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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Maximum principles for second order elliptic systems with deviating arguments

IOAN CRISTIAN CHIFU

ABSTRACT. The purpose of this paper is to give some boundness properties for the solutions of an elliptic system with deviating arguments, by using the tool of maximum principles.

1. INTRODUCTION

Let $\Omega, D \subset \mathbb{R}^n, \Omega \subset D$ two domains. Let us consider the following second order elliptic operators with deviating argument:

$$L_p u := \sum_{i,j=1}^{n} a_{ij}^{p} \frac{\partial^2 u_p}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j^p \frac{\partial u_p}{\partial x_j} + c^p u_p + \sum_{l=1}^{s} c_{pl} u_l \left(g_l\right), p = \overline{1, s},$$

where $a_{ij}^p, b_j^p, c^p, c_{pl} : \overline{\Omega} \to \mathbb{R}, i, j = \overline{1, n}, p, l = \overline{1, s}$, $g_l : \overline{\Omega} \to \overline{D}, l = \overline{1, s}$. Suppose that the matrix $\left[a_{ij}^p\right], p = \overline{1, s}$, is positive defined.

Definition 1.1. A function $u \in C^2\left(\Omega, \mathbb{R}^s\right) \cap C\left(\overline{D}, \mathbb{R}^s\right)$ satisfies the *vector maximum principle* if there exists a component, u_p, of u with the following properties:

(i) $\left\{ \max_{x \in \overline{D}} u_p(x) = M > 0, u_p(x_o) = M \right\} \Longrightarrow \left\{ x_o \in \overline{D} \setminus \Omega \right\}$;

(ii) $u \leq M$.

Our results extend some previous theorems of Hartman [1], Lihtarnikov [2] and Rus [5], [6] (where the case s=1 is treated), as well as some theorems of Protter [4], Rus [7], etc, where the classical elliptic operator is considered.

2. MAXIMUM PRINCIPLES

In what follows we shall give some maximum principles.

Theorem 2.1. *Let $u \in C^2\left(\Omega, \mathbb{R}^s\right) \cap C\left(\overline{D}, \mathbb{R}^s\right)$ be a solution of $L_p u \geq 0, p = \overline{1, s}$. Suppose that:*

(i) *$\Omega \subset D \subset \mathbb{R}^n$, are two bounded domains;*

(ii) *L_p is an elliptic operator on Ω, $p = \overline{1, s}$;*

(iii) *$c^p + \sum_{l=1}^{s} c_{pl} < 0, c_{pl} \geq 0,\ p, l = \overline{1, s}$;*

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(iv) $u \nleq 0$.

Under these hypotheses u satisfies the vector maximum principle.

Proof. Suppose that there exists $x_k^o \in \Omega$ such that $\max_{x \in \overline{D}} u_k(x) = u_k(x_k^o) = M_k$, $k = \overline{1,s}$. Let $M_p = \max\{M_1, ..., Ms\}$. Because of (iv) it is obvious that $M_p > 0, u_p \leq M_p$ and $u_p(x_p^o) = M_p$. So, for u_p, the conditions from Definition 1 are satisfied. We shall show now that $x_p^o \in \overline{D} \setminus \Omega$. If $x_p^o \in \Omega$, then for u_p we have

$$0 \leq L_p u_p(x_p^o) \leq c^p(x_p^o) u_p(x_p^o) + \sum_{l=1}^{s} c_{pl}(x_p^o) u_l(g_l(x_p^o))$$
$$\leq \left(c^p(x_p^o) + \sum_{l=1}^{s} c_{pl}(x_p^o)\right) M_p < 0,$$

a contradiction. □

Theorem 2.2. *Let $u \in C^2(\Omega, \mathbb{R}^s) \cap C(\overline{D}, \mathbb{R}^s)$ be a solution of*

$$L_p u + f_p(\cdot, u_1, ..., u_s) \geq 0, p = \overline{1,s},$$

where $f_p \in C(\overline{\Omega} \times R^s)$. Under the hypotheses of Theorem 2.1, if

(i) $\forall t, s \in \mathbb{R}^s$, $t \leq s \Longrightarrow f_p(x,t) \leq f_p(x,s), \forall x \in \Omega$, $p = \overline{1,s}$;

(ii) $f_p(x, r, ..., r) \leq 0, \forall r \in \mathbb{R}, \forall x \in \Omega$, $p = \overline{1,s}$;

(iii) $u \nleq 0$,

then u satisfies the vector maximum principle.

Proof. Let $\max_{\overline{D}} u_k = M_k, k = \overline{1,s}$ and let $M_p = \max\{M_1, ..., Ms\}$.
Because of (iii) it is obvious that $M_p > 0$, and there exists $x_o \in \overline{D}$ such that $u_p(x_o) = M_p, u_p \leq M_p$.
We will show now that $x_o \in \overline{D} \setminus \Omega$. We have

$$L_p u(x_o) \geq -f_p(x_o, u_1(x_o), ..., u_s(x_o)) \geq -f_p(x_o, M_p, ..., M_p) \geq 0.$$

Starting from here and following the proof of Theorem 2.1 we obtain that $x_o \in \overline{D} \setminus \Omega$. In conclusion u satisfies the vector maximum principle. □

3. Boundness results

In what follows we shall apply Theorem 2.1 to prove the boundness for the solutions of an second order elliptic system with deviating arguments in the case $\Omega = D = \mathbb{R}^n$.

