CARPATHIAN J. MATH.
23 (2007), No. 1 - 2, 141 - 148

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

A boundary value problem for some functional - differential equations, via Picard operators

VIORICA MUREŞAN

ABSTRACT. In this paper we use Picard operators' technique (see I. A. Rus [21] - [23], [26], [27] and [31]) to obtain existence, uniqueness and data dependence results for the solution of a boundary value problem for a functional-differential equation with linear modification of the argument.

1. INTRODUCTION

Many problems from physics, chemistry, astronomy, biology, engineering, social sciences lead to mathematical models described by functional - differential equations (see [12] - [15], [17], [18], [24], [34], [35]). The theory of these equations has developed very much.

For the monographs in the field of functional differential equations we quote here [1], [2], [4], [5], [7], [9], [10], [15], [16], [20], [21], [27].

The boundary value problems for differential and functional-differential equations have been considered in many monographs and papers (see [3], [5], [8], [15], [16], [27], [30], [31], [36]).

In this paper we use Picard operators' technique and results given by I. A. Rus in [21]-[23], [25]-[28] and [31] to obtain existence, uniqueness and data dependence results for the solution of a boundary value problem for a functional-differential equation with linear modification of the argument.

We remark here that the class of Picard operators and the class of weakly Picard operators were introduced by Profesor Ioan A. Rus and many results in these fields have been obtained by him and by his Ph D students (see [32]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by F_A the fixed point set of A.

Definition 1.1. (Rus [26]) A is a **Picard operator** if there exists $x^* \in X$ such that:

1) $F_A = \{x^*\}$;

2) the successive approximation sequence $(A^n(x_o))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 1.2. (Rus [25]) A is a **weakly Picard operator** it the sequence $(A^n(x_o))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and its limit (which may depend on X_0) is a fixed point of A.

Received: 23.11.2006; In revised form: 18.01.2007; Accepted: 19.02.2007
2000 *Mathematics Subject Classification.* 34K15, 47H10.
Key words and phrases. *Functional-differential equations, fixed points.*

2. A BOUNDARY VALUE PROBLEM, VIA PICARD OPERATORS

Consider the following problem:

$$-y'' = f(x, y(x), y(\lambda x)), \; x \in [a,b], \; 0 < a < b, \; 0 < \lambda < 1, \; \lambda b \geq a \tag{2.1}$$

$$y\big|_{[0,a]} = \varphi, \; y(b) = \beta, \tag{2.2}$$

where $f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C[0,a]$ and $\beta \in \mathbb{R}$.

By a solution of (2.1) and (2.2) we mean a function $y \in C[0,b]$ which satisfies the equation (2.1) and the conditions (2.2).

As it is well known (see [19], [27], [31]) such kind of problems are equivalent with Fredholm functional-integral equations. The problem (2.1) and (2.2) is to (2.3) equivalent, where

$$y(x) = \begin{cases} \varphi(x), & x \in [0,a] \\ \frac{x-a}{b-a}\beta + \frac{b-x}{b-a}\varphi(a) + \int_a^b G(x,s) \; f(s, y(s), y(\lambda s))\, ds, & \text{if } x \in [a,b] \end{cases} \tag{2.3}$$

and

$$G : [a,b] \times [a,b] \to \mathbb{R}$$

is the Green function defined by

$$G(x,s) = \begin{cases} \dfrac{(s-a)(b-x)}{b-a}, & s \leq x \\ \dfrac{(x-a)(b-s)}{b-a}, & s \geq x. \end{cases}$$

To prove existence and uniqueness for the solution of (2.3), we apply the following well known result:

Theorem 2.1. (Contraction principle) *Let $(X(d))$ be a complete metric space and $A : X \to X$ a contraction. Then A is a Picard operator.*

We have

Theorem 2.2. *We suppose that:*
(i) *$f \in C([a,b] \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C[0,a]$ and $\beta \in \mathbb{R}$;*
(ii) *there exists $L > 0$ such that*

$$|f(x,u,v) - f(x,\overline{u},\overline{v})| \leq L(|u - \overline{u}| + |v - \overline{v}|),$$

for all $x \in [a,b]$ and $u, \overline{u}, v, \overline{v} \in \mathbb{R}$;
(iii) $\frac{2L(b-a)^2}{4} < 1$.
Then the equation (2.3) *has in $C[0,b]$ a unique solution and this solution can be obtained by the successive approximation method starting from any element of $C[0,b]$.*

