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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

On asymptotic behaviors for linear skew-evolution semiflows in Banach spaces

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ABSTRACT. In this paper we define the notion of linear skew-evolution semiflow in Banach spaces. We give several characterizations of some asymptotic behaviors, as stability, instability and dichotomy. The obtained results are generalizations in the nonuniform case of some well-known results on asymptotic behaviors of linear differential equations.

1. PRELIMINARIES AND DEFINITIONS

The progress made in the last years in the study of the asymptotic behavior of linear skew-product semiflows has been used in the theory of evolution equations in infinite dimensional spaces. The approach from point of view of asymptotic properties for the evolution semigroup associated to the linear skew-product semiflows was essential.

In this paper we define the concept of skew-evolution semiflow, more appropriate for the study of asymptotic behaviors of evolution equations by means of evolution operators. A unified treatment for nonuniform behaviors as stability, instability and dichotomy is given.

Let X be a metric space.

Definition 1.1. A mapping $\varphi : \mathbb{R}_+ \times X \to X$ is called a *semiflow* on X if it satisfies the following properties

(s_1) $\varphi(0, x) = x$ for all $x \in X$

(s_2) $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for all $(t, s, x) \in \mathbb{R}^2_+ \times X$.

Example 1.1. Let V_1 and V_2 be two Banach spaces and let us consider that the set defined by $X = \mathcal{C}(\mathbb{R}_+ \times V_1, V_2)$ of all continuous functions $x : \mathbb{R}_+ \times V_1 \to V_2$ is given with the topology of uniform convergence of bounded sets. The mapping

$$\varphi : \mathbb{R}_+ \times X \to X, \varphi(t, x) = x_t,$$

where $x_t : \mathbb{R}_+ \times V_1 \to V_2$ is defined by $x_t(s, v_1) = x(t + s, v_1)$ is a semiflow on X.

Let X be a metric space, V a Banach space, $\mathfrak{B}(V)$ the space of all bounded linear operators on V and let us denote $E = \mathbb{R}^2_+ \times X$ respectively $Y = X \times V$.

Definition 1.2. A mapping $\Phi : E \to \mathfrak{B}(V)$ is called a *2-parameter cocycle* over the semiflow $\varphi : \mathbb{R}_+ \times X \to X$ on E if it satisfies the following properties

(c_1) $\Phi(t, t, x) = x$ for all $t \geq 0$ and all $x \in X$

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(c_2) $\Phi(t,t_0,x)=\Phi(t,s,\varphi(s,x))\Phi(s,t_0,x)$ for all $t,s,t_0\in\mathbb{R}_+$, $t\geq s\geq t_0$ and all $x\in X$

(c_3) there exist a nondecreasing function $f:\mathbb{R}_+\to\mathbb{R}_+^*$ such that

$$\|\Phi(t,t_0,x)\|\leq f(t-t_0) \tag{1.1}$$

for all $(t,t_0)\in\mathbb{R}_+^2$, $t\geq t_0$ and all $x\in X$.

Definition 1.3. A function $\xi:\mathbb{R}_+^2\times Y\to Y$ defined by

$$\xi(t,s,x,v)=(\varphi(t,x),\Phi(t,s,x)v) \tag{1.2}$$

for all $(t,s,x,v)\in\mathbb{R}_+^2\times Y$, where Φ is a 2-parameter cocycle over the semiflow φ on E, is called *linear skew-evolution semiflow* on Y.

Example 1.2. Let $F:\{(t,s)\in\mathbb{R}_+^2, t\geq s\}\to\mathfrak{B}(V)$ be an evolution operator on V and X a metric space. The pair $\xi_F=(\varphi,\Phi_F)$ where φ is a semiflow on X and $\Phi_F:\mathbb{R}_+^2\times X\to\mathfrak{B}(V)$ is given by $\Phi_F(t,s,x)=F(t,s)$ is a linear skew-evolution semiflow on Y.

Definition 1.4. A linear skew-evolution semiflow $\xi=(\varphi,\Phi)$ is said to be *strongly measurable* if for every $(t_0,x,v)\in\mathbb{R}_+\times Y$ the mapping $t\mapsto\Phi(t,t_0,x)v$ is measurable.