Definition 3.2. A function $u \in C^2(\mathbb{R}^n, \mathbb{R}^s)$ is said to be bounded if all its components are bounded.

Theorem 3.3. *Suppose that*

(i) $a_{ij}^p, b_j^p, c^p, c_{pl}$ *are bounded in* $\mathbb{R}^n$, $i, j = \overline{1,n}$, $p, l = \overline{1,s}$;

(ii) L_p *is uniform elliptic in* $\mathbb{R}^n$, $p = \overline{1,s}$;

(iii) $c^p + \sum_{l=1}^{s} c_{pl} \leq -\delta^2 < 0, c_{pl} \geq 0, \delta \in \mathbb{R} \setminus \{0\}$, $p, l = \overline{1,s}$;

(iv) $g_p \in C(\mathbb{R}^n, \mathbb{R}^s)$, $p = \overline{1,s}$;

(v) *there exists a matrix function* $v \in C^2(\mathbb{R}^n,\ M_s(\mathbb{R}))$ *such that:*
  (a) $v(x) > 0, \forall\, x \in \mathbb{R}^n \setminus \{0\}$, $v(0) = I_s$;
  (b) $Lv(x) \leq 0, \forall\, x \in \mathbb{R}^n$;
  (c) $\lim\limits_{\|x\| \to \infty} v_{pl}(x) = +\infty$, $p, l = \overline{1, s}$.

Under these hypotheses we have:

(1) $u \in C^2(\mathbb{R}^n, \mathbb{R}^s)$ *is upper bounded and if* $Lu \geq 0$ *then* $u \leq 0$ *in* $\mathbb{R}^n$;
(2) $u \in C^2(\mathbb{R}^n, \mathbb{R}^s)$ *is lower bounded and if* $Lu \leq 0$ *then* $u \geq 0$ *in* $\mathbb{R}^n$.

Proof. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) > 0$ and let $h_p > 0$ be such that $0 < h_p < u_p(x_0)$, $p = \overline{1, s}$. Let us consider the function

$$w(x) = (u(x_0) - h)\, v(x - x_0),$$

where $h = (h_1, ..., h_s)$. We have the followings relations:

$$w_p(x) = \sum_{l=1}^{s} (u_l(x_0) - h_l)\, v_{pl}(x - x_0);$$

$$(u_p - w_p)(x_0) = h_p > 0;$$

$$L_p(u - w) = L_p u - \sum_{l=1}^{s} (u_l(x_0) - h_l)\, L_{pl} v \geq 0.$$

Hence, we observe that for $u_p - w_p$ we have:

$$L_p(u - w)(x) \geq 0, \forall x \in \mathbb{R}^n;$$

$$(u_p - w_p)(x_0) = h_p > 0 \Longrightarrow u - w \nleq 0;$$

$$\lim_{\|x\| \to \infty} (u_p - w_p)(x) = -\infty.$$

Using Theorem 2.1 we obtain that $u - w$ satisfies the vector maximum principle in the sense of Definition 1.1. Hence $u - w$ cannot attain its positive maximum at a point of $\mathbb{R}^n$. On the other hand, because $\lim\limits_{\|x\| \to \infty} (u_p - w_p)(x) = -\infty$, $u_p - w_p$ is negative in a neighborhood of $+\infty$. Hence we obtain that there exists $x_1 \in \mathbb{R}^n$ such that $u_p(x) \leq u_p(x_1)$ and $u_p(x_1) > 0$, $p = \overline{1, s}$. □

Theorem 3.4. *Let* $f_p : \mathbb{R}^n \to \mathbb{R}$ *be bounded functions and* $f_p \geq 0, p = \overline{1, s}$. *Under the conditions of Theorem* 3.3, *the system* $L_p u = f_p, p = \overline{1, s}$, *has at most one bounded solution* $u \in C^2(\mathbb{R}^n, \mathbb{R}^s)$.

Proof. Let $u^1, u^2 \in u \in C^2(\mathbb{R}^n, \mathbb{R}^s)$ two solutions of the system and let $v = u^1 - u^2$. v is a solution of the system $L_p v = 0, p = \overline{1, s}$. Now we apply Theorem 3.3. □

Theorem 3.5. *Let* $f_p : \mathbb{R}^n \to \mathbb{R}$ *be bounded functions and* $f_p \geq 0, p = \overline{1, s}$. *Under the conditions of Theorem* 3.3, *if* $u \in C^2(\mathbb{R}^n, \mathbb{R}^s)$ *is a solution of the system* $L_p u = f_p, p = \overline{1, s}$, *then*

$$\sup_{\mathbb{R}^n} |u_p(x)| \leq \frac{1}{\delta^2} \sup_{\mathbb{R}^n} |f_p(x)|, p = \overline{1, s}.$$

Proof. Consider the following functions:

$$v_p^1(x) = \sup_{\mathbb{R}^n} f_p(x) - \delta^2 u_p(x);$$
$$v_p^2(x) = \sup_{\mathbb{R}^n} f_p(x) + \delta^2 u_p(x).$$

Let $\alpha_k = \sup_{\mathbb{R}^n} f_k(x)$ and $\alpha_p = \max\{\alpha_1, ..., \alpha_s\}$. The conclusion of the theorem follows by Theorem 3.3 and the fact that $L_p v^1 \leq 0$ and $L_p v^2 \geq 0$. □

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Babes - Bolyai University
Department of Business
Horea 7, 400174 Cluj-Napoca, România
E-mail address: cochifu@tbs.ubbcluj.ro