Proof. Consider the operator $A : (C[0,b], \|\cdot\|_C) \to (C[0,b], \|\cdot\|_C)$ (here $\|\cdot\|_C$ is the Chebyshev's norm on $C[0,b]$ defined by $\|y\|_C = \max_{x\in[0,b]} |y(x)|$), where

$$(A(y))(x) = \begin{cases} \varphi(x), & \text{if } x \in [0,a] \\ \frac{x-a}{b-a}\beta + \frac{b-x}{b-a}\varphi(a) + \int_a^b G(x,s) \; f(s, y(s), y(\lambda s))\, ds, & \text{if } x \in [a,b]. \end{cases}$$

We have:

$$|\left(A\left(y\right)\right)\left(x\right)-\left(A\left(z\right)\left(x\right)\right)|=0,\ \text{ for all } x\in\left[0,a\right]$$

and

$$\left(A\left(y\right)\right)\left(x\right)-\left(A\left(z\right)\left(x\right)\right)|\leq 2L\left(\max_{x\in[a,b]}\int_a^b G\left(x,s\right)ds\right)\left\|y-z\right\|_C,$$

for all $x\in[a,b]$.

It follows that (see [5])

$$\left\|A\left(y\right)-A\left(z\right)\right\|_C\leq\frac{2L\left(b-a\right)^2}{4}\left\|y-z\right\|_C,\ \text{ for all} y,z\in C\left[0,b\right].$$

Because of (iii), A is a contraction. So A is a Picard operator. □

Now, we consider the following problem:

(2.4) $$-y''\left(x\right)=g\left(x,y\left(x\right),\ y\left(\lambda x\right)\right),\ x\in\left[a,b\right],\ 0<\lambda<1,\ \lambda b\geq a,$$

(2.5) $$y\big|_{[0,a]}=\varphi,\ y\left(b\right)=\beta,$$

where $g\in C\left(\left[a,b\right]\times\mathbb{R}\times\mathbb{R}\right)$. Here $\varphi\in C\left[0,a\right]$ and $\beta\in\mathbb{R}$ are the same as in the problem (2.1) and (2.2).

Next, we will use the following result:

Theorem 2.3. (Data dependence theorem) (Rus [27]) *Let (X,d) be a complete metric space and $A,B:X\to X$ two operators. We suppose that:*
(i) *A is α-contraction and $F_A=\{x_A^*\}$;*
(ii) *B has fixed points and $x_B^*\in F_A$;*
(iii) *there exists $\delta>0$ such that*

$$d\left(A\left(x\right),\ B\left(x\right)\right)\leq\delta,\ \textit{ for all }\ x\in X.$$

Then

$$d\left(x_A^*,\ x_B^*\right)\leq\frac{\delta}{1-\alpha}.$$

We have

Theorem 2.4. *We support that all the conditions in Theorem 2.2 are satisfied and $y^*\in C\left[0,b\right]$ is the unique solution of (2.3). Let $z^*\in C\left[0,b\right]$ a solution of the problem (2.4) and (2.5). If there exists $\eta>0$ such that*

$$\left|f\left(x,u,v\right)-g\left(x,u,v\right)\right|\leq\eta,\ \textit{ for all }\ x\in\left[0,b\right]\ \textit{ and } u,v\in\mathbb{R},$$

then

$$\left\|y^*-z^*\right\|_C\leq\frac{\eta\cdot\frac{(b-a)^2}{8}}{1-2L\frac{(b-a)^2}{4}}.$$

Proof. Let $B:C\left[0,b\right]\to C\left[0,b\right]$, defined by

$$\left(B\left(y\right)\right)\left(x\right)=\begin{cases}\varphi\left(x\right), & \text{if }\ x\in\left[0,a\right]\\ \frac{x-a}{b-a}\beta+\frac{b-x}{b-a}\varphi\left(a\right)+\int_a^b G\left(x,s\right)g\left(s,y\left(s\right),y\left(\lambda s\right)\right)ds, & \text{if }\ x\in\left[a,b\right].\end{cases}$$

