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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Generalized pseudo-metrics and fixed points in probabilistic metric spaces

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ABSTRACT. We illustrate the generalized (pseudo-)metric method, revealed by Cain and Kasriel, to demonstrate fixed point theorems in probabilistic metric spaces. Our fundamental tool is the fixed point alternative.

1. THE EXTENDED FUNCTIONALS ρ_G

Let Δ^+ ($\mathcal{D}^+$) denote the set of all distribution functions of nonnegative extended (real) random variables, which are nondecreasing and left-continuous on $(0, \infty)$ (and have limit 1 at ∞). Suppose we are given a probabilistic metric $\mathcal{F} : S \times S \to \mathcal{D}^+$. As usual, the values $\mathcal{F}(p, q)$ are denoted by F_{pq}, $F_{pq}(0) = 0$ and, for the sake of convenience, one sets $F_{pq}(\infty) = 1$. Let $\mathcal{G}^+$ be the family of all nondecreasing functions $G : [0, \infty] \to [0, 1]$ with $G(0) = 0$ and $G(\infty) = 1$. For any $G \in \mathcal{G}^+$, we consider the mapping $\rho_G : S \times S \to [0, \infty]$, given by

$$\rho_G(p, q) = inf\{a \mid a \in (0, \infty] \text{ and } F_{pq}(ax) \geq G(x)\ ,\ \forall x > 0\}. \tag{1.1}$$

Notice that (1.1) is equivalent to the following Minkowski-type formula:

$$\rho_G(p, q) = \inf\{a \mid a \in (0, \infty] \text{ and } F_{pq} \geq a \circ G\}, \tag{1.2}$$

where

$$a \circ G(x) := G\left(\frac{x}{a}\right),\ \forall\, x > 0, a > 0.$$

Obviously, the elements of $\mathcal{D}^+ \subset \Delta^+$ are the only left-continuous functions $G \in \mathcal{G}^+$. The following theorem shows that it is useful and always possible to suppose G left-continuous on $(0, \infty)$, that is, in Δ^+. Recall that $\ell^- H(t) := \sup_{s<t}\{H(s)\}$ for $t \in [0, \infty)$,.

Theorem 1.1. *Let G be a fixed element in Δ^+.*
(i) *If $\rho_G(p, q) \in (0, \infty)$, then*

$$F_{pq}(\rho_G(p, q)x) \geq G(x),\ \forall\, x > 0.$$

(ii) *If $H \in \mathcal{G}^+$ and $\ell^- H = G$, then $\rho_H = \rho_G$.*

Proof. Suppose that $a = \rho_G(p, q) \in (0, \infty)$. Then, for every $x, \varepsilon > 0$, we have

$$F_{pq}(ax) = F_{pq}\left((a + \varepsilon)\frac{ax}{a + \varepsilon}\right) \geq G\left(\frac{ax}{a + \varepsilon}\right).$$

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Hence, by the left-continuity of G, $F_{pq}(ax) \geq G(x)$ and (i) holds.
In order to prove (ii), observe that (G is automatically left continuous at any finite point and) $G \leq H$, hence $\rho_G \leq \rho_H$. Now for $a = \rho_G(p,q) \in [0,\infty)$ and any $\varepsilon > 0$, we have:

$$F_{pq}[(a+\varepsilon)x] = F_{pq}\left[\left(a+\frac{\varepsilon}{2}\right)\frac{a+\varepsilon}{a+\frac{\varepsilon}{2}}x\right] \geq G\left(\frac{a+\varepsilon}{a+\frac{\varepsilon}{2}}x\right)$$
$$= H^-\left(x+\frac{\varepsilon}{2a+\varepsilon}x\right) \geq H(x).$$

Therefore $\rho_H(p,q) \leq a$, so that we always have the equality $\rho_H = \rho_G$. □

2. Some fundamental properties of ρ_G

Theorem 2.2. *For every $G, H \in \mathcal{G}^+$, we have:*
(i) *ρ_G is an extended semipseudometric on S;*
(ii) *If $G \leq H$ then*

$$\rho_G(p,q) \leq \rho_H(p,q),\ \forall\, p,q \in S;$$

(iii) *If $(S,\mathcal{F},T)$ is a Menger space and $G \leq \mathbf{T}(H,H)$, then*

$$\rho_G(p,q) \leq \rho_{\mathbf{T}(H,H)}(p,q) \leq \rho_H(p,r) + \rho_H(r,q),\ \forall\, p,q,r \in S;$$

