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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

On the approximation of surfaces with negative Gauss curvature using surfaces attached to the monogenous functions

LIDIA ELENA KOZMA

ABSTRACT. This article analyses the possibility of approximating the surfaces with negative Gauss curvature of form: $(S) : \bar{r} = (f(t), M(t,v), N(t,v))$ or $(f(v), M(t,v), N(t,v))$ with surfaces of form $(S_m) : \bar{r} = (y, U(x,y), V(x,y))$ or $(x, U(x,y), V(x,y))$, where $U(x,y) + iV(x,y) = F(z)$ is a monogenous function in $D \subset \mathbb{R}^2$, simple connected.

1. INTRODUCTION

The minimal surfaces (having the total curvature negative) constitute a very interesting subject because of the applying possibilities in various domains.

Two surfaces being given (S_1) and (S_2)

$$(S_1) : \bar{r} = \bar{r}_1(u,v); \quad (S_2) : \bar{r} = \bar{r}_2(u^*, v^*)$$

reported to the curve-linear coordinates (u,v) and (u^*, v^*), we can establish a bi-univocal, bi-continuous correspondence between two regulate portions of each surface. This correspondence is a transformation of form:

$$(T) : \begin{cases} u^* = u^*(u,v) \\ v^* = v^*(u,v) \end{cases}$$

with functions $u^*(u,v)$ and $v^*(u,v)$ bi-univocal and continuous and the functional determinant $\frac{D(u^*,v^*)}{D(u,v)} \neq 0$, finite in the domain from the variables plan (u,v).

It is known that in the case when (T) is an isometric transformation, the total curvature is kept. The system which conditions the fact that transformation T is isometric is of form (see [1]).

$$\begin{cases} E^* \left(\dfrac{\partial u^*}{\partial u}\right)^2 + 2F^* \dfrac{\partial u^*}{\partial u} \cdot \dfrac{\partial v^*}{\partial u} + G^* \dfrac{\partial v^*}{\partial u} = E \\ E^* \dfrac{\partial u^*}{\partial u} \cdot \dfrac{\partial u}{\partial v} + F^* \left(\dfrac{\partial u^*}{\partial u} \cdot \dfrac{\partial v}{\partial v} + \dfrac{\partial u^*}{\partial v} \cdot \dfrac{\partial v^*}{\partial u}\right) + G^* \dfrac{\partial u^*}{\partial u} \cdot \dfrac{\partial v^*}{\partial v} = F \\ E^* \dfrac{\partial u^*}{\partial v} + 2F^* \dfrac{\partial u^*}{\partial v} \cdot \dfrac{\partial v^*}{\partial v} + G^* \left(\dfrac{\partial v^*}{\partial v}\right) = G. \end{cases} \tag{1.1}$$

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The above system is in general incompatible. We wonder if there is the possibility of a transformation of coordinates through which a surface of negative Gauss curvature could be transformed into a surface of form

$$(S_m) : \bar{r} = (y, U(x,y), V(x,y)), \quad (x,y) \in D \subset \mathbb{R}^2, \ D - \text{simple connected.} \tag{1.2}$$

where $U(x,y) + iV(x,y) = F(x,y)$ is a monogenous function.

May there be a surface with negative Gauss curvature of form

$$(S) : \bar{r} = (L(u,v), M(u,v), N(u,v)), \quad (u,v) \in D^* \subset \mathbb{R}^2.$$

In [2] we made the transformation of the form:

$$(T) : \begin{cases} y = L(u,v) \\ x = x(u,v) \end{cases}$$

unknown till now, it will be determined from the condition that function

$$F(x,y) = M(u(x,y), v(x,y)) + iN(u(x,y), v(x,y))$$

is monogenous.

If the requests imposed on this condition take place, then any surface which fulfills these requests can be transformed into a monogenous surface of form (1.2).

These requests, although they lead to a less difficult system than system (1.1), do not always take place. In these conditions we will use an approximating method.

In this paper we intend to analyze the possibility of approximating a surface of form

$$\begin{array}{l} (S_1) : \bar{r} = (f(t), M(t,v), N(t,v)) \quad \text{or} \\ (S_2) : r = (g(v), M(t,v), N(t,v)) \end{array} \tag{1.3}$$

with monogenous surfaces of form (1.2).

Which is the advantage of approximating a surface with negative curvature of form (S_1) or (S_2) through monogenous surfaces of form (1.2)?

In [3], we extended the identity Principle for holomorphic functions in the case of surfaces of form (1.2) under the form following theorem:

Theorem 1.1. *If a surface of form* (1.2) *attached to a monogenous function is known in sequence of points on the surface with the sequence being convergent towards a point of the surface, then the surface is known for the entire monogenity domain of the attached function.*

The approximation of a surface of form (1.3) (S_1) or (S_2) by monogenous surfaces would allow the approximate determination of surfaces (1.2) knowing a line of points convergent on these surfaces.