We have

$$|(A(y))(x) - (B(y))(x)| = 0, \text{ for all } x \in [0, a]$$

and $|(A(y))(x) - (B(y))(x)| \leq \eta \max_{x\in[a,b]} \int_a^b G(x,s)\,ds = \eta \frac{(b-a)^2}{8}$, for all $x \in [a, b]$ (see [5], [27]).

It follows that

$$\|A(y) - B(y)\|_C \leq \eta \frac{(b-a)^2}{8}.$$

We apply Theorem 2.3. □

3. THE DEPENDENCE WITH RESPECT TO PARAMETER

In order to study the smooth dependence with respect to parameter, we consider the following boundary value problem with parameter:

(3.6) $-y''(x) = f(x, y(x), y(\eta x); \mu),\ x \in [a, b],\ 0 < a < b,\ 0 < \lambda < 1,\ \lambda b \geq a.$

(3.7) $$y_{|[0,a]} = \varphi,\ y(b) = \beta$$

where $f \in C([a, b]) \times \mathbb{R} \times \mathbb{R},\ \varphi \in C[0, a],\ \beta \in \mathbb{R},\ \mu \in J$ and $J \subset \mathbb{R}$ is a compact interval.

By using the same method as in Theorem 2.2 we obtain the following result:

Theorem 3.5. *We suppose that:*

(i) $f \in C([a, b] \times \mathbb{R} \times \mathbb{R}),\ \varphi \in C[0, a],\ \beta \in \mathbb{R}$ *and* $\mu \in J,\ J \subset \mathbb{R}$ *is compact interval;*

(ii) *there exists* $L_f > 0$ *such that:*

$$\left| \frac{\partial f(x, u_1, u_2; \mu)}{\partial u_i} \right| \leq L_g, \textit{for all } x \in [a, b],\ u_i \in \mathbb{R}, i = 1, 2 \textit{ and } \mu \in J;$$

(iii) $\frac{2L_f(b-a)^2}{4} < 1.$

Then the problem (3.6) *and* (3.7) *has in* $C[0, b]$ *a unique solution* $y^* \in (x, \mu)$ *which is the uniform limit of the successive approximations sequence.*

Now, we prove that $y^*(x, \cdot) \in C^1(J)$, for all $x \in [0, b]$. In order to do this, we need the following result:

Theorem 3.6. (Fiber contraction theorem) (Hirsch, Pugh [11], Rus [28]) *Let* (X, d) *be a metric space,* (Y, ρ) *be a complete metric space and* $T : X \times Y \to X \times Y$. *We suppose that:*

(i) $T(x, y) = (T_1(x),\ T_2(x, y));$

(ii) $T_1 : X : X$ *is a weakly Picard operator;*

(iii) *there exists* $c \in]0, 1[$ *such that*

$$\rho(T_2(x, y), T_2(x, z)) \leq c\rho(y, z), \textit{ for all } x \in X \textit{ and all } y, z \in Y.$$

Then the operator T *is weakly Picard operator. Moreover, if* T_1 *is Picard operator, then* T *is Picard operator.*

Consider the equation

(3.8) $$-y''(x, \mu) = f(x, y(x, \mu), y(\lambda x; \mu); \mu),\ x \in [a, b],\ 0 < \lambda < 1,\ \mu \in J.$$

Proof. Consider the operator

$$B : C([0,b] \times J) \to C([0,b] \times J)$$

where

$$(B(y))(x;\mu) := \begin{cases} \varphi(x), \text{ if } x \in [0,a], \ \mu \in J \\ \frac{x-a}{b-a}\beta + \frac{b-x}{b-a}\varphi(a) + \int\limits_a^b G(x,s) f(s, y(s;\mu), \ y(\lambda s); u)\, ds, \\ \text{if } x \in [a,b], \mu \in J \end{cases}$$