Remark 1.1. Let $\xi=(\varphi,\Phi)$ be a linear skew-evolution semiflow on Y and $\lambda\in\mathbb{R}$. We denote by Φ_λ the mapping $\Phi_\lambda:E\to\mathfrak{B}(V)$ defined by

$$\Phi_\lambda(t,t_0,x)=e^{-\lambda(t-t_0)}\Phi(t,t_0,x). \tag{1.3}$$

Then $\xi_\lambda=(\varphi,\Phi_\lambda)$ is also a linear skew-evolution semiflow called the *shifted skew-evolution semiflow* on Y.

2. Stability for linear skew-evolution semiflows

Definition 2.5. A linear skew-evolution semiflow $\xi=(\varphi,\Phi)$ is said to be *stable* if there exists a function $N:\mathbb{R}_+\to\mathbb{R}_+^*$ such that

$$\|\Phi(t,t_0,x)\|\leq N(t_0) \tag{2.4}$$

for all $t\geq t_0\geq 0$ and all $x\in X$.

Proposition 2.1. *Let us consider a strongly measurable linear skew-evolution semiflow $\xi=(\varphi,\Phi)$. If there exists a function $M:\mathbb{R}_+\to\mathbb{R}_+^*$ such that*

$$\int_{t_0}^{t}\|\Phi(s,t_0,x)v\|\,ds\leq M(t_0)\,\|v\| \tag{2.5}$$

for all $t\geq t_0\geq 0$, all $x\in X$ and all $v\in V$, then ξ is stable.

Proof. As a first step let us consider $t_0\geq 0$, $t\geq t_0+1$ and $x\in X$. We denote

$$K=\int_0^1\frac{1}{f(u)}du$$

where function f is given as in relation (1.1) of Definition 1.2. Following relations hold

$$\int_0^1 \frac{\|\Phi(t,t_0,x)v\|}{f(u)}du = \int_{t-1}^t \frac{\|\Phi(t,t_0,x)v\|}{f(t-s)}ds$$
$$\leq \int_{t-1}^t \|\Phi(s,t_0,x)v\| \, ds \leq \int_{t_0}^t \|\Phi(s,t_0,x)v\| \, ds \leq M(t_0) \, \|v\| \, .$$

Hence

$$\|\Phi(t,t_0,x)v\| \leq K^{-1} M(t_0) \, \|v\| \, .$$

As a second step, if $t_0 \leq t \leq t_0 + 1$ and $v \in V$, we have

$$\|\Phi(t,t_0,x)v\| \leq f(1) \, \|v\| \, .$$

If we consider the function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ defined as

$$N(t_0) = K^{-1} M(t_0) + f(1)$$

we obtain that the skew-evolution semiflow $\xi = (\varphi, \Phi)$ is stable. □

Definition 2.6. A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *exponentially stable* if there exist a function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\nu > 0$ such that

$$\|\Phi(t,t_0,x)v\| \leq N(t_0) e^{-\nu(t-t_0)} \, \|v\| \tag{2.6}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Example 2.3. Let us consider the Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t > t_0 \\ x(t_0) = x_0 \end{cases}$$

where

$$A(t)x = \dot{\alpha}(t)x$$

and

$$\alpha(t) = -3t + t\cos t.$$

The skew-evolution semiflow $\xi = (\varphi, \Phi)$, where φ is a semiflow and the 2-parameter cocycle is solution of the Cauchy problem, given by

$$\Phi(t,t_0,x) = e^{\alpha(t)-\alpha(t_0)} x$$

is exponentially stable with $N(t_0) = e^{2t_0}$ and $\nu = 2$.

Proposition 2.2. *The strongly measurable linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is exponentially stable if and only if there exist a function $M : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\alpha > 0$ such that*

$$\int_{t_0}^t e^{\alpha(\tau - t_0)} \, \|\Phi(\tau,t_0,x)v\| \, d\tau \leq M(t_0) \, \|v\| \tag{2.7}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Proof. Necessity. As the skew-evolution semiflow $\xi = (\varphi, \Phi)$ is exponentially stable there exist $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and $\nu > 0$ such that

$$\|\Phi(t,s,x)v\| \leq N(s)e^{-\nu(t-s)}\|v\|$$

for all $t \geq s \geq 0$, $x \in X$ and all $v \in V$. We consider α such that $\nu = 2\alpha$. Following relations hold

$$\int_{t_0}^{t} e^{\alpha(\tau-t_0)}\|\Phi(\tau,t_0,x)v\|\,d\tau \leq N(t_0)\|v\|\int_{t_0}^{t} e^{\alpha(\tau-t_0)}e^{-\nu(\tau-t_0)}d\tau \leq M(t_0)\|v\|,$$

where we have denoted

$$M(t_0) = \frac{N(t_0)}{\alpha}.$$

Sufficiency. Let f be the function given by Definition 1.2.

