for every*** *function $f : G \to G$ satisfying the inequality*

$$d(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon, \forall x, y \in G,$$

there exists an automorphism a of *G with*

$$d(f(x), a(x)) < k\varepsilon, \forall x \in G\ ?$$

D. H. Hyers gave an affirmative answer to the question of Ulam for Cauchy equation, in Banach spaces. Then T. Aoki, D. Bourgin as well as other authors considered the stability problem with unbounded Cauchy differences (see e.g. [12] and [17]). Their results include the following

Theorem 1.1. (Hyers-Aoki-Gajda). *Suppose that E is a real normed space, F is a real Banach space and $f : E \to F$ is a given function, such that the following condition holds*

$$(\mathbf{1_p}) \qquad \|f(x+y) - f(x) - f(y)\|_F \leq \theta(\|x\|_E^p + \|y\|_E^p), \forall x, y \in E,$$

for some $p \in [0, \infty)\backslash\{1\}$. Then there exists a unique additive function $a : E \to F$ such that

$$(\mathbf{2_p}) \qquad \|f(x) - a(x)\|_F \leq \frac{2\theta}{|2 - 2^p|} \|x\|_E^p, \forall x \in E.$$

This phenomenon is called *generalized Ulam-Hyers stability* and has been extensively investigated for different functional equations. It is worth mentioning that almost all proofs used the idea conceived by D. H. Hyers. Namely, the additive

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function $a : E \to F$ is constructed, starting from the given function f, by the following formulae

$$(\mathbf{2_{p<1}}) \qquad a(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \text{ if } p < 1;$$

$$(\mathbf{2_{p>1}}) \qquad a(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \quad \text{if } p > 1.$$

This method is called *the direct method* or *Ulam's method*. It is often used to construct a solution of a given functional equation and is seen to be a powerful tool for studying the stability of many functional equations (cf. [12] and [17] for details). On the other hand, in [20], [8] and [6] a *fixed point method* was proposed, by showing that many theorems concerning the stability of Cauchy and Jensen equations are consequences of the fixed point alternative. Subsequently, the method has been successfully used, e.g., in [7], [9], [11], [21] or [19].

Our aim is to highlight *generalized stability results* (of Hyers-Aoki-Bourgin type) for some functional equations obtained by using the fixed point alternative. The method introduces a metrical context and better clarifies the ideas of stability, which is seen to be plainly related to some fixed point of a suitable operator: our control conditions (below denoted by $(\mathbf{H_i})$ and $(\mathbf{H_i^*})$ or $(\mathbf{H_j})$ and $(\mathbf{H_j^*})$), are responsible for the following fundamental facts: They ensure

1) the *contraction property* of a Schröder type operator J and

2) the first two successive approximations, f and Jf, to be at a *finite distance*. Moreover,

3) they force the fixed point of J to be a *solution of the initial equation*.

Some illustrative applications to concrete (quadratic and monomial) functional equations are also given.

2. A SIMPLE FIXED POINT METHOD

For the sake of convenience, we recall the following.

Theorem 2.2. (The alternative of fixed point [18], see also [23], ch. 5]). *Suppose we are given a complete generalized metric space $(\mathcal{E}, d)$ -i.e. one for which d may assume infinite values- and a strictly contractive mapping $A : \mathcal{E} \to \mathcal{E}$, with the Lipschitz constant L. Then, for each given element $f \in \mathcal{E}$,*

either

(A_1) $d(A^n f, A^{n+1} f) = +\infty \quad , \forall n \geq 0,$

or

(A_2) *There exists a natural number n_0 such that*

(A_{20}) $d(A^n f, A^{n+1} f) < +\infty, \forall n \geq n_0$;

(A_{21}) *The sequence $(A^n f)$ is convergent to a fixed point f^* of A ;*

(A_{22}) *f^* is the unique fixed point of A in the set $\mathcal{E}^\star = \{g \in \mathcal{E},\ d\left(A^{n_0} f, g\right) < +\infty\}$;*

(A_{23}) $d\left(g, f^*\right) \leq \frac{1}{1-L} d\left(g, Ag\right), \forall g \in \mathcal{E}^\star.$

Remark 2.1. The fixed points f^*, if any, need not be uniquely determined *in the whole space $\mathcal{E}$* and do depend on the initial guess f.

