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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Approximate fixed point theorems for weak contractions on metric spaces

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ABSTRACT. Some existence results concerning approximate fixed points of the weak contractions introduced by V. Berinde on a metric space (not necessarily complete) are given. We also prove some quantitative theorems regarding the set of approximate fixed points for two subclasses of weak contractions, one of them being the class of Rus-Reich operators.

1. INTRODUCTION

There are plenty of problems in applied mathematics which can be solved by means of fixed point theory. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required, but that of "nearly" fixed points. Another type of practical situation that lead to this approximation is when the conditions that have to be imposed in order to guarantee the existence of fixed points are far too strong for the real problem one has to solve.

It is then natural to introduce the concepts of *$\varepsilon-$fixed point* (or *approximate fixed point)*, which is a "nearly" fixed point, and that of mapping with the *approximate fixed point property* and to formulate a proper theory regarding them.

In the paper [1], starting from the article of Tijs, Torre and Branzei [6], we have considered several types of operators on metric spaces, namely operators satisfying Banach, Kannan, Chatterjea and Zamfirescu type conditions. The first three of these are independent conditions (see [10]). A more general type of condition for which we could formulate our results was the Ciric type condition with $k \in]0, \frac{1}{2}[$.

Still the most general type of operators considered there were the weak contractions introduced in [3], which generalize all the contraction conditions mentioned above, as shown there.

In our paper [1] we have already proved some approximate fixed point results for weak contractions, but there is a quantitative theorem which requires a special condition on the operator, and this restricts the class of weak contractions for which it holds. We will try to get inside this class and to see how much we can enlarge it. In this respect we will use Rus-Reich type contraction conditions. But first we should recall some notions and results.

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2. Weak contractions and approximate fixed points

Let (X, d) be a metric space. Note that none of the results concerning approximate fixed points requires the completeness of the space.

Definition 2.1. *Let $f : X \to X$, $\varepsilon > 0$, $x_0 \in X$. An ellement x_0 is called an $\varepsilon-$**fixed point** (or **approximate fixed point) of** f provided that*

$$d(f(x_0), x_0) < \varepsilon.$$

For a given $\varepsilon > 0$, we will denote the set of all $\varepsilon-$fixed points of f by:

$$Fix_\varepsilon(f) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } f\}.$$

Definition 2.2. *Let $f : X \to X$. Then f has **the approximate fixed point property** if*

$$\forall \varepsilon > 0,\ Fix_\varepsilon(f) \neq \emptyset.$$

Regarding this we have given in [1] the following lemmas.

Lemma 2.1. *Let (X, d) be a metric space, $f : X \to X$ such that f is asymptotically regular, i.e.*

$$d(f^n(x), f^{n+1}(x)) \to 0 \text{ as } n \to \infty, \text{ for all } x \in X.$$

Then f has the approximate fixed point property.

In the following, by $\delta(A)$ for a set $A \neq \emptyset$ we will understand the diameter of the set A, i.e.

$$\delta(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Lemma 2.2. *Let (X, d) be a metric space, $f : X \to X$ an operator and $\varepsilon > 0$. We assume that:*

$i)$ *f has the approximate fixed point property;*

$ii)$ *for each $\eta > 0$, there exists $\varphi(\eta) > 0$ such that*

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta), \text{ for all } x, y \in Fix_\varepsilon(f).$$

Then:

$$\delta(Fix_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

For the detailed proof of the previous lemmas, see [1]. We have used Lemmas 2.1, 2.2 in order to obtain approximate fixed point results for the contractive operators we have mentioned. Now we will only consider a most general class of contractive operators.

Definition 2.3 ([3])**.** *A mapping $f : X \to X$ is called **weak contraction** if there exist $\alpha \in]0, 1[$ and $L \geq 0$ such that*

$$d(f(x), f(y)) \leq \alpha d(x, y) + L d(y, f(x)), \text{ for all } x, y \in X.$$

Remark 2.1. It is not difficult to show that any operator satisfying Banach, Kannan, Chatterjea, Zamfirescu or Ciric (with the constant k in $]0, \frac{1}{2}[$) type conditions is a weak contraction (see [3]).

Using the previously given lemmas we can prove the following results.

Theorem 2.1. *Let (X, d) be a metric space and $f : X \to X$ a weak contraction. Then f has the approximate fixed point property.*

Proof. We show that f is asymptotically regular.