For $t \geq t_0 + 1$ following relations hold

$$\|\Phi(t,t_0,x)v\|\,e^{\alpha(t-t_0)}\int_0^1 \frac{e^{-\alpha u}}{f(u)}du = e^{\alpha(t-t_0)}\|\Phi(t,t_0,x)v\|\int_{t-1}^{t}\frac{e^{-\alpha(t-\tau)}}{f(t-\tau)}d\tau$$

$$= \int_{t-1}^{t}\frac{\|\Phi(t,t_0,x)v\|}{f(t-\tau)}e^{\alpha(\tau-t_0)}d\tau \leq \int_{t-1}^{t}\|\Phi(\tau,t_0,x)v\|\,e^{\alpha(\tau-t_0)}d\tau \leq M(t_0)\|v\|.$$

If $t \in [t_0, t_0+1)$ we have

$$\|\Phi_{-\alpha}(t,t_0,x)v\| \leq f(1)e^{\alpha}\|v\|.$$

We obtain

$$\|\Phi_{-\alpha}(t,t_0,x)v\| \leq \widetilde{M}(t_0)\|v\|$$

where we have denoted

$$\widetilde{M}(t_0) = f(1)e^{\alpha} + M(t_0)[\int_0^1 \frac{e^{-\alpha u}}{f(u)}du]^{-1}.$$

It follows that

$$\|\Phi(t,t_0,x)v\| \leq \widetilde{M}(t_0)e^{-\alpha(t-t_0)}\|v\|$$

for $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$. Hence the linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is exponentially stable. □

3. Instability for linear skew-evolution semiflows

Definition 3.7. A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *unstable* if there exists a function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$N(t)\|\Phi(t,t_0,x)v\| \geq \|\Phi(s,t_0,x)v\| \tag{3.8}$$

for all $t \geq s \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Proposition 3.3. *Let $\xi = (\varphi, \Phi)$ be a strongly measurable linear skew-evolution semiflow. If there exists a function $M : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that*

$$\int_{t_0}^{t}\|\Phi(s,t_0,x)v\|ds \leq M(t)\|\Phi(t,t_0,x)v\| \tag{3.9}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$ then ξ is unstable.

Proof. If we denote by

$$\frac{1}{K} = \int_0^1 \frac{du}{f(u)},$$

where the function f is given by (1.1), then, for $t \geq s \geq t_0 \geq 0, x \in X, v \in V$, we have

$$\begin{aligned} \frac{\|\Phi(s,t_0,x)v\|}{K} &= \int_{s-1}^{s} \frac{\|\Phi(s,t_0,x)v\|}{f(s-\tau)} d\tau \\ &\leq \int_{s-1}^{s} \|\Phi(\tau,t_0,x)v\| d\tau \leq M(t)\|\Phi(t,t_0,x)v\| \end{aligned}$$

and hence we obtain the conclusion. □

Definition 3.8. A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *exponentially unstable* if there exist a function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\nu > 0$ such that

$$N(t)\|\Phi(t,t_0,x)v\| \geq e^{\nu(t-t_0)}\|v\| \tag{3.10}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Example 3.4. Let us consider the Cauchy problem

$$\begin{cases} \dot{x}(t) = B(t)x(t), \quad t > t_0 \\ x(t_0) = x_0 \end{cases}$$

where

$$B(t)x = \dot{\beta}(t)x$$

and

$$\beta(t) = 2t - 3t\cos t.$$

The linear skew-evolution semiflow $\xi = (\varphi, \Phi)$, where φ is a semiflow and the 2-parameter cocycle is solution of the Cauchy problem, given by

$$\Phi(t,t_0,x) = e^{\beta(t)-\beta(t_0)}x$$

is exponentially unstable with $N(t) = e^{6t}$ and $\nu = 5$.

