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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Fixed points for multivalued operators with respect to a w-distance on metric spaces

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ABSTRACT. In this paper we first recall the concept of w-distance on a metric space. Then, we prove a fixed point theorem for contractive type multi-valued operators in terms of a w-distance.

1. INTRODUCTION

In 1996 O. Kada, T. Suzuki and W. Takahashi introduced the concept of w-distance. They gave some examples of w-distance, as well as, some generalizations of Caristi's fixed point theorem, Ekeland's variational principle and the Takahashi's nonconvex minimization theorem, see [1]. Also, some fixed point theorems for multi-valued operators on a complete metric space endowed with a w-distance were established in T. Suzuki, W. Takahashi [3].

Let (X, d) be a complete metric space. Let $T : X \to P(X)$ be a multi-valued operator. Define the function $f : X \to \mathbb{R}$ as $f(x) = D(x, T(x))$. For a positive constant $b \in (0, 1)$ define the set $I_b^x \subset X$ as:

$$I_b^x = \{y \in T(x) \mid bd(x, y) \leq D(x, T(x))\}.$$

In 2006 Y. Feng and S. Liu proved the following theorem, see [2].

Theorem 1.1. *Let (X, d) be a complete metric space $T : X \to P_{cl}(X)$ be a multi-valued operator. If there exists a constant $c \in (0, 1)$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying $D(y, T(y)) \leq cd(x, y)$. Then T has a fixed point in X provided $c < b$ and f is lower semicontinuous on X.*

The purpose of this paper is to extend the above fixed point result for multi-valued operators from [2] in terms of a w-distance on a complete metric space.

2. NOTATION AND BASIC NOTATIONS

Let (X, d) be a complete metric space. We will use the following notations:
$P(X)$ - the set of all nonempty subsets of X;
$P_{cl}(X)$ - the set of all nonempty closed subsets of X;
$P_{b,cl}(X)$ - the set of all nonempty bounded and closed, subsets of X;
$D : P(X) \times P(X) \to \mathbb{R}_+$,

$$D(Z, Y) = \inf\{d(x, y) : x \in Z \,, y \in Y\}, \; Z \subset X$$

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the gap between two nonempty sets.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[1]) as follows:

Definition 2.1. Let (X, d) be a metric space, $p : X \times X \to [0, \infty)$ is called w-distance on X if the following axioms are satisfied :

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
(2) for any $x \in X : p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \delta$.

Let us give some examples of w-distance (see [1]).

Example 2.1. Let (X, d) be a metric space. Then the metric d is a w-distance on X.

Example 2.2. Let X be a normed linear space with norm $||\cdot||$. Then the function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = ||x|| + ||y||$ for every $x, y \in X$ is a w-distance.

Example 2.3. Let (X, d) be a metric space and let $g : X \to X$ a continuous operator. Then the function $p : X \times Y \to [0, \infty)$ defined by

$$p(x, y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ is a w-distance.

For the proof of the main results we need the following lemma (see [3]).

Lemma 2.1. *Let X be a metric space with metric d, p be a w-distance in X,* $\{x_n\}, \{y_n\}$ *be two sequences in X,* $\{\alpha_n\}, \{\beta_n\}$ *be sequence in* $[0, \infty)$ *converging to 0 and* $x, y \in X$. *Then the following hold:*

(i) *if* $p(x_n, y) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$;
(ii) *if* $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to z;*
(iii) *if* $p(x_n, x_m) \leq \alpha_n$ *for any* $m, n \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*
(iv) *if* $p(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

3. Main result

Definition 3.2. Let $T : X \to P(X)$ be a multi-valued operator, $p : X \times X \to [0, \infty)$ be a w-distance on X. Define function $f : X \to \mathbb{R}$ as $f(x) = D_p(x, T(x))$, where $D_p(x, T(x)) = inf\{p(x, y) : y \in T(x)\}$.

For a positive constant $b \in (0, 1)$ define the set $I_b^x \subset X$ as follows:

$$I_b^x = \{y \in T(x) \mid bp(x, y) \leq D_p(x, T(x))\}.$$

We will present now a fixed point theorem for multi-valued operators on a complete metric space endowed with a w-distance.

Theorem 3.2. *Let* (X, d) *be a complete metric space,* $T : X \to P_{cl}(X)$ *a multi-valued operator,* $p : X \times X \to [0, \infty)$ *be a w-distance on X and* $b \in (0, 1)$.

Suppose that:

(i) *there exists* $c \in (0,1)$, *with* $c < b$, *such that for any* $x \in X$ *there is* $y \in I_b^x$ *satisfying*

$$D_p(x, T(x)) \leq cp(x, y);$$

(ii) $f : X \to \mathbb{R}$, $f(x) = D_p(x, T(x))$ *is lower semicontinuous.*
Then T has a fixed point in X.