(iv) *If $(S,\mathcal{F},T)$ is a Menger space and $\mathbf{T}(G,G) = G$, then ρ_G is an extended **pseudometric**. In particular, this is true for every Menger space $(S,\mathcal{F},Min)$.*
If, in addition, $G \in \mathcal{D}^+$, then
(v) *ρ_G is an extended semimetric on S;*
(vi) *The ρ_G-topology is stronger than the $\mathcal{F}-$topology;*
(vii) *(S,ρ_G) is complete for each complete Menger space $(S,\mathcal{F},T)$ with*

$$\lim_{b\to 1} T(a,b) = a, \forall a \in [0,1];$$

(viii) *If $(S,\mathcal{F},T)$ is a Menger space and $\mathbf{T}(G,G) = G$, then ρ_G is an extended **metric**.*

Proof. (vi) Suppose that $\{p_n\}$ is ρ_G-convergent to p and let $\varepsilon > 0$ and $\lambda \in (0,1)$ be given. Since $G \in \mathcal{D}^+$, then there exists x_0 such that $G(x_0) > 1-\lambda$. For $a < \frac{\varepsilon}{x_0}$, we choose n_0 such that $\rho_G(p_n,p) < a$ for all $n \geq n_0$. Therefore

$$F_{p_np}(\varepsilon) \geq F_{p_np}(ax_0) \geq G(x_0) > 1-\lambda,\ \forall n \geq n_0$$

and we see that $\{p_n\}$ is $\mathcal{F}-$convergent to p.

(vii) Suppose that $\{p_n\}$ is ρ_G-Cauchy and $(S,\mathcal{F})$ is complete. As above, we obtain that $\{p_n\}$ is $\mathcal{F}$-Cauchy and then there exists $p \in S$ such that $\{p_n\}$ is $\mathcal{F}$-convergent to p. Let $a,\ \delta > 0$ be given. Then there exists n_0 such that $F_{p_np_{n+m}}(ax) \geq G(x)$ for all $n > n_0$, all $m \geq 1$ and each x. Let $n > n_0$ and $x > 0$ be fixed. Since

$$F_{p_np}((a+\delta)x) \geq T(F_{p_np_{n+m}}(ax), F_{p_{n+m}p}(\delta x))$$
$$\geq T(G(x), F_{p_{n+m}p}(\delta x)),$$

then, by letting $m \to \infty$, $F_{p_np}((a+\delta)x) \geq T(G(x),1) = G(x)$. Therefore $\rho_G(p_n,p) \leq a+\delta$, $\forall\, n \geq n_0$ and we can see that $\{p_n\}$ is ρ_G-convergent. □

Remark 2.1. It is easy to see that

$$G(x) = \varepsilon_1(x) = \begin{cases} 0, & x \le 1 \\ 1, & x > 1 \end{cases} \Rightarrow \rho_G(p,q) = inf\{x \mid F_{pq}(x) = 1\}.$$

Although ρ_G is an extended metric, it need not be nontrivial: Every metric space (S,d) is seen to be a Menger space under $T_M = Min$ if we set $F_{pq}(x) = \frac{x}{x+d(p,q)}$, $\forall x \ge 0$. Clearly, $\rho_G(p,q) < \infty \Leftrightarrow p = q$. On the other hand, (S,d) can as well be regarded as a Menger space under T_M with $F_{pq}(x) = \varepsilon_{d(p,q)}(x) = G\left(\frac{x}{d(p,q)}\right)$, for $d(p,q) \ne 0$ and it is easily seen that $\rho_G(p,q) = d(p,q)$ in this case. Notice that the situation is reversed for $G(x) = \frac{x}{x+1}$, $\forall\, x \ge 0$.

The following result shows that the condition $\lim_{t\to\infty} G(t) = 1$ is essential for the conclusions (vi)-(vii) in Theorem 2.2. At the same time it gives a very useful family of distance-type functionals (compare with [26], § 8.2).