2.1. **A simple proof of Theorem 1.1.** We consider the set

$$\mathcal{E} := \{g : E \to F,\ p \cdot g(0) = 0\}$$

and introduce the *generalized metric* $d = d_p : \mathcal{E} \times \mathcal{E} \to [0, +\infty]$ by the formula

$$d_p(g,h) = \sup_{x \neq 0} \frac{\|g(x) - h(x)\|_F}{\|x\|_E^p}$$

It is well-known and easy to verify that $(\mathcal{E}, d)$ *is complete*.

Now we will consider the (linear) mapping

$$A : \mathcal{E} \to \mathcal{E},\ Ag(x) := \frac{1}{q} g(qx)$$

where $q = 2$ if $p < 1$, and $q = 2^{-1}$ if $p > 1$. Obviously, for any $g, h \in \mathcal{E}$:

$$\frac{\|Ag(x) - Ah(x)\|_F}{\|x\|_E^p} = \frac{1}{q} \frac{\|g(qx) - h(qx)\|_F}{\|x\|_E^p} = q^{p-1} \frac{\|g(qx) - h(qx)\|_F}{\|qx\|_E^p} \leq q^{p-1} d(g,h),$$

for all $g, h \in \mathcal{E}$. Therefore

$$d(Ag, Ah) \leq q^{p-1} d(g,h), \forall g, h \in \mathcal{E},$$

so that A is *strictly contractive* with the Lipschitz constant $L = q^{p-1}$.
Now, if we set $y = x$ in the hypothesis (1_p), then

$$\|2f(x) - f(2x)\| \leq 2\theta \|x\|^p, \forall x \in E,$$

which says that

$$\begin{aligned} d(f, Af) &\leq \theta < \infty, \text{ for } p < 1, \text{ and} \\ d(f, Af) &\leq \frac{\theta}{2^{p-1}} < \infty, \text{ for } p > 1 \end{aligned}$$

Using the fixed point alternative we obtain the existence of a mapping $a : \mathcal{E} \to \mathcal{E}$ such that:

1^0 a is a fixed point of A:

$$a(2x) = 2a(x), \forall\, x \in E. \tag{2.1}$$

The mapping a is the unique fixed point of A in the set

$$\mathcal{F} = \{g \in \mathcal{E},\ d(f,g) < +\infty\}.$$

This means that a is the unique mapping $g : E \to F$ verifying *both* $(2.1) - (2.2)$, where

$$\exists\, c \in [0, \infty) \text{ such that } \|g(x) - f(x)\|_F \leq c \|x\|_E^p,\ \forall\, x \in E. \tag{2.2}$$

2^0 $d(A^n f, a) \to 0$, which implies

$$\lim_{n \to \infty} \frac{f(q^n x)}{q^n} = a(x), \quad \forall\, x \in E. \tag{2.3}$$

Therefore at least one of the statements $(\mathbf{2_{p<1}})$ and $(\mathbf{2_{p>1}})$ is seen to be true.

3^0 $d(f,a) \leq \frac{1}{1-q^{p-1}} d(f, Af) \leq \frac{2\theta}{|2-2^p|}$, that is the inequality $(\mathbf{2_p})$ takes place.