The problem (3.8) and (3.7) is equivalent to the following functional-integral equation:

$$y(x;\mu) = \begin{cases} \varphi(x), \text{ if } x \in [0,a], \ \mu \in J \\ \frac{x-a}{b-a}\beta + \frac{b-x}{b-a}\varphi(a) + \int\limits_a^b G(x,s) f(s, y(s;\mu), \ y(\lambda s); u)\, ds, \\ \text{if } x \in [a,b], \mu \in J \end{cases} \tag{3.9}$$

□

We have

Theorem 3.7. *We suppose that:*
(i) $f \in C([a,b] \times \mathbb{R} \times \mathbb{R} \times J), \varphi \in C[0,a], \beta \in \mathbb{R}$;
(ii) *there exists* $L_f > 0$ *such that*

$$\left| \frac{\partial f(x, u_1, u_2; \mu)}{\partial u_i} \right| \le L_f, \textit{ for all } x \in [a,b], u_i \in \mathbb{R}, \ i = 1,2 \textit{ and } \mu \in J;$$

(iii) $\frac{2Lf(b-a)^2}{4} < 1.$
Then:
(a) *the problem* (3.8) *and* (3.7) *has a unique solution* $y^* = y^*(x;\mu)$ *in* $C \in [0,b] \times J$;
(b) $y^*(x,.) \in C^1(J)$, *for all* $x \in [0,b]$.

Proof. Consider the operator

$$B : C([0,b] \times J) \to C([0,b] \times J)$$

where

$$(B(y))(x;\mu) := \begin{cases} \varphi(x), \text{ if } x \in [0,a], \ \mu \in J \\ \frac{x-a}{b-a}\beta + \frac{b-x}{b-a}\varphi(a) + \int\limits_a^b G(x,s) f(s, y(s;\mu), y(\lambda s); u)\, ds, \\ \text{if } x \in [a,b], \mu \in J. \end{cases}$$

Let $X := C([0,b] \times J)$ be and we denote by $||.||_1$ the Chebyshev's norm on $C([0,b] \times J)$.

It is obviously that, if the conditions (i)-(iii) in Theorem 3.7 are satisfied, then the operator B is a Picard operator. Let $y^*(x,\mu)$ its unique fixed point. We suppose that there exists $\frac{\partial y^*}{\partial \mu}$. Then, from (3.10) we have that

$$\frac{\partial y^*(x;\mu)}{\partial \mu} = 0 \text{ if } x \in [0,a]$$

and

$$\frac{\partial y^*(x;\mu)}{\partial \mu} = \int_a^b G(x,s)\frac{\partial f(s,y^*(s;\mu),y^*(\lambda s;\mu);\mu)}{\partial u_1}\cdot\frac{\partial y^*(s;\mu)}{\partial \mu}ds$$
$$+\int_a^b G(x,s)\frac{\partial f(s,y^*(s;\mu),y^*(\lambda s;\mu);\mu)}{\partial u_2}\cdot\frac{\partial y^*(\lambda s;\mu)}{\partial \mu}ds$$
$$+\int_a^b G(x,s)\frac{\partial f(s,y^*(s;\mu),y^*(\lambda s;\mu)\mu)}{\partial \mu}ds,\ \text{if } x\in[0,b],\mu\in J.$$

The above relationships suggests us to consider the following operator:

$$C: X\times X\to X,\ (y,z)\to C(y,z),$$

where

$$C(y,z)(x;\mu):=0,\ \text{if } x\in[0,a],\ \mu\in J$$

and

$$C(y,z)(x;\mu):=\int_a^b G(x,s)\frac{\partial f(s,y(s;\mu),y(\lambda s;\mu);\mu)}{\partial u_1}\cdot z(s;\mu)\,ds$$
$$+\int_a^b G(x,s)\frac{\partial f(s,y(s;\mu),y(\lambda s;\mu);\mu)}{\partial u_2}\cdot z(\lambda s;\mu)\,ds$$
$$+\int_a^b G(x,s)\frac{\partial f(s,y(s;\mu),y(\lambda s;\mu);\mu)}{\partial u}ds,\ \text{if } x\in[a,b] \text{ and } \mu\in J.$$

So, we have the triangular operator

$$A: X\times X\to X\times X,\quad (y,z)\to(B(y),C(y,z)),$$

where B is a Picard operator and $C(y,\cdot): X\to X$ is an α - contraction with $\alpha=\frac{2L_f(b-a)^2}{4}$.

