CARPATHIAN J. MATH.
23 (2007), No. 1 - 2, 77 - 80

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Properties of the solution of an integral equation with modified argument

MARIA DOBRIŢOIU

ABSTRACT. In this paper we will use the technique of Picard operators in order to obtain two properties of the solution of the following nonlinear integral equation

$$x(t) = \int_a^b K(t, s, x(s), x(g(s)), x(a), x(b))ds + f(t), \ \ t \in [a, b].$$

1. INTRODUCTION

We consider the nonlinear integral equation

$$x(t) = \int_a^b K(t, s, x(s), x(g(s)), x(a), x(b))ds + f(t), \ \ t \in [a, b]. \tag{1.1}$$

In the paper [2] has been studied the existence and uniqueness, data dependence and the approximation of the solution of this integral equation.

The integral equations of this type have been also studied in [1], [3], [4], [5], [6], [7], [8], [9].

In this paper we will study two properties of the solution of this nonlinear integral equation, via Picard operators technique.

2. NOTATIONS AND PRELIMINARIES

Let X be a nonempty set, d a metric on X and $A : X \to X$ an operator. In this paper we shall use the following notations:

$$F_A := \{x \in X |\ A(x) = x\} \text{ - the fixed point set of } A$$
$$A^{n+1} := A \circ A^n, \quad A^0 := 1_X, \quad A^1 := A, \quad n \in \mathbb{N}.$$

Definition 2.1. (Rus [4] or [5]) Let (X, d) be a metric space. An operator $A : X \to X$ is a **Picard operator** if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$;
(b) the sequence $(A^n(x_0))_{n\in N}$ converges to x^*, for all $x_0 \in X$.

Definition 2.2. (Rus [4] or [5]) Let (X, d) be a metric space. An operator $A : X \to X$ is a **weakly Picard operator** if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

Received: 13.11.2006; In revised form: 18.01.2007; Accepted: 19.02.2007
2000 *Mathematics Subject Classification.* 45G10, 47H10.
Key words and phrases. *Nonlinear integral equation, Picard operator, subsolution.*

If A is a weakly Picard operator, then we consider the following operator

$$A^{\infty} : X \to X, \quad A^{\infty}(x) = \lim_{n\to\infty} A^n(x),\ x \in X.$$

It is clear that $A^{\infty}(X) = F_A$.

In section 3 we will use the Picard operators technique in order to obtain two properties of the solutions of the integral equation (1.1) and we will need the following results (see [4], [5], [8]).

Let $\leq$ be an order relation on X.

Lemma 2.1. (Rus [8]) *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator, such that:*

(i) *the operator A is increasing ;*
(ii) *A is a weakly Picard operator.*

Then the operator A^{∞} is increasing.

Lemma 2.2. (Comparison abstract lemma) *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ three operators, such that:*

(i) *$A \leq B \leq C$;*
(ii) *A, B, C are weakly Picard operators;*
(iii) *the operator B is increasing.*

Then

$$x \leq y \leq z \implies A(x) \leq B(y) \leq C(z).$$

Remark 2.1. Let A, B, C be the operators defined in the comparison abstract lemma. Moreover, we suppose that

$$F_B = \{x_B^*\},$$

i.e. B is Picard operator. Then we have

$$A^{\infty}(x) \leq x_B^* \leq C^{\infty}(x), \quad for \quad all \quad x \in X.$$

But

$$A^{\infty}(X) = F_A,$$

and

$$C^{\infty}(X) = F_C.$$

Therefore we have

$$F_A \leq x_B^* \leq F_C.$$

Lemma 2.3. (Abstract Gronwall lemma) (Rus [5]) *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ anoperator. We suppose that:*

(i) *A is Picard operator;*
(ii) *the operator A is increasing.*

If we denote with x_A^* the unique fixed point of A, then

(a) $x \leq A(x) \ \Rightarrow\ x \leq x_A^*$;
(b) $x \geq A(x) \ \Rightarrow\ x \geq x_A^*$.

3. Main results

Using the Picard operators technique for the integral equations, we have obtained two properties of the solution of the integral equation (1.1).

Theorem 3.1. *We suppose that:*

(i) $K \in C([a,b]\times[a,b]\times\mathbb{R}^4)$, $f \in C[a,b]$ *and* $g \in C([a,b],[a,b])$;

(ii) $K(t,s,\cdot,\cdot,\cdot,\cdot)$ *is increasing for all* $t,s \in [a,b]$;

(iii) *there exists* $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| \leq$$

$$\leq L\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|\right),$$

for all $t,s\in[a,b]$, $u_i,v_i\in\mathbb{R}$, $i=\overline{1,4}$;

(iv) $4L(b-a)<1$.