Finaly, as it is well-known, the additivity of a follows immediately from $(\mathbf{1_p})$ and (2.3):
If in $(\mathbf{1_p})$ we replace x by $q^n x$ and y by $q^n y$, then we obtain

$$\left\| \frac{f(q^n\,(x+y))}{q^n} - \frac{f(q^n x)}{q^n} - \frac{f(q^n y)}{q^n} \right\|_F \leq L^n \theta(\|x\|_E^p + \|y\|_E^p), \forall\, x, y \in E,$$

and, letting $n \to \infty$, we get

$$a(x+y) = a\,(x) + a\,(y)\,, \forall\, x, y \in E.$$

3. A general fixed point method

Firstly, we prove an auxiliary stability result (compare with [1], [3] and [14]) for the single variable equation $w \circ g \circ \eta = g$, where

1. the unknown is a mapping $g : G \to Y$.
2. η is a self-mapping of the Abelian group G;
3. w is a Lipschitzian self-mapping (with Lipschitz constant ℓ_w) of the $\beta-$normed space Y, where, as usual, a mapping $||\cdot||_\beta : Y \to \mathbb{R}_+$, with $\beta \in (0,1]$, is called a $\beta-$*norm* iff it has the properties $(n_\beta^I):\ ||y||_\beta = 0 \Leftrightarrow y = 0$, $(n_\beta^{II}):\ ||\lambda \cdot y||_\beta = |\lambda|^\beta \cdot ||y||_\beta$, and $(n_\beta^{III}):\ ||y+z||_\beta \leq ||y||_\beta + ||z||_\beta$, for all $y, z \in Y$, and $\lambda \in \mathbb{K}$.

Theorem 3.3. *Suppose that $f : G \to Y$ satisfies*

$$(\mathbf{C}_\psi) \qquad \|(w \circ f \circ \eta)(x) - f(x)\|_\beta \leq \psi(x),\ \forall\, x \in G,$$

with some fixed mapping $\psi : G \to [0,\infty)$. If there exists $L < 1$ such that

$$(\mathbf{H}_\psi) \qquad \ell_w \cdot (\psi \circ \eta)(x) \leq L\psi(x), \forall\, x \in G,$$

then there exists a ***unique mapping*** *$c : G \to Y$ which satisfies both*
the equation

$$(\mathbf{E}_{w,\eta}) \qquad (w \circ c \circ \eta)(x) = c(x), \forall\, x \in G$$

and ***the estimation***

$$(\mathbf{Est}_\psi) \qquad \|f(x) - c(x)\|_\beta \leq \frac{\psi(x)}{1-L}, \forall\, x \in G.$$

Namely, the solution mapping c can be acquired through the Hyers method:

$$c(x) = \lim_{n\to\infty} (w^n \circ f \circ \eta^n)\,(x), \forall\, x \in G.$$

Proof. Let us consider the set $\mathcal{E} := \{g : G \to Y\}$ and introduce a *complete generalized metric* on $\mathcal{E}$ (as usual, $inf\ \emptyset = \infty$):
$(\mathbf{GM}_\psi)$

$$d\,(g,h) = d_\psi\,(g,h) = \inf\left\{K \in \mathbb{R}_+, \|g\,(x) - h\,(x)\|_\beta \leq K\psi(x), \forall\, x \in G\right\}.$$

Now, define the mapping

$$(\mathbf{OP}) \qquad J : \mathcal{E} \to \mathcal{E}, Jg\,(x) := (w \circ g \circ \eta)(x).$$

Step I. Using the hypothesis $(\mathbf{H}_\psi)$ it follows that J *is strictly contractive* on $\mathcal{E}$. Indeed, for any $g, h \in \mathcal{E}$ we have:

$$d(g,h) < K \Longrightarrow \|g\,(x) - h\,(x)\|_\beta \leq K\psi(x), \forall x \in G$$

and

$$\|Jg(x) - Jh(x)\|_\beta = \|w\left(g(\eta(x))\right) - w\left(h(\eta(x))\right)\|_\beta \le \ell_w \cdot \|g(\eta(x)) - h(\eta(x))\|_\beta .$$

Therefore

$$\|Jg(x) - Jh(x)\|_\beta \le \ell_w \cdot K \cdot \psi(\eta(x)) \le K \cdot L \cdot \psi(x), \forall\, x \in G$$

so that $d\left(Jg, Jh\right) \le LK$, which implies

$$(\mathbf{CC_L}) \qquad d\left(Jg, Jh\right) \le Ld\left(g, h\right), \forall\, g, h \in \mathcal{E}.$$

This says that J is a *strictly contractive* self-mapping of $\mathcal{E}$, with the constant $L < 1$.