By using Fiber contraction theorem (Theorem 3.2) we obtain that A is a Picard operator. So, the sequences

$$y_{n+1}:=B(y_n),\ n\in\mathbb{N}$$
$$z_{n+1}:=C(y_n,z_n),\ n\in\mathbb{N}$$

converge uniformly (with respect to $x\in[0,b],\ \mu\in J$) to $(y^*,z^*)\in F_A$, for all $y_0,z_0\in C([0,b]\times J)$. If we take, $y_0=0,\ z_0=\frac{\partial y_0}{\partial\mu}=0$, then $z_1=\frac{\partial y_1}{\partial \mu}$.

By mathematical induction method we have that

$$z_n=\frac{\partial y_n}{\partial\mu},\quad for\ \ all\quad n\in\mathbb{N}^*,$$

Thus $(y_n)_{n\in\mathbb{N}}$ converges uniformly to y^* and $\left(\frac{\partial y_n}{\partial\mu}\right)_{n\in\mathbb{N}}$ converges to z^*. By using a Weierstrass argument we have that $\frac{\partial y^*}{\partial\mu}$ exists and $\frac{\partial y^*}{\partial\mu}=z^*$. $\square$

Remark 3.1. By the same arguments as above we have that, if $f(x,\cdot,\cdot,\cdot) \in C^k(\mathbb{R}\times\mathbb{R}\times J)$, for all $x\in[0,b]$, then $y^*(x;\cdot)\in C^k(J)$, for all $x\in[0,b]$.

Remark 3.2. The following boundary value problem with both delayed and advanced arguments vas studied by I. A. Rus in [30]:

$$-y''(x) = f(x, y(x), y(g(x)), y(h(x))), \text{ if } x\in[a,b]) \tag{3.10}$$

$$\begin{cases} y(x) = \varphi(x), & \text{if } x\in[a_1,a] \\ y(x) = \psi(x), & \text{if } x\in[b,b_1], \end{cases} \tag{3.11}$$

where $a_1 \le a < b \le b_1,\ g,h\in C([a,b],[a_1,b_1]),\varphi\in C[a_1,a],\ \psi\in C[b,b_1]$ and $f\in C\left([a,b]\times\mathbb{R}^3\right)$.