Let $x \in X$. Then

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq \alpha d(f^{n-1}(x), f^n(x)) + Ld(f^n(x), f^n(x)) =$$

$$= \alpha d(f^{n-1}(x), f^n(x))$$

So

$$d(f^n(x), f^{n+1}(x)) \leq \alpha d(f^{n-1}(x), f^n(x)) \leq ... \leq \alpha^n d(x, f(x)).$$

As $\alpha \in]0, 1[$, we obtain that

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Therefore by Lemma 2.1, it follows that f has the approximate fixed point property. □

Theorem 2.2. *Let (X, d) be a metric space and $f : X \to X$ a weak contraction satisfying the condition $\alpha + L < 1$.*

Then:

$$\delta(Fix_\varepsilon(f)) \leq \frac{(2 + L)\varepsilon}{1 - \alpha - L}, \textit{ for each } \varepsilon > 0.$$

Proof. We want to use Lemma 2.2. We already know that f has the approximate fixed point property, so it is enough to show it satisfies condition (ii) in this lemma.

Let $\varepsilon > 0$ and $x, y \in Fix_\varepsilon(f)$.

Let $\eta > 0$ such that

$$d(x, y) - d(f(x), f(y)) \leq \eta.$$

Then we have

$$d(x, y) \leq d(f(x), f(y)) + \eta \leq \alpha d(x, y) + Ld(y, f(x)) + \eta \leq$$

$$\leq \alpha d(x, y) + L\,[d(x, y) + d(x, f(x))] + \eta \leq (\alpha + L)d(x, y) + Ld(x, f(x)) + \eta.$$

As x is an $\varepsilon-$fixed point, we obtain that

$$d(x, y) \leq (\alpha + L)d(x, y) + L\varepsilon + \eta,$$

so

$$d(x, y) \leq \frac{L\varepsilon + \eta}{1 - \alpha - L}.$$

Taking $\varphi(\eta) = \dfrac{L\varepsilon + \eta}{1 - \alpha - L} > 0$ we can apply Lemma 2.2 and we get that

$$\delta(Fix_\varepsilon(f)) \leq \frac{(2 + L)\varepsilon}{1 - \alpha - L}, \text{ for each } \varepsilon > 0.$$

So the diameter of the set of $\varepsilon-$fixed points goes to 0 when ε goes to 0. □

3. Subclasses of weak contractions and approximate fixed points

Following the proof of Theorem 2.2, it is easy to see why we had to impose the condition

$$\alpha + L < 1,$$

namely in order to ensure that

$$\varphi(\eta) = \frac{L\varepsilon + \eta}{1 - \alpha - L} > 0.$$

But this reduces the generality of our result, although the contraction condition is very general. It is then natural to wonder if there might not be a less restrictive condition to be imposed on the operator in order to enlarge the family of weak contractions for which our results still hold.

A direction is suggested by the very form of our contraction condition and by the restriction we had to impose, namely to consider an operator which is a weak contraction and besides it has the sum of its coefficients less than 1. The suitable candidates are the Rus-Reich type operators.

Definition 3.4. *Let (X, d) be a metric space and $f : X \to X$. We say that f is a* ***Rus-Reich operator*** *if there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq a\, d(x, y) + b\, d(x, f(x)) + c\, d(y, f(y)), \text{ for all } x, y \in X.$$

Remark 3.2. In a complete metric space any Rus-Reich operator has a unique fixed point. This inspires us to formulate a result which ensures the existence of approximate fixed points for any Rus-Reich operator defined on a metric space, not necessarily complete. In our paper it is more natural to obtain this via weak contractions. But first we have to prove the next result.

Proposition 3.1. *Any Rus-Reich operator $f : X \to X$ is a weak contraction.*

Proof. Let $f : X \to X$ be a Rus-Reich operator, with $a, b, c \in \mathbb{R}_+$, with $a+b+c < 1$ such that

$$d(f(x), f(y)) \leq a\, d(x, y) + b\, d(x, f(x)) + c\, d(y, f(y)), \text{ for all } x, y \in X.$$

We have to show that there exist $\alpha \in]0, 1[$ and $L \geq 0$ such that

$$d(f(x), f(y)) \leq \alpha\, d(x, y) + L\, d(y, f(x)), \text{ for all } x, y \in X.$$

We can write

$$d(f(x), f(y)) \leq a\, d(x, y) + b\, d(x, f(x)) + c\, d(y, f(y)) \leq$$
$$\leq a\, d(x, y) + b[d(x, y) + d(y, f(x))] + c[d(y, f(x)) + d(f(x), f(y))] \leq$$
$$\leq (a + b)d(x, y) + (b + c)d(y, f(x)) + c\, d(f(x), f(y)), \text{ for all } x, y \in X.$$

Thus we obtain

$$d(f(x), f(y)) \leq \frac{a + b}{1 - c} d(x, y) + \frac{b + c}{1 - c} d(y, f(x)), \text{ for all x,y} \in X.$$

Now taking $\alpha = \dfrac{a + b}{1 - c}$ and $L = \dfrac{b + c}{1 - c}$ the conditions $\alpha \in]0, 1[$ and $L \geq 0$ are fulfilled, so f is a weak contraction. □

It is then easy to prove the following result.