Step II. Obviously, $d\left(f, Jf\right) < \infty$. In fact, using the relation $(\mathbf{C}_\psi)$ it results that $d\left(f, Jf\right) \le 1$.

Step III. We can apply the fixed point alternative (see [18], [23] or [6]), and we obtain the existence of a mapping $c : G \to Y$ such that:

- c is a fixed point of J, that is

$$(\mathbf{E}_{w,\eta}) \qquad (w \circ c \circ \eta)(x) = c(x)\ , \forall\, x \in G.$$

The mapping c is the unique fixed point of J in the set

$$\mathcal{F} = \left\{g \in \mathcal{E},\ d\left(f, g\right) < \infty\right\}.$$

This says that c is the unique mapping verifying *both* $(\mathbf{E}_{w,\eta})$ and (3.4), where

$$(3.4) \qquad \exists K < \infty \text{ such that } \|c\left(x\right) - f\left(x\right)\|_\beta \le K\psi(x), \forall\, x \in G.$$

- $d(J^n f, c) \xrightarrow[n\to\infty]{} 0$, which implies

$$c(x) = \lim_{n\to\infty}\left(w^n \circ c \circ \eta^n\right)(x), \forall\, x \in G.$$

- $d(f, c) \le \frac{1}{1-L} d\left(f, Jf\right)$, which implies the inequality

$$d(f, c) \le \frac{1}{1-L},$$

that is $(\mathbf{Est}_\psi)$ is seen to be true. □

3.1. **Applications to the quadratic equation.** We will consider the following equation, where the "unknowns" are functions $f : X \to Y$, between two vector spaces:

$$(3.5) \qquad f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad \text{for all } x, y \in X.$$

Since the real functions defined on $\mathbb{R}$ by $x \to kx^2$ satisfy the equation (3.5), the above functional equation is called **quadratic**. Every solution of a quadratic functional equation is called a *quadratic function*.

Recall that a 2-divisible group is an Abelian group $(X, +)$ such that for any $x \in X$ there exists a unique $a \in X$ with the property $x = 2a$; this unique element a is denoted by $\frac{x}{2}$.

Theorem 3.4. *Let X be a 2-divisible group, Y a Banach space, and* $r_i = \begin{cases} 2, i = 0 \\ \frac{1}{2}, i = 1 \end{cases}$. *Suppose that the mapping $f : X \to Y$ satisfies the condition $f\left(0\right) = 0$ and an inequality of the form*

$$(\mathbf{Q}_\varphi) \qquad \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_Y \le \varphi(x, y), \forall\, x, y \in X,$$

where $\varphi : X \times X \to [0, \infty)$ is a given function.

If there exists $L = L(i) < 1$ such that the mapping

$$x \to \Omega(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

has the property

$$(\mathbf{H}_i) \qquad \Omega(x) \leq L \cdot r_i^2 \cdot \Omega\left(\frac{x}{r_i}\right), \forall\, x \in X,$$

and if the mapping φ has the property

$$(\mathbf{H_i^*}) \qquad \lim_{n\to\infty} \frac{\varphi\left(r_i^n x, r_i^n y\right)}{r_i^{2n}} = 0, \forall\, x, y \in X,$$

then there exists a unique quadratic mapping $q : X \to Y$ which satisfies the functional equation (3.5) and the inequality

$$(\mathbf{Est}_i) \qquad \|f(x) - q(x)\|_Y \leq \frac{L^{1-i}}{1-L}\Omega(x), \forall\, x \in X.$$