Proposition 3.4. *The strongly measurable linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is exponentially unstable if and only if there exist a function $M : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\beta > 0$ such that*

$$\int_{t_0}^{t} e^{\beta(t-\tau)} \|\Phi(\tau,t_0,x)v\| \, d\tau \leq M(t) \|\Phi(t,t_0,x)v\| \tag{3.11}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Proof. Necessity. As ξ is exponentially unstable, there exist a function $N : \mathbb{R}_+ \to \mathbb{R}_+^*$ and a constant $\nu > 0$ such that

$$e^{\nu(t-s)}\|\Phi(s,t_0,x)v\| \leq N(t)\|\Phi(t,t_0,x)v\| \tag{3.12}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$. Let us consider β such that $\nu = 2\beta$. We have

$$\int_{t_0}^{t} e^{\beta(t-\tau)} \left\| \Phi(\tau, t_0, x)v \right\| d\tau$$
$$\leq N(t) \int_{t_0}^{t} e^{\beta(t-\tau)} e^{-\nu(t-\tau)} \left\| \Phi(t, t_0, x)v \right\| d\tau \leq M(t) \left\| \Phi(t, t_0, x)v \right\|,$$

where

$$M(t) = \frac{N(t)}{\beta}.$$

Sufficiency. Let f be given by Definition 1.2.

For $t \geq s \geq t_0 \geq 0, x \in X, v \in V$ we have

$$\left\| \Phi(s, t_0, x)v \right\| e^{\beta(t-s)} \int_0^1 \frac{e^{\beta u}}{f(u)} du = \int_{s-1}^{s} \left\| \Phi(s, t_0, x)v \right\| \frac{e^{\beta(t-s)}}{f(s-\tau)} e^{\beta(s-\tau)} d\tau$$
$$\leq \int_{s-1}^{s} \left\| \Phi(\tau, t_0, x)v \right\| e^{\beta(t-\tau)} d\tau \leq M(t) \left\| \Phi(t, t_0, x)v \right\|.$$

We obtain

$$\left\| \Phi(s, t_0, x)v \right\| \leq N(t) e^{-\beta(t-s)} \left\| \Phi(t, t_0, x)v \right\|,$$

where

$$N(t) = M(t) [\int_0^1 \frac{e^{\beta u}}{f(u)}]^{-1}$$

and hence ξ is exponentially unstable. □

4. Dichotomy for linear skew-evolution semiflows

Definition 4.9. A mapping $P : Y \to Y$ is said to be a *projector* if P is continuous and has the form

$$P(x, v) = (x, P(x)v) \tag{4.13}$$

where $P(x)$ is a linear projection on $Y_x = \{x\} \times V$, $x \in X$.

Remark 4.2. The function $P(x) : Y_x \to Y_x$ is a bounded linear mapping with the property $P(x)P(x) = P^2(x) = P(x)$ for all $x \in X$.

Definition 4.10. The mapping $Q : Y \to Y$ given by $Q(x, v) = (x, v - P(x)v)$, where P is a projector on Y, is called the *complementary projector* to P on Y.

Definition 4.11. A projector P on Y is said to be invariant if one has

$$P(\varphi(t, x))\Phi(t, s, x) = \Phi(t, s, x)P(x) \tag{4.14}$$

for all $t \geq s \geq 0$ and all $x \in X$.

Definition 4.12. The linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *dichotomic* on Y if there exist functions $N_1, N_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$, a projector P and its complementary projector Q such that

(d_1) projectors P and Q are invariant on Y

(d_2) for every $x \in X$ projections $P(x)$ and $Q(x)$ commute and

$$P(x)Q(x) = 0$$

(d_3) following inequalities hold

$$\|\Phi(t,t_0,x)P(x)v\| \leq N_1(t_0)\,\|\Phi(s,t_0,x)P(x)v\| \tag{4.15}$$

$$\|\Phi(s,t_0,x)Q(x)v\| \leq N_2(t)\,\|\Phi(t,t_0,x)Q(x)v\| \tag{4.16}$$

for all $t \geq s \geq t_0 \geq 0$, $x \in X$ and all $v \in V$.