References

[1] Andras, S., *Fredholm - Volterra integral equations*, E.D.P. Bucuresti, 2005 (In Romanian)
[2] Azbelov, N. V., Maksimov, V. P. and Rahmatulina, L. F., *Introduction to functional - differential equations theory*, MIR, Moscow, 1991 (In Russian)
[3] Bailey, P. B., Shampine, L. F. and Waltman, P. E., *Nonlinear two point boundary value problems*, Acad. Press, New York, 1968
[4] Bellman, R. E. and Cooke, K. L., *Differential - difference equations*, Acad. Press, New York, 1963
[5] Bernfeld, S. R. and Lakshmikantham, V., *An introduction to nonlinear boundary value problems*, Acad. Press, New York, 1974
[6] Conti, R., *Recent trends in the theory of boundary value problem for ordinary differential equations*, Boll. U. M. I., **22** (1967), 135-178
[7] Elsgoltz, L. F. and Norkin, S. B., *Introduction to the theory of differential equations with deviating arguments*, MIR, Moskow, 1971 (In Russian)
[8] Grinum, L. J. and Schmitt, K., *Boundary value problems for differential equations with deviating arguments*, Aequationes Math., **4** (1970), 176-190
[9] Hale, J. K., *Theory of functional - differential equations*, Springer Verlag, 1977
[10] Hale, J. K., Sjoerd M. Verduyn Lunel, *Introduction to functional - differential equations*, Springer Verlag, New York, 1993
[11] Hirsch, M. W. and Pugh, C. C., *Stable manifolds and hyperbolic sets*, Proc. Symp. in Pure Math., **14** (1970), 133-163
[12] Iserles, A., *On the generalized pantograph functional - differential equation*, European J. Appl. Math. **4** (1992), 1-38
[13] Kato, T. and Mc Leod, J. B., *The functional - differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. , 77, **6** (1971), 891-937
[14] Lakhmikantham, V. (ed), *Trends in the theory and practice of nonlinear differential equations*, Marcel Dekker, New York, 1984
[15] Muresan, V., *Differential equations with affine modification of the argument*, Transilvania Press, Cluj-Napoca, 1997 (In Romanian)
[16] Muresan, V., *Functional - integral equations*, Ed. Mediamira, Cluj-Napoca, 2003
[17] Ockendon, J. R., *Differential equations and industry,* The Math. Scientist, **5** (1980), no. 1, 1-12
[18] Ockendon, J. R. and Tayler, A. B., *The dynamics of a curent collection system for an electric locomotive*, Proc. Poy. Soc. London, A 322, 1971, 447-468
[19] Piccinini, L. C., Stampacchia, G. and Vidossich, G., *Ordinary differential equations in* $\mathbb{R}^n$ Springer - Verlag, 1984
[20] Precup, R., *Integral and nonlinear equations*, "Babeş-Bolyai" University, Cluj-Napoca, 1993 (In Romanian).
[21] Rus, I. A., *Principles and pplications of the fixed point theory*, Ed. Dacia, Cluj-Napoca, 1979 (In Romanian)
[22] Rus, I. A., *Metrical fixed point theorems*, Univ. of Cluj-Napoca, 1979
[23] Rus, I. A., *Picard mappings: results and problems*, "Babeş-Bolyai" University, Cluj-Napoca, Seminar on fixed point theory, Preprint **6** (1987), 55-64

[24] Rus, I. A., *A delay integral equation from biomathematics*, "Babeş-Bolyai" University, Cluj-Napoca, Preprint **3** (1989), 87-90
[25] Rus, I. A., *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, 34, **4** (1993), 769-773
[26] Rus, I. A., *Picard operators and applications*, "Babeş-Bolyai" of Cluj-Napoca, Preprint **3** (1996)
[27] Rus, I. A., *Differential equations, integral equations and dynamical systems*, Transilvania Press, Cluj-Napoca, 1996 (In Romanian)
[28] Rus, I. A., *A fiber generalized contraction theorem and applications*, Mathematica, **41**(**64**), (1999), No. 1, 85-90
[29] Rus, I. A., *Weakly Picard operators and applications*, "Babeş-Bolyai" University, Cluj-Napoca, Seminar on fixed point theory, **2** (2001), 41-58
[30] Rus, I. A., *Functional differential of mixed type, via weakly Picard operators*, Seminar on fixed point theory, Cluj-Napoca, **3** (2002), 335-346
[31] Rus, I. A., *Picard operators and applications*, Scientiae Mathematicae Japonica, **58** (2003), No. 1, 191 - 219
[32] Rus, I. A., Petrusel, A. and Petrusel, G., *Fixed Point Theory: 1950 - 2000 Romanian Contributions*, House of the Book Science, Cluj-Napoca, 2002
[33] Rustichini, A., *Functional differential equations of mixed type: The linear autonomous case*, J. Dynam. Diff. Eqs., **1** (1989), 121-143
[34] Terjeki, J., *Representation of the solutions to linear pantograph equations*, Acta Sci. Math., (Szeged), **60** (1995), 705 - 713
[35] Tomaras, A., *Oscillation of an equation relevant to an industrial problem*, Bull. Austral. Math. Soc., **12** (1975), 425 - 43
[36] Trif, D., *On the Dirichlet problem for differential equations with modified argument*, "Babeş-Bolyai" University, Cluj-Napoca, Preprint **8** (1988), 31 - 36

Faculty of Economics
Department of Statistics, Economical Analysis, Forecasting and Mathematics
"Babeş-Bolyai" University of Cluj-Napoca
M. Kogălniceanu St. 1, Romania
E-mail address: vmuresan@econ.ubbcluj.ro