Proof. Note that $r_0 = 2$ if $(\mathbf{H}_0)$ holds, and $r_1 = 2^{-1}$ if $(\mathbf{H}_1)$ holds.
If the hypothesis $(\mathbf{H}_0)$ holds, and we set $x = y$ in the relation $(\mathbf{Q}_\varphi)$, then we see that

$$(\mathbf{Q_{\Omega,0}}) \qquad \left\|\frac{f\,(2x)}{4} - f(x)\right\|_Y \leq \frac{\Omega(2x)}{4}, \forall\, x \in X.$$

If the hypothesis $(\mathbf{H}_1)$ holds, and we replace both x and y by $\frac{x}{2}$ in the relation $(\mathbf{Q}_\varphi)$, then we obtain

$$(\mathbf{Q_{\Omega,1}}) \qquad \left\|4f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \Omega(x), \forall\, x \in X.$$

Now we can apply Theorem 3.3 (for $\beta = 1$), with $w : X \to Y$, $\eta : X \to X$, $\psi : X \to [0, \infty)$,

$$w(x) := \frac{x}{{r_i}^2},\ \eta(x) := r_i x,\ \psi(x) := \frac{\Omega(2^{1-i}x)}{2^{2(1-i)}}$$

and $r_i = \begin{cases} 2, i = 0 \\ \frac{1}{2}, i = 1 \end{cases}$. Clearly, $l_w = \frac{1}{r_i^2}$ and, by using $(\mathbf{Q_{\Omega,i}})$ and the hypothesis $(\mathbf{H}_i)$, we obtain that $(\mathbf{C}_\psi)$ and $(\mathbf{H}_\psi)$ hold.

Then there exists a unique mapping $q : X \to Y$,

$$(3.6) \qquad q(x) := \lim_{n\to\infty} \left(w^n \circ q \circ \eta^n\right)(x) = \lim_{n\to\infty} \frac{q\left({r_i}^n x\right)}{{r_i}^{2n}}, \forall\, x \in X,$$

which satisfies the following equation

$$(w \circ q \circ \eta)(x) = q(x) \Leftrightarrow q\,(2x) = 2^2 q\,(x)\,,\ \forall\, x \in X$$

and the inequality

$$\|f(x) - q(x)\|_Y \leq \frac{\psi(x)}{1-L} = \frac{1}{1-L} \cdot \frac{\Omega(2^{1-i}x)}{2^{2(1-i)}} \leq \frac{L^{1-i}}{1-L} \cdot \Omega(x), \forall\, x \in X.$$

The statement that q is a quadratic mapping follows immediately from $(\mathbf{Q}_\varphi)$ and (3.6): If in $(\mathbf{Q}_\varphi)$ we replace x by ${r_i}^n x$ and y by ${r_i}^n y$, then we obtain

$$\left\| \frac{f({r_i}^n (x+y))}{{r_i}^{2n}} + \frac{f({r_i}^n (x-y))}{{r_i}^{2n}} - 2\frac{f({r_i}^n x)}{{r_i}^{2n}} - 2\frac{f({r_i}^n y)}{{r_i}^{2n}} \right\|_Y \leq \frac{\varphi({r_i}^n x, {r_i}^n y)}{{r_i}^{2n}},$$

for all x, y in X.
Taking into account the hypothesis $(\mathbf{H}_\mathbf{i}^*)$ and letting $n \to \infty$, we get

$$q(x+y) + q(x-y) - 2q(x) - 2q(y) = 0, \forall\, x, y \in X.$$

which ends the proof. □

3.2. **The general case of the monomoial equation and $\beta-$normed spaces.** For an Abelian group X and a vector space Y consider the difference operators defined, for each $y \in X$ and any mapping $f : X \to Y$, in the following manner:

$$\Delta_y^1 f(x) := f(x+y) - f(x), \text{ for all } x \in X,$$

and, inductively, $\Delta_y^{n+1} = \Delta_y^1 \circ \Delta_y^n,$ for all $n \geq 1.$

A mapping $f : X \to Y$ is called a *monomial function of degree N* if it is a solution of the *monomial functional equation*.

$$\Delta_y^N f(x) - N! f(y) = 0, \ \forall\, x, y \in X. \tag{3.7}$$

Notice that the monomial equation of degree 1 is exactly the Cauchy equation, while for N=2 the monomial equation has the form $f(x+2y) - 2f(x+y) + f(x) - 2f(y) = 0$, which is equivalent to the well-known quadratic functional equation.