Proposition 2.1. *Let* $\lambda \in (0,1)$ *and* $G(t) = \begin{cases} 0, & t \le 1 \\ 1-\lambda, & t > 1 \end{cases}$. *Then*

(i) $\rho_G(p,q) = \rho_\lambda(p,q) := inf\{t \mid F_{pq}(t) \ge 1-\lambda\}$;

(ii) $\{\rho_\lambda\}_{\lambda\in(0,1)}$ *is a family of semipseudometrics and generates the* $\mathcal{F}-$ *uniformity:*

$$\rho_\lambda(p,q) < \varepsilon \Rightarrow F_{pq}(\varepsilon) \ge 1-\lambda \text{ and } F_{pq}(\varepsilon) > 1-\lambda \Rightarrow \rho_\lambda(p,q) < \varepsilon;$$

(iii) *If* $(S,\mathcal{F},T)$ *is a Menger space and* $T(1-\mu,1-\mu) \ge 1-\lambda$, *then*

$$\rho_\lambda(p,q) \le \rho_\mu(p,r) + \rho_\mu(r,q),\ \forall p,q,r \in S;$$

(iv) *If* $T(1-\lambda,1-\lambda) = 1-\lambda$, *then* ρ_λ *is a pseudometric for every Menger space* $(S,\mathcal{F},T)$.

Proof. (i) Let $\varepsilon > 0$ be given and recall that F_{pq} is nondecreasing. Then $F_{pq}(\rho_\lambda(p,q)+\varepsilon) \ge 1-\lambda$ and

$$F_{pq}((\rho_\lambda(p,q)+\varepsilon)t) \ge 1-\lambda,\ \forall\, t > 1.$$

This implies

$$\rho_\lambda(p,q) + \varepsilon \ge \rho_G(p,q),\ \forall \varepsilon > 0,$$

so that $\rho_\lambda(p,q) \ge \rho_G(p,q)$.

On the other hand, we have

$$F_{pq}(\rho_G(p,q)+2\varepsilon) = F_{pq}\left((\rho_G(p,q)+\varepsilon)\frac{\rho_G(p,q)+2\varepsilon}{\rho_G(p,q)+\varepsilon}\right) \ge G\left(\frac{\rho_G(p,q)+2\varepsilon}{\rho_G(p,q)+\varepsilon}\right) = 1-\lambda.$$

Therefore

$$\rho_G(p,q) + 2\varepsilon \ge \rho_\lambda(p,q),\ \forall\, \varepsilon > 0,$$

whence $\rho_G(p,q) \ge \rho_\lambda(p,q)$.

The assertions (ii)-(iv) are easily verified. □

Example 2.1. Let (S,d) be a metric space and, for any additive generator g of the continuous and Archimedean t-norm $T = T_g$, set

$$F_{pq}(x) = g^{(-1)}\left(\frac{d(p,q)}{x}\right),\ \forall\, x > 0,\ p,q \in S.$$

Then:
(i) $(S, \mathcal{F}, T)$ is a nonArchimedean Menger space for which T is the best t-norm;
(ii) $(S, \mathcal{F}, T_M)$ is a Menger space and $\rho_\lambda(p,q) = \frac{d(p,q)}{g(1-\lambda)}$ for each λ.

Lemma 2.1. ([14], [16]) *Let $(b_n)_{n\geq 1} \subset (0,1)$ be a strictly increasing sequence such that $\lim_{n\to\infty} b_n = 1$.*

(i) *For every $G \in \mathcal{D}^+$, the function G^* defined by*

$$G^*(t) = \begin{cases} 0 & iff \quad G(t) \leq b_1 \\ b_n & iff \quad G(t) \in (b_n, b_{n+1}] \\ 1 & iff \quad G(t) = 1 \end{cases} \tag{2.3}$$

is also in $\mathcal{D}^+$. Moreover, $G^* \leq G$.

(ii) *Let $(S, \mathcal{F}, T)$ be a (complete) Menger space and $\mathcal{F}^*(p,q) = (F_{pq})^*$. If*

$$T(b_n, b_n) = b_n \text{ and } a, b > b_n \Rightarrow T(a,b) > b_n, \ \forall n \geq 1, \tag{2.4}$$

then $(S, \mathcal{F}^*, T_M)$ is a (complete) Menger space with the same (ε, λ)-topology.