We note one of the important properties of surfaces (1.2): the fact that these surfaces have negative Gauss curvature, and for calculating the curvature we deducted the following relation (see [4]),

$$K = -\frac{|F''(z)|^2}{|F'(z)|^2[1 + |F'(z)|^2]^2} \tag{1.4}$$

where $F(z) = U(x,y) + iV(x,y)$ monogenous on $D \subset \mathbb{R}^2$

$$F'(z) = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$
$$F''(z) = \frac{\partial^2 F}{\partial z^2} = \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 U}{\partial x^2} + i\frac{\partial^2 V}{\partial x^2}\,.$$

2. The analyze of transforming surfaces of form (1.3) into surfaces S_m of form (1.2)

We formulate the theorem:

Theorem 2.2. *Surfaces of form* (S_1) *or* (S_2)

$$(S_1) : \bar{r} = (y, U(x,y), V(x,y))$$
$$(S_2) : \bar{r} = (x, U(x,y), V(x,y))$$

with $U(x,y) + iV(x,y)$ *a monogenous function on* $D \subset \mathbb{R}^2$*, have the same Gauss curvature.*

Proof. The results are obtained immediately from the direct calculation of coefficients E, F, G of the two surfaces. If we note E_1, F_1, G_1 the coefficients of the first fundamental form for surface (S_1), E_2, F_2, G_2 for surface (S_2), then:

$$E_2 = E_1 + 1$$
$$F_1 = F_2 = 0$$
$$G_2 = G_1 - 1$$

which, replaced in the known calculation formula of the Gauss curvature [1] for $F = 0$, will give the same curvature as given in relation (1.4). □

We will analyze as follows the generalization of the above mentioned theorem for surfaces of the following form:

$$\begin{aligned}(S_1^*) &: \bar{r} = (f(y), U(x,y), V(x,y)) \\ (S_2^*) &: r = (g(x), U(x,y), V(x,y))\end{aligned} \tag{2.5}$$

having $U + iV$ monogenous are surfaces of negative Gauss curvature, different from the Gauss curvature given by formula (1.3).

Indeed, coefficient $\bar{F} = \bar{r}_x \cdot \bar{r}_y = 0$, if we regard the Cauchy-Riemann monogenity conditions.

For (S_1^*)

$$(S_1^*) : \begin{cases} \bar{r}_x = (0, U_x, V_x) \\ \bar{r}_y = (f'(y), U_y, V_y) \end{cases}$$

and $\bar{r}_x \cdot \bar{r}_y = U_x U_y + V_x V_y = U_x U_y - U_y U_x = 0$.

In the same way, for (S_2^*)

$$(S_2^*) : \begin{cases} \bar{r}_x = (g'(x), U_x, V_x) \\ \bar{r}_y = (0, U_x, V_y) \end{cases}$$

one obtains $\bar{r}_x \cdot \bar{r}_y = 0$.

For beginning we will consider the following two particulary cases of surfaces with negative Gauss curvature. We consider the following cases:

(2.6) Case I $\quad (S_1) = \bar{r} = (f(u), u\cos v, u\sin v)$

(2.7) Case II $$(S_2) : \bar{r} = (f(t), a \sin t \cos v, a \sin t \sin v),$$

two surfaces which have negative Gauss curvature.

We intend to determine the monogenous surfaces of negative Gauss curvature which can approximate these surfaces.

Case I. $\bar{r} = (f(u), u \cos v, u \sin v)$, where $(u, v) \in D \subset \mathbb{R}^2$.

We do the transformation: $y = f(u)$, in the condition where we have $u = f^{-1}(y)$ with

$$\left[f^{-1}(y)\right]_y' \neq 0. \tag{2.8}$$

We'll determine $v = v(x, y)$ from the Cauchy-Riemann conditions applied to functions

$$\begin{cases} U = f^{-1}(y) \cos v(x, y) \\ V = f^{-1}(y) \sin v(x, y). \end{cases}$$

We obtain the system:

$$\begin{cases} f^{-1}(y)\, v_y = 0 \\ v_x = -\frac{\left[f^{-1}(y)\right]_y'}{f^{-1}(y)} \end{cases} \tag{2.9}$$

from where $v_y = 0$ and thus $v = v(x)$.

The compatibility of system (2.6) will impose v_x =constant = k and we obtain $v = kx + C$, C = constant.

Condition

$$-\frac{\left[f^{-1}(y)\right]_y'}{f^{-1}(y)} = constant \tag{2.10}$$

will request the reconsidering of relation (2.5) and thus $\frac{\left[f^{-1}(y)\right]_y'}{f^{-1}(y)} = k_1$; $\left[\ln f^{-1}(y)\right]' = k_1$ and $\ln f^{-1}(y) = k_1 y + k_2$

$$f^{-1}(y) = e^{k_1 y + k_2}, \tag{2.11}$$

which is not necessarily the function given by relation (2.5).