In the sequel, the positive integer N will be fixed.

Let X be a 2-divisible group, let Y be a (real) complete β-normed space and assume we are given a function $\varphi : X \times X \to [0, \infty)$ with the following property:

$$\lim_{m\to\infty} \frac{\varphi\left(r_j^m x, r_j^m y\right)}{r_j^{mN\beta}} = 0, \forall\, x, y \in X, \text{ for } r_j := 2^{1-2j}, j \in \{0, 1\}. \tag*{$\left(\mathbf{H}_\mathbf{j}^*\right)$}$$

Theorem 3.5. *Suppose the mapping $f : X \to Y$, with $f(0) = 0$, verifies **the control condition***

$$\|\Delta_y^N f(x) - N! \cdot f(y)\|_\beta \leq \varphi(x, y)\ , \ \forall\, x, y \in X. \tag{3.8}$$

If there exists a positive constant $L < 1$ such that the mapping

$$x \to \Theta(x) = \frac{1}{(N!)^\beta} \left(\varphi(0, x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \varphi\left(\frac{ix}{2}, \frac{x}{2}\right) \right)\ , \ \forall\, x \in X,$$

satisfies the inequality

$$\Theta(r_j x) \leq L \cdot r_j^{N\beta} \cdot \Theta\left(x\right), \forall\, x \in X, \tag*{$(\mathbf{H}_\mathbf{j})$}$$

*then there exists a **unique monomial mapping** $g : X \to Y$ with the following fitting property:*

$$\|f(x) - g(x)\|_\beta \leq \frac{L^{1-j}}{1-L} \Theta(x), \forall\, x \in X. \tag*{$(\mathbf{Est}_\mathbf{j})$}$$

For the proof of our theorem, we need the following fundamental lemma (cf. [10]:

Lemma 3.1. *Let us consider an Abelian group X, a $\beta-$normed linear space Y and a mapping $\varphi : X \times X \to [0, \infty)$. If the function $f : X \to Y$ satisfies* (3.8) *then, for all $x \in X$,*

$$\left\| \frac{f(2x)}{2^N} - f(x) \right\|_\beta \le \frac{1}{2^{N\beta} \cdot (N!)^\beta} \cdot \left(\varphi(0, 2x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \varphi(ix, x) \right). \tag{3.9}$$

Proof of Theorem 3.5. For j=0, by Lemma 3.1, we have, $\forall x \in X$:

$(\mathbf{Q_{\Theta,0}})$

$$\left\| \frac{f(2x)}{2^N} - f(x) \right\|_\beta \le \frac{1}{2^{N\beta} \cdot (N!)^\beta} \left(\varphi(0, 2x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \varphi(ix, x) \right) = \frac{\Theta(2x)}{2^{N\beta}}.$$

For j=1, one can show, for all $x \in X$

$(\mathbf{Q_{\Theta,1}})$

$$\left\| f(x) - 2^N f\left(\frac{x}{2}\right) \right\|_\beta \le \frac{1}{(N!)^\beta} \left(\varphi(0, x) + \sum_{i=0}^{N} \begin{pmatrix} N \\ N-i \end{pmatrix} \cdot \varphi\left(\frac{ix}{2}, \frac{x}{2}\right) \right) = \Theta(x).$$

Now we can apply Theorem 3.3, with $w : X \to Y$, $\eta : X \to X$, $\psi : X \to [0, \infty)$,

$$w(x) := \frac{x}{{r_j}^N},\ \eta(x) := r_j x,\ \psi(x) := \frac{\Theta(2^{1-j}x)}{2^{N\beta(1-j)}}$$

and $r_j = \begin{cases} 2, j = 0 \\ \frac{1}{2}, j = 1 \end{cases}$. Cleary, $l_w = \frac{1}{r_j^{N\beta}}$ and, by using $(\mathbf{Q_{\Theta,j}})$ and the hypothesis $(\mathbf{H}_j)$, we obtain that $(\mathbf{C}_\psi)$ and $(\mathbf{H}_\psi)$ hold.