Proposition 2.2. *Suppose that the sequence $(b_n)_{n\geq 1} \subset (0,1)$ is strictly increasing to 1 and that $(S, \mathcal{F}, T)$ is a Menger space under a $t-$ norm T with $T(b_n, b_n) = b_n$ for all n.*

(i) *If we define*

$$r_n(p,q) := inf\{t \mid F_{pq}(t) \geq b_n\} = \rho_{1-b_n},$$

then $\{r_n\}_{n\geq 1}$, is a family of pseudometrics generating the $\mathcal{F}-$uniformity.

(ii) *The same conclusion holds for the Nishiura family $\{\delta_n\}_{n\geq 1}$, given by*

$$\delta_n(p,q) = inf\{t \mid F_{pq}(t) > b_n\},$$

provided that the condition (2.4) is verified.

Remark 2.2. Notice that the two-place functions

$$\rho^\star_\lambda(p,q) := inf\{t \mid F^\star_{pq}(t) \geq 1 - \lambda\},$$

generated by $\mathcal{F}^\star$, verify the equality

$$\rho^\star_\lambda = r^\star_n, \ \forall \lambda \in [1 - b_n, 1 - b_{n-1})$$

and, in fact, $\delta_n = r^\star_n$.

3. Remarks on some fixed point theorems

Recall that a Sehgal-contraction, or a $\mathbf{B}-$*contraction*, on a probabilistic metric space $(S, \mathcal{F})$ is a self-mapping A of S such that

$$F_{ApAq}(Lx) \geq F_{pq}(x), \ \forall p, q \in S \tag{BC_L}$$

for some $L \in (0,1)$ and every x. The proof of the following result is straightforward:

Lemma 3.2. (i) *Every Sehgal-contraction on $(S, \mathcal{F})$ is a Banach-contraction on (S, ρ_G), for every G. Namely, if $A : S \to S$ verifies the condition (BC_L), then*

$$\rho_G(Ap, Aq) \leq L\rho_G(p,q), \forall p, q \in S.$$

(ii) *A **B**-contraction on a PM space has at most one fixed point.*

Let $p_0 \in S$. As usual, the *sequence of iterates* of p_0 under A is the sequence $\{p_n\}$ defined inductively via $p_n = Ap_{n-1}$ for every positive integer n and the function G_{p_0} is defined via

$$G_{p_0}(x) = inf\{F_{p_0 p_m}(x) : m \text{ is a positive integer}\}$$

for each real x where $\{p_n\}$ is the sequence of iterates of p_0 under A.

3.1. **A method of type Maia for Sherwood's alternative.** The following new proof of the Sherwood alternative (cf. [28], Theorem 3.1) is given by remarking the fact that the probabilistic metric is "dominated" by a suitable semimetric and is showing off the full meaning of Sherwood's result.

Theorem 3.3. *Let $(S, \mathcal{F}, T)$ be a complete Menger space with T left-continuous. Let A be a **B**-contraction on S. Then,* ***either***

(i) *A has a unique fixed point*

or

(ii) *For every $p_0 \in S$, $sup\{G_{p_0}(x) : x$ is real$\} < 1$.*

Proof. Suppose there exists $p_0 \in S$ such that $sup\{G_{p_0}(x) : x \text{ is real}\} = 1$. Then $G = \ell^- G_{p_0} \in \mathcal{D}^+$, by hypothesis. Let us consider the set $Q = \{q \in S,\ \rho_G(p_0, q) \leq 1\}$. For $q_n \in Q$ and $q_n \stackrel{\mathcal{F}}{\to} q$, that is, $F_{q_n q}(t) \to 1,\ \forall\, t > 0$, we can write:

$$F_{p_0 q}(x) \geq T(F_{p_0 q_n}(x-t), F_{q_n q}(t) \geq T(G(x-t), F_{q_n q}(t)) \to T(G(x-t), 1) = G(x-t)$$

for all $t \in (0, x)$. Therefore, by the left-continuity of G, $F_{p_0 q}(x) \geq G(x),\ \forall x > 0$, showing that $\rho_G(p_0, q) \leq 1$ and so $q \in Q$. Thus $(Q, \mathcal{F}, T)$ is a complete Menger space. By Theorem 2.2 we have that (Q, ρ_G) is a **bounded** complete semimetric space. Since, by construction, $p_m \in Q$ for all m, then the strict ρ_G-contraction A has a (unique) fixed point in Q and the theorem follows. □

Remark 3.3. In fact, $F_{p_n p_{n+m}}(x) \geq F_{p_0 p_m}(x/L^n) \geq G(x/L^n)$ and $\rho_G(p_n, p_{n+m}) \leq L^n \leq 1$.