In this conditions, if x *is a subsolution of the equation* (1.1) *and* $x^* \in C[a,b]$ *is the unique solution of this integral equation, then*

$$x \leq x^*.$$

Proof. We consider the operator $A_K : C[a,b] \to C[a,b]$, defined by

$$A_K(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t),\ \ t\in[a,b]. \tag{3.1}$$

From condition (ii) it results that the operator A_K is increasing. From conditions (i), (iii) and (iv) it results that the operator A_K is an α-contraction with the coefficient $\alpha = 4L\,(b-a)$ and therefore it results that the operator A_K is a Picard operator.

The conditions of the *abstract Gronwall lemma* being satisfied, it results the conclusion of the theorem

$$x \leq x^*$$

and the proof is complete. □

Now we consider the integral equation (1.1) corresponding to the functions K_i, f_i, $i=1,2,3$ and g.

We have:

Theorem 3.2. *We suppose that the functions* K_i, f_i, $i=1,2,3$ *and* g *satisfy the following conditions:*

(i) $K_i \in C([a,b]\times[a,b]\times\mathbb{R}^4)$, $f_i\in C[a,b]$, $i=1,2,3$ *and* $g\in C([a,b],[a,b])$;

(ii) $K_2(t,s,\cdot,\cdot,\cdot,\cdot)$ *is increasing for all* $t,s\in[a,b]$;

(iii) $K_1 \leq K_2 \leq K_3$ *and* $f_1\leq f_2\leq f_3$;

(iv) *there exists* $L_i>0$ *such that*

$$|K_i(t,s,u_1,u_2,u_3,u_4) - K_i(t,s,v_1,v_2,v_3,v_4)|$$
$$\leq L_i\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|+|u_4-v_4|\right),$$

$i=1,2,3$, *for all* $t,s\in[a,b]$, $u_j,v_j\in\mathbb{R}$, $j=\overline{1,4}$;

(v) $4L_i(b-a)<1$, $i=1,2,3$.

Let x_i^* be the unique solution of the equation (1.1) *corresponding to* K_i *,* f_i*,* $i = 1, 2, 3$ *and* g*.*

Then

$$x_1^* \le x_2^* \le x_3^*.$$

Proof. We consider the operators $A_i : C[a, b] \to C[a, b]$, $i = 1, 2, 3$, defined by

$$A_i(x)(t) := \int_a^b K_i(t, s, x(s), x(g(s)), x(a), x(b))ds + f_i(t), \ \ t \in [a, b], \ \ i = 1, 2, 3.$$

From condition (ii) it results that the operator A_2 is increasing, from condition (iii) we have

$$A_1 \le A_2 \le A_3,$$

and from conditions (i), (iv) and (v) it results that the operators A_i are α_i-contractions with the coefficients $\alpha_i = 4L_i\,(b - a)$, $i = 1, 2, 3$ and therefore the operators A_i are Picard operators.

By the *comparison abstract lemma* it results that

$$x_1 \le x_2 \le x_3 \Longrightarrow A_1^\infty(x_1) \le A_2^\infty(x_2) \le A_3^\infty(x_3),$$

but A_1, A_2, A_3 are Picard operators and then by remark 2.1 it results the conclusion of the theorem

$$x_1^* \le x_2^* \le x_3^*.$$

The proof is complete. □

REFERENCES

[1] Dobriţoiu, M., *An integral equation with modified argument*, Studia Univ. "Babeş-Bolyai", Cluj-Napoca, Mathematica, **XLIX**, nr. 3/2004, 27-33

[2] Dobriţoiu, M., *Analysis of an integral equation with modified argument*, Studia Univ. "Babeş-Bolyai", Cluj-Napoca, Mathematica, **51**, nr. 1/2006, 81–94

[3] Rus, I. A., *Generalized contractions*, Univ. "Babeş-Bolyai" of Cluj-Napoca, Preprint No. 3, 1983, 1-130

[4] Rus, I. A., *Weakly Picard mappings*, Comment. Math. Univ. Caroline, **34**, 3(1993), 769-773

[5] Rus, I. A., *Picard operators and applications*, "Babeş-Bolyai" University of Cluj-Napoca, Preprint No. 3, 1996

[6] Rus, I. A., *Fiber Picard operators on generalized metric spaces and applications*, Scripta Scient. Math., Tomus I, Fasciculus II, anno MCMXCIX, 1999, 326-334

[7] Rus, I. A., *A Fiber generalized contraction theorem and applications*, Academie Roumaine Filiale de Cluj-Napoca, Mathematica, Tome **41**(64), No. 1, 1999, 85-90

[8] Rus, I. A., *Weakly Picard operators and applications*, Seminar on Fixed Point Theory, "Babeş-Bolyai" University of Cluj-Napoca, **2**, 2001, 41-58

[9] Şerban, M. A., *Application of fiber Picard operators to integral equations*, Bul. Ştiinţ. Univ. Baia Mare, Seria B, Fasc. Matematică-Informatică, **XVIII** (2002), No. 1, 119-128

UNIVERSITY OF PETROŞANI,
PETROŞANI, ROMANIA
E-mail address: mariadobritoiu@yahoo.com