Then there exists a unique mapping $g : X \to Y$,

$$g(x) := \lim_{m\to\infty} (w^m \circ g \circ \eta^m)(x) = \lim_{m\to\infty} \frac{g({r_j}^m x)}{{r_j}^{mN}}, \forall x \in X, \tag{3.10}$$

which satisfies the following equation

$$(w \circ g \circ \eta)(x) = g(x) \Leftrightarrow g(2x) = 2^N g(x),\ \forall\, x \in X$$

and the inequality

$$\|f(x) - g(x)\|_\beta \le \frac{\psi(x)}{1 - L} = \frac{1}{1 - L} \cdot \frac{\Theta(2^{1-j}x)}{2^{N\beta(1-j)}} \le \frac{L^{1-j}}{1 - L} \cdot \Theta(x), \forall\, x \in X.$$

We show that g is a *monomial function of degree* N. To this end, we replace x with $r_j^m x$ and y with $r_j^m y$ in relation (3.8), then divide the obtained relation by r_j^{mN} and we obtain

$$\left\| \frac{\Delta^N_{r_j^m y} f(r_j^m x)}{r_j^{mN}} - N! \frac{f(r_j^m y))}{r_j^{mN}} \right\|_\beta \le \frac{\varphi(r_j^m x, r_j^m y)}{r_j^{mN\beta}},\ \forall\, x, y \in X.$$

On the other hand,

$$\begin{aligned}\frac{\Delta^N_{r_j^m y} f(r_j^m x)}{r_j^{mN}} &= \sum_{k=0}^{N}(-1)^{N-k}\left(\begin{array}{c}N\\ k\end{array}\right)\frac{f(r_j^m x + k r_j^m y)}{r_j^{mN}} = \\ &= \sum_{k=0}^{N}(-1)^{N-k}\left(\begin{array}{c}N\\ k\end{array}\right) g_m(x+ky) = \\ &= \Delta_y^N g_m(x), \forall\, x, y \in X.\end{aligned}$$

And we get

$$\left|\left|\Delta_y^N g_m(x) - N! \cdot g_m(y)\right|\right|_\beta \leq \frac{\varphi(r_j^m x, r_j^m y)}{r_j^{mN\beta}}\ ,\ \forall x, y \in X.$$

By letting $m \to \infty$ in the above relation and using $\left(\mathbf{H}_\mathbf{j}^*\right)$, we obtain

$$\Delta_y^N g(x) - N! \cdot g(y) = 0,\ \forall x, y \in X. \qquad \square$$

Remark 3.2. For $N = 1$ in Theorem 3.5, we obtain a generalized stability result for the additive Cauchy equation and functions with values in complete β-normed spaces, with

$$\Theta(x) = \varphi\,(0, x) + \varphi\left(0, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \forall\, x \in X.$$

If $N = 2$ in the above theorem it results (as in Theorem 3.4 for $\beta = 1$) that the quadratic functional equation (again for functions with values in complete β-normed spaces) is stable in the Ulam-Hyers-Bourgin sense, with

$$\Theta(x) = \frac{1}{2^\beta}\left(\varphi\,(0, x) + \varphi\left(0, \frac{x}{2}\right) + 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)\right), \forall x\ \in X.$$

It is worth noting that the estimations obtained directly for particular values of N (as in [6], [7], [8] or [11]) are generally better than those resulting from $(\mathbf{Est_j})$, which in its turn **is applicable for all** N.

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