3.2. **The case of Hadžić-type triangular norms.** By using the alternative of fixed point, we can prove the following result (compare with [3], [10] and [4]– Chapter 3):

Theorem 3.4. *Let A be a Sehgal-contraction on the complete Menger space $(S, \mathcal{F}, T)$ and suppose that T has a sequence of idempotents, strictly increasing to 1. Then A has a unique fixed point $p^\star$ and, for each $p \in S$, $p^\star = \lim_{n\to\infty} A^n p$ in the $(\varepsilon, \lambda)-$topology.*

Proof. Let $p \in S$ and $G := (F_{pAp})^\star$(see Lemma 2.1.(i)). Since $T(G(x), G(x)) = G(x)$ for all x, then (S, ρ_G) is a complete generalized metric space (by Theorem 2.2) and A is strictly ρ_G-contractive (by Lemma 3.2). As $\rho_G(p, Ap) \leq 1 < \infty$, we can apply the fixed point alternative (see [8], [20]): the sequence $A^n(p)$ converges to a fixed point $p^\star$ of A, in the metric ρ_G, and so in the $(\varepsilon, \lambda)-$topology. Clearly $p^\star$ is uniquely determined and globally attractive. □

4. Comments

It is worth noting that, in [1], G. L. Cain & R. Kasriel proved the Sehgal's result (for $T = Min$) by using the *Nishiura pseudo-metrics* d_λ, where

$$d_\lambda(p,q) = inf\{t \mid F_{pq}(t) > 1-\lambda\} \quad (p,q \in S,\ 0 < \lambda < 1).$$

In fact, the family $\{d_\lambda\}_{\lambda\in(0,1)}$ generates the $(\varepsilon,\lambda)-$topology on S and

$$d_\lambda(Ap,Aq) \le Ld_\lambda(p,q),\ \forall\, p,q \in S.$$

for every $\lambda \in (0,1)$. Hence one can apply the Banach contraction principle in the uniform space $(S,\{d_\lambda\}_{\lambda\in(0,1)})$. As it was pointed out in [16], the above method can be used in the case of triangular norms of type Hadžić and Budinčević, by using the pseudometrics δ_n (see Proposition 2.2 (ii)). Instead, by using the pseudometrics r_n in the setting of complete probabilistic (b_n)-structures, we have been able (see [10]) to prove a very general fixed point result (including Theorem 3.4).

Remark 4.4. For if $d_\lambda(p,q) < r$ then $F_{pq}(r) > 1-\lambda$ and the contraction condition (BC_L) implies $F_{ApAq}(Lr) > 1-\lambda$, which shows that $d_\lambda(Ap,Aq) < Lr$.

Remark 4.5. Let $(S,\mathcal{F},Min)$ be a complete Menger space and suppose that, for some $G \in \mathcal{D}^+$, the ρ_G-topology and the $(\varepsilon,\lambda)-$topology are identical. Then, for every Sehgal-contraction A on S, we have:

(i) For every $p \in S$, $A^n p$ is convergent to the unique fixed point of A;

(ii) For each $p \in S$ there exists $n \ge 0$ such that $F_{A^npA^{n+1}p}(x) \ge G(x),\ \forall\, x$.

Indeed, the first assertion follows from Theorem 3.4. The second assertion follows from the fixed point alternative, since (i) is always true. In fact, these assertions indicate to a certain extent the behavior of the values of $\mathcal{F}$, as in the following

Example 4.2. For $\beta > 0$, let $G(x) = \begin{cases} 0, & x \le 1 \\ 1-\frac{1}{x^\beta}, & x > 1 \end{cases}$. It is easy to see that $\rho_G(p,q) = \sup\ \alpha^{\frac{1}{\beta}} d_\alpha(p,q)$. If ρ_G induces the (ε,λ)-topology on S, then for each $p \in S$ there exists $m \ge 0$ such that $F_{A^npA^{n+1}p}(x) \ge 1-\frac{1}{x^\beta}, \quad \forall x \ge 1\ ,\ \forall n \ge m.$

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