Theorem 3.3. *Let (X, d) be a metric space and $f : X \to X$ a Rus-Reich type operator, in the sense of Definition 3.1. Then f has the approximate fixed point property.*

Proof. By Proposition 3.1, f is also a weak contraction. So, according to Theorem 2.1, f has the approximate fixed point property. □

Now it is interesting to see what happens with the quantitative result given in Theorem 2.2, which holds only for weak contractions with $\alpha + L < 1$. The following question appears naturally: are Rus-Reich type operators among the weak contractions with $\alpha + L < 1$? If not, can we still prove a similar result regarding Rus-Reich operators?

If we continue the idea in the proof of Proposition 3.1, where we took

$$\alpha = \frac{a+b}{1-c} \text{ and } L = \frac{b+c}{1-c},$$

we see that $\alpha + L < 1$ implies

$$\frac{a+2b+c}{1-c} < 1,$$

equivalent to

$$a + 2b + 2c < 1,$$

which does not hold for any Rus-Reich type operator. So only some Rus-Reich type operators, namely those for which the above condition holds, fulfill the conditions in Theorem 2.2. Still this does not mean that we cannot formulate an analogous result concerning all Rus-Reich type operators, and this is given in the following.

Theorem 3.4. *Let (X, d) be a metric space and $f : X \to X$ a Rus-Reich type operator, in the sense of Definition 3.1. Then:*

$$\delta(Fix_\varepsilon(f)) \le \frac{(b+c+2)\varepsilon}{1-a}, \textit{ for each } \varepsilon > 0.$$

Proof. Again we shall use Lemma 2.2. We already know that f has the approximate fixed point property, so it only has to fulfill condition (ii) in the above mentioned lemma.

Let $\varepsilon > 0$ and $x, y \in Fix_\varepsilon(f)$.

We take $\eta > 0$ such that

$$d(x, y) - d(f(x), f(y)) \le \eta.$$

Then we can write:

$$d(x, y) \le d(f(x), f(y)) + \eta \le a\, d(x, y) + b\, d(x, f(x)) + c\, d(y, f(y)) + \eta.$$

As $x, y \in Fix_\varepsilon(f)$, this implies:

$$d(x, y) \le a\, d(x, y) + b\varepsilon + c\varepsilon + \eta,$$

so

$$d(x, y) \le \frac{(b+c)\varepsilon + \eta}{1-a}.$$

Now taking $\varphi(\eta) = \dfrac{(b+c)\varepsilon + \eta}{1-a}$ we obviously have that $\varphi(\eta) > 0$ and, by Lemma 2.2, we obtain that

$$\delta(Fix_\varepsilon(f)) \leq \frac{(b+c+2)\varepsilon}{1-a}, \text{ for each } \varepsilon > 0,$$

so $\delta(Fix_\varepsilon(f)) \to 0$ as $\varepsilon \to 0$. □

As we see, Rus-Reich type operators are "good" for our study, although not all of them are weak contractions with $\alpha + L < 1$ as required by Theorem 2.2. At this point one could wonder if they are not more general than this particular type of weak contractions and so Theorem 2.2 could be contained in Theorem 3.2.

This question can be answered only in the negative, as shown in the following.

We suppose that all weak contractions with $\alpha + L < 1$ are Rus-Reich type operators. We have that

$$\begin{aligned} d(f(x), f(y)) &\leq \alpha\, d(x,y) + L\, d(y, f(x)) \leq \\ &\leq \alpha\, d(x,y) + L[d(x,y) + d(x, f(x))] = \\ &= (\alpha + L) d(x,y) + L\, d(x, f(x)). \end{aligned}$$

Now taking

$$a = \alpha + L,\ b = L,\ c = 0,$$

we have to impose $a + b + c < 1$, that is

$$\alpha + 2L < 1,$$

which obviously does not hold for all weak contractions with $\alpha + L < 1$.

In conclusion, weak contractions with $\alpha + L < 1$ and Rus-Reich operators are two subclasses of weak contractions for which it is possible to show that the $\varepsilon-$fixed points set is non-empty and has its diameter going to 0 when ε goes to 0. These contraction conditions are not independent, there exist operators contained in both classes, neither are they anyhow subordinated: there are Rus-Reich operators which are not weak contractions with $\alpha + L < 1$, and viceversa.

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