Theorem 4.1. *The strongly measurable linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is dichotomic on Y if there exist the functions $N_1, N_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$, the constants $\nu_1, \nu_2 > 0$, a projector P and its complementary projector Q such that*

(i) *projectors P and Q are invariant on Y*
(ii) *for every $x \in X$ projections $P(x)$ and $Q(x)$ commute and*

$$P(x)Q(x) = 0$$

(iii) *the following inequalities hold*

$$\int_{t_0}^{t} \|\Phi(s,t_0,x)P(x)v\| ds \leq N_1(t_0)\|P(x)v\| \tag{4.17}$$

$$\int_{t_0}^{t} \|\Phi(s,t_0,x)Q(x)v\| ds \leq N_2(t)\|\Phi(t,t_0,x)Q(x)v\| \tag{4.18}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Proof. It follows according to Definition 4.12, Definition 2.5, Definition 3.7, Proposition 2.1 and Proposition 3.3. □

Definition 4.13. The linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *exponentially dichotomic* on Y if there exist functions $N_1, N_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$, a projector P and its complementary projector Q such that

(ed_1) projectors P and Q are invariant on Y
(ed_2) for every $x \in X$ projections $P(x)$ and $Q(x)$ commute and

$$P(x)Q(x) = 0$$

(ed_3) following inequalities hold

$$e^{\nu_1(t-s)}\,\|\Phi(t,t_0,x)P(x)v\| \leq N_1(t_0)\,\|\Phi(s,t_0,x)P(x)v\| \tag{4.19}$$

$$e^{\nu_2(t-s)}\,\|\Phi(s,t_0,x)Q(x)v\| \leq N_2(t)\,\|\Phi(t,t_0,x)Q(x)v\| \tag{4.20}$$

for all $t \geq s \geq t_0 \geq 0$, $x \in X$ and all $v \in V$.

Remark 4.3. Without any loss of generality, one can consider

$$N(t) = \max\{N_1(t), N_2(t)\} \text{ and } \nu = \min\{\nu_1, \nu_2\}$$

and we will call N and ν *characteristics for the exponential dichotomy* of the linear skew-product semiflow ξ.

Example 4.5. Let us consider $V = \mathbb{R}^2$ endowed with the norm

$$\|(v_1, v_2)\| = |v_1| + |v_2|.$$

The mapping

$$\Phi : \mathbb{R}_+ \times X \to \mathfrak{B}(V),$$

defined by

$$\Phi(t, t_0, x)(v_1, v_2) = (e^{\alpha(t)-\alpha(t_0)} v_1, e^{\beta(t)-\beta(t_0)} v_2)$$

where $\alpha(t)$ and $\beta(t)$ are given as in Example 2.3, respectively Example 3.4, is a 2-parameter cocycle on $X \times V$.

We consider the projections

$$P(x)(v_1, v_2) = (v_1, 0),$$

$$Q(x)(v_1, v_2) = (0, v_2).$$

The linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is exponentially dichotomic with characteristics

$$N(t_0) = e^{6t_0} \text{ and } \nu = 2$$

Theorem 4.2. *The strongly measurable linear skew-evolution semiflow $\xi = (\varphi, \Phi)$ is exponentially dichotomic on Y if there exist the functions $M_1, M_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$, a constant $\alpha > 0$, a projector P and its complementary projector Q such that*

(i) *projectors P and Q are invariant on Y*
(ii) *for every $x \in X$ projections $P(x)$ and $Q(x)$ commute and*

$$P(x)Q(x) = 0$$

(iii) *following inequalities hold*

$$\int_{t_0}^{t} e^{\alpha(\tau-t_0)} \|\Phi(\tau, t_0, x)P(x)v\| \, d\tau \leq M_1(t_0) \|P(x)v\| \tag{4.21}$$

$$\int_{t_0}^{t} e^{\alpha(t-\tau)} \|\Phi(\tau, t_0, x)Q(x)v\| d\tau \leq M_2(t) \|\Phi(t, t_0, x)Q(x)v\| \tag{4.22}$$

for all $t \geq t_0 \geq 0$, all $x \in X$ and all $v \in V$.

Proof. It follows according to Definition 4.13, Definition 2.6, Definition 3.8, Proposition 2.2 and Proposition 3.4. □

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