2.1. **Example for surfaces of form (2.6) in the case I. Example A.** May there be the surface of form I in the case when:

$$f(u) = a \log\left(u + \sqrt{u^2 - a^2}\right), \ a > 0$$

and the function obtained from here will be

$$f^{-1}(y) = \frac{1}{2}\left(e^{\frac{y}{a}} + a^2 e^{-\frac{y}{a}}\right) \tag{2.12}$$

different from the function obtained in relation (2.7) as a result of Cauchy-Riemann conditions.

In this case we will consider an approximation of surface

$$(S_a) : \bar{r}_a = \left(a \log\left(u + \sqrt{u^2 - a^2}\right), u \cos v, u \sin v\right) \tag{2.13}$$

with a monogenous surface (attached to a monogenous function) of form (1.2)

$$(S_m) : \bar{r}_m = (y, e^{k_1 y + k_2} \cos(-k_1 x + b), e^{k_1 y + k_2} \sin(-k_1 x + b)) \tag{2.14}$$

with $k_1 = -k$, $k_1, k_2, b-$ real constants.

The monogenous function $f(z) = U(x, y) + iV(x, y)$ will be in this case:

$$F(x, y) = e^{k_1 y + k_2} \cdot e^{i(-k_1 x + b)} = e^{k_1 y + k_2 + i(-k_1 x + b)}. \tag{2.15}$$

The Gauss curvature of the monogenous surface (10) will be calculated using formula (1.3) and it will be

$$K = -\frac{k_1^2}{\left[1 + k_1^2 e^{2(k_1 y + k_2)}\right]^2}. \tag{2.16}$$

A possible expression of the error used to approximate surface (2.9) through surface (2.11) is given by the difference:

$$|f(u) - f^*(u)| = \left| a \log\left(u + \sqrt{u^2 - a^2}\right) - \frac{\ln u - k_2}{k_1} \right| \tag{2.17}$$

where we noted $f^*(u) = \frac{\ln u - k_2}{k}$.

The relation (2.17) is equivalent to the difference of inversions

$$|f^{-1}(y) - f^{*^{-1}}(y)| = \left| \frac{1}{2}\left(e^{\frac{y}{a}} + a^2 e^{-\frac{y}{a}}\right) - e^{k_1 y + k_2} \right|. \tag{2.18}$$

Case II.

$$\bar{r} = (f(t), a \sin t \cos v, a \sin t \sin v). \tag{2.19}$$

Doing the transformation $y = f(t)$ in the existence conditions of the inversion function $t = f^{-1}(y)$, with $\left[f^{-1}(y)\right]' \neq 0$ we obtain

$$\begin{cases} U = a \sin f^{-1}(y) \cos v(x, y) \\ V = a \sin f^{-1}(y) \sin v(x, y) \end{cases} \tag{2.20}$$

The Cauchy-Riemann conditions for functions U and V from relation (2.19) will give:

$$\begin{cases} 0 = a \sin f^{-1}(y) v_y \\ -a \sin f^{-1}(y) v_x = a \cos f^{-1}(y) \left[f^{-1}(y)\right]', \end{cases} \tag{2.21}$$

and thus $v_y = 0$, $v_x = -\frac{\left[f^{-1}(y)\right]'}{tg f^{-1}(y)}$ or

$$\begin{cases} v = k_1 x + k_2 \\ \left[f^{-1}(y)\right]' \operatorname{ctg} f^{-1}(y) = constant = k_1 \end{cases} \tag{2.22}$$

from where

$$y = \frac{1}{k_1} \ln \sin t. \tag{2.23}$$

We say that transformation $y = f(t)$ is exact if relation

$$y = f(t) = \frac{1}{k_1} \ln \sin t,$$

takes place, where $f(t)$ is the function from relation (2.19) given for a surface of form II.

In a contrary case, there will only be an approximation of the surface of form II with the monogenous surface (attached to a monogenous function).

Solving an example for surfaces of form (2.7), in case II.

Example B. Let there be a surface of form II

$$r = \left(a \left(\cos t + \log tg \frac{t}{2} \right), a \sin t \cos v, a \sin t \sin v \right). \tag{2.24}$$

Using the transformation:

$$y = a \left(\cos t + \log tg \frac{t}{2} \right) \quad \text{or} \quad f^{-1}(y) = k \ln \sin f^{-1}(y) \tag{2.25}$$

we obtain an exact value of the monogenous surface, function given in relation (2.25) should be $y = \frac{1}{k_1} \ln \sin t$ from relation (2.23).