Proof. Since $T(x) \subset P_{cl}(X)$ then, for any $x \in X$, I_b^x is nonempty for any constant $b \in (0,1)$. For any initial point $x_0 \in X$, there is $x_1 \in I_b^{x_0}$ such that $D_p(x_1, T(x_1)) \leq cp(x_0, x_1)$. For any $x_1 \in X$ there is $x_2 \in I_b^{x_1}$ such that $D_p(x_2, T(x_2)) \leq cp(x_1, x_2)$. We obtain an iterative sequence $\{x_n\}_{n=0}^{\infty}$ where $x_{n+1} \in I_b^{x_n}$ and $D_p(x_{n+1}, T(x_{n+1}))$ $\leq cp(x_n, x_{n+1})$, for $n = 0, 1, 2, \ldots$. We will verify that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore

$$D_p(x_{n+1}, T(x_{n+1})) \leq cp(x_n, x_{n+1}),\ n = 0, 1, 2, \ldots \tag{3.1}$$

On the other hand $x_{n+1} \in I_b^{x_n}$ implies:

$$bp(x_n, x_{n+1}) \leq D_p(x_n, T(x_n)),\ n = 0, 1, 2, \ldots \tag{3.2}$$

By (3.2) it follows $p(x_n, x_{n+1}) \leq D_p(x_n, T(x_n))$, $n = 0, 1, 2, \ldots$.

Using (3.1) we obtain

$$D_p(x_{n+1}, T(x_{n+1})) \leq c\frac{1}{b} D_p(x_n, T(x_n)), n = 0, 1, 2, \ldots \tag{3.3}$$

By (3.1) we have

$$D_p(x_n, T(x_n)) \leq cp(x_{n-1}, x_n),\ n = 0, 1, 2, \ldots$$
$$D_p(x_{n+1}, T(x_{n+1})) \leq cp(x_n, x_{n+1}),\ n = 0, 1, 2, \ldots$$

We replace in (3.3) and we obtain

$$cp(x_n, x_{n+1}) \leq \frac{c}{b} cp(x_{n-1}, x_n),\ n = 0, 1, 2, \ldots$$

If we divide by c we obtain

$$p(x_n, x_{n+1}) \leq \frac{c}{b} p(x_{n-1}, x_n),\ n = 0, 1, 2, \ldots$$

then

$$p(x_{n+1}, x_{n+2}) \leq \frac{c}{b} p(x_n, x_{n+1}), n = 0, 1, 2, \ldots \tag{3.4}$$

We must prove that

$$p(x_n, x_{n+1}) \leq \frac{c^n}{b^n} p(x_0, x_1),\ n = 0, 1, 2, \ldots$$
$$D_p(x_n, T(x_n)) \leq \frac{c^n}{b^n} D_p(x_0, T(x_0)),\ n = 0, 1, 2, \ldots$$

We know that $p(x_n, x_{n+1}) \leq \frac{c}{b} p(x_{n-1}, x_n)$, $n = 0, 1, 2, \ldots$. Hence, we get that $p(x_n, x_{n+1}) \leq \frac{c^n}{b^n} p(x_0, x_1)$, $n = 0, 1, 2, \ldots$

Since $D_p(x_n, T(x_n)) \leq \frac{c}{b} D_p(x_{n-1}, T(x_{n-1}))$, $n = 0, 1, 2, \ldots$ we obtain

$$D_p(x_{n-1}, T(x_{n-1})) \leq \frac{c}{b} D_p(x_{n-2}, T(x_{n-2})),\ n = 0, 1, 2, \ldots$$

We replace and obtain

$$D_p(x_n, T(x_n)) \le \frac{c}{b}\frac{c}{b} D_p(x_{n-2}, T(x_{n-2})),\ n = 0, 1, 2, \dots$$

Therefore

$$D_p(x_n, T(x_n)) \le \frac{c^2}{b^2} D_p(x_{n-2}, T(x_{n-2})),\ n = 0, 1, 2, \dots$$
$$D_p(x_{n-2}, T(x_{n-2})) \le \frac{c}{b} D_p(x_{n-3}, T(x_{n-3})),\ n = 0, 1, 2, \dots$$

Then we have

$$D_p(x_n, T(x_n)) \le \frac{c^3}{b^3} D_p(x_{n-3}, T(x_{n-3})),\ n = 0, 1, 2, \dots$$

We obtain that $D_p(x_n, T(x_n)) \le \frac{c^n}{b^n} D_p(x_0, T(x_0))$, $n = 0, 1, 2, \dots$
Then, for $m, n \in \mathbb{N}$, $m > n$ and $a = \frac{c}{b}$ we have

$$\begin{aligned} p(x_m, x_n) &\le p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \cdots + p(x_{n+1}, x_n) \\ &\le a^{m-1} p(x_0, x_1) + a^{m-2} p(x_0, x_1) + \cdots + a^n p(x_0, x_1) \\ &\le \frac{a^n}{1-a} p(x_0, x_1). \end{aligned}$$

Using Lemma 2.1 it follows that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. X is a complete space, then there is a $x \in X$ such that $\{x_n\}_{n=0}^{\infty}$ converge to x. We assert that x is a fixed point of T. The sequence $\{f(x_n)\}_{n=0}^{\infty} = \{D_p(x_n, T(x_n))\}_{n=0}^{\infty}$ is decreasing and from the above construction it converge to 0. Since f is lower semicontinuous we have

$$0 \le f(x) \le \lim_{n\to\infty} f(x_n) = 0.$$

Since $f(x) = 0$. From $T(x) \in P_{cl}(X)$ we have that $x \in T(x)$. Then T has a fixed point in X. $\square$

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