The monogenous function for $y = f(t)$ given by relation (2.25) is

$$F(x, y) = ae^{c_1 y} \cdot e^{i(k_1 x + k_2)} = ae^{i(k_1 z + k_2)}, \tag{2.26}$$

where we note $c_1 = -k_1$.

The monogenous surface corresponding to function (2.26) is

$$(S_m) : \bar{r} = (y, ae^{-k_1 y} \cos(k_1 x + k_2), ae^{-k_1 y} \sin(k_1 x + k_2). \tag{2.27}$$

The error of approximation of surface (2.24) through a monogenous surface can be considered given by the distance:

$$\varepsilon(t) = \left| -\frac{1}{k} \ln \sin t - a \left(\cos t + \log tg \frac{t}{2} \right) \right|. \tag{2.28}$$

In this example, choosing $a = \frac{1}{k}$, we obtain

$$\begin{aligned} \varepsilon(t) &= a | \ln(1 + \cos t) - \cos t | \\ &= a \left| -\frac{(\cos t)^2}{2} + \frac{(\cos t)^3}{3} - \frac{(\cos t)^4}{4} + \dots \right| \\ &< a \left| \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \right| = a \ln 2. \end{aligned}$$

The Gauss curvature of the monogenous surface (2.27) is noted K_m and in this case it will be

$$K_m = -\frac{1}{a^2 [1 + (e^{c_1 y})^2]^2} = -\frac{1}{a^2 [1 + \sin^2 t]^2}, \quad t \neq 0. \tag{2.29}$$

The Gauss curvature of the given surface under the form (2.24) is

$$K = -\frac{1}{a^2}. \tag{2.30}$$

We can formulate the following theorem:

Theorem 2.3. *To any surface of form*

$$(S_G) : \bar{r} = (f(t), M(t, v), N(t, v)), \tag{2.31}$$

with $(t, v) \in D^* \subset \mathbb{R}^2$ *having negative Gauss curvature, there can be associated a monogenous surface of form*

$$(S_m) : \bar{r} = (y, U(x, y), V(x, y)) \tag{2.32}$$

where $(x, y) \in D \subset \mathbb{R}^2$ *and* $F(z) = U(x, y) + iV(x, y)$*, is a monogenous function on domain* $D \subset \mathbb{R}^2$*, simple connected.*

The above mentioned association has the following meaning:

Case I. Surface (S_G) transforms punctually exactly into surface (S_m) using the system of transformations (T).

$$(T): \begin{cases} y = f(t) \ \text{ if exist } \ t = f^{-1}(y) \ \text{ with } \ [f^{-1}(y)]' \neq 0 \\ v = v(x, f(t)) \end{cases} \tag{2.33}$$

and if functions $f(t);\ v(x, f(t))$ satisfies the Cauchy-Riemann monogenity conditions

$$\begin{cases} U_x = V_y \\ U_y = -V_x \end{cases} \tag{2.34}$$

applied to functions

$$\begin{cases} U(x, y) = M(f^{-1}(y), v(x, f^{-1}(y)) \\ V(x, y) = N(f^{-1}(y), v(x, f^{-1}(y)). \end{cases} \tag{2.35}$$

Remark 2.1. Generally system (2.28) of Cauchy-Riemann conditions allows the determination of function $y = f^*(t)$, not necessarily equal to the expression of $f(t)$ from the equation of the given surface (2.25).

In the above mentioned case I we consider $f^*(t) = f(t)$. We note that the determination of function $v = v(x, y)$ takes place as a result of system (2.34) only when the conditions of the implicit functions' theorem are satisfied. System (2.34) applied to functions U and V from (2.35) allow the calculation of partial derivatives v_x and v_y and not directly of function $v(x, y)$.

Case II. If from system (2.34) we obtain a function $y = f^*(t)$ different from function $f(t)$ given in equation (2.25) of surface (S_G), then the monogenous surface of form

$$(S_m) : \bar{r} = (f^*(t), U(x, f^*(t)), V(x, f^*(t))) \tag{2.36}$$

approximates surface (S_G) given by relation (2.31).

In all cases the functions $M(t, v),\ N(t, v) \in C^2(D^*)$.

The proof of the theorem results from the direct calculation.

Remark 2.2. An expression of the approximation error of surface (S_G) from relation (2.31) with the help of the monogenous surface (S_m) given by relation (2.36) can be expressed using the Euclidean distance $\varepsilon(t) = |f^*(t) - f(t)|$ in each point $(x, y) = (x, f^*(t))$ of the monogenous surface. See relation (2.17) from example (A) and relation (2.28) for example (B).

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NORTH UNIVERSITY OF BAIA MARE
DEPARTMENT OF MATHEMATICS AND
COMPUTER SCIENCE
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
E-mail address: kozma_lidia@yahoo.com