CARPATHIAN J. MATH.
23 (2007), No. 1 - 2, 156 - 164

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

On the asymptotic behaviour of the number of maximum points of a simple random walk

EUGEN PĂLTĂNEA

ABSTRACT. For a sequence $(X_i)_{i\geq 1}$ of independent and identically distributed random variables, taking the values -1, 0 and 1, we define $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$, for $k \geq 1$. We study the asymptotic behaviour of the sequence of random variables $(Q_n)_{n\geq 1}$, where Q_n indicates the number of absolute maximum points of the simple random walk $S_0, S_1, \cdots, S_n$. The paper extends some results of Dwass [2], Révész [11], Katzenbeisser and Panny [7], [8].

1. INTRODUCTION

Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables, $X_i \in \{-1, 0, 1\}$ with $\mathbb{P}\{X_i = 1\} = \alpha$, $\mathbb{P}\{X_i = 0\} = \beta$ and $\mathbb{P}\{X_i = -1\} = \gamma$, where $\alpha, \gamma > 0$. The paper deals with the simple random walk S_k in the sense of Cox and Miller [1], $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$ for $k = 1, 2, \cdots$. We study the related random variable Q_n defined by:

$$Q_n = \text{card}\{\, k \in \mathbb{N} :\ 0 \leq k \leq n, S_k = M_n\},$$

where $M_n = \max\{S_0, S_1, \cdots, S_n\}$, for $n \geq 1$. Obviously Q_n corresponds to the number of times where the sequence $\{S_0, S_1, \cdots, S_n\}$ reaches its maximum. We also investigate the conditioned random variable $[\, Q_n \,|\, S_n = 0\,]$, under the assumption that $\mathbb{P}\{S_n = 0\} > 0$.

The papers [2], [11] and [7] deal with the classical case $\beta = 0$ and $\alpha = \gamma = \frac{1}{2}$. The general case is considered by Katzenbeisser and Panny [8], but for an alternative definition Q'_n. The difference is due to the treatment of consecutive maxima (comprising one or more 0-steps). Thus, Q_n counts all points belonging to a consecutive maximum, whereas for Katzenbeisser, Panny's Q'_n a consecutive maximum contributes only 1. Remark that the results for Q_n are not directly obtained from the corresponding results on Q'_n, except for the case $\beta = 0$.

The main results of the present paper are given in Theorems 4.1, 4.3 and 4.4. Thus, Theorem 4.1 deals with the probability function for $\mathbb{P}\{Q_n = r, S_n = k\}$. Theorem 4.3 gives exact and asymptotic expressions for $\mathbb{E}(Q_n|S_n = 0)$:

$$\mathbb{E}(Q_n \mid S_n = 0) = \frac{\mathbb{P}\{S_{n+1} = 1\}}{\mathbb{P}\{S_1 = 1\}\mathbb{P}\{S_n = 0\}} \;=\; 2 + \frac{\beta}{\sqrt{\alpha\gamma}} + O\left(n^{-1}\right), \text{ as } n \to \infty.$$

Received: 22.10.2006; In revised form: 09.01.2007; Accepted: 19.02.2007
2000 *Mathematics Subject Classification.* 60G50, 60G70, 62E15, 62E20.
Key words and phrases. *random walk, distribution function, generating function, extremes, asymptotic behaviour.*

Theorem 4.4 deals with $\mathbb{E}(Q_n)$ including its asymptotic behavior:

$$\mathbb{E}(Q_n) = \frac{1}{\max\{\alpha, \gamma\}} + O\left(n^{-\frac{1}{2}}\right), \text{ as } n \to \infty.$$

2. THE DISTRIBUTION OF S_n

Let n be a positive integer. The random variable S_n takes integer values in the interval $[-n, n]$ and its distribution can be formulated by means of ordinary trinomial coefficients (see [8]).

Lemma 2.1. *For an integer k, such that $-n \leq k \leq n$, we have:*

$$\mathbb{P}\{S_n = k\} = \sum_{\substack{a+b+c=n;\\ a-c=k}} \binom{n}{a, b, c} \alpha^a \beta^b \gamma^c, \tag{2.1}$$

with the convention $0^0 = 1$.

Let us denote $\varphi(z) = \alpha z + \beta + \gamma z^{-1}$ the generating function of X_1 and let

$$[z^k]\ \{\ f(z)\ \}$$

as be the coefficient of z^k into Laurent's development near the origin of the rational complex function f.

From the theorem of residues we obtain:

$$[z^k]\ \{\ f(z)\ \} = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{f(z)}{z^{k+1}}\, dz, \tag{2.2}$$

where $\rho > 0$.

Thus, the following useful integral formula for $\mathbb{P}\{S_n = k\}$ holds (see [8]).

Lemma 2.2. *For any integer k,*

$$\mathbb{P}\{S_n = k\} = [z^k]\ \{\varphi^n(z)\} = \frac{1}{\pi} \left(\frac{\alpha}{\gamma}\right)^{\frac{k}{2}} \int_0^\pi (\beta + 2\sqrt{\alpha\gamma}\ \cos t)^n\ \cos kt\ dt. \tag{2.3}$$

Let us consider the following sequence:

$$p_n := \mathbb{P}\{S_n = 0\} = \frac{1}{\pi} \int_0^\pi (\beta + 2\sqrt{\alpha\gamma} \cos\ t)^n\ dt,\ n \in \mathbb{N}. \tag{2.4}$$

From the integral formula of p_n given by (2.4) it results that the sequence $(p_n)_{n \geq 1}$ satisfies the following reccurent formula:

$$p_{n+2} = \beta \frac{2n+3}{n+2}\, p_{n+1} + (4\alpha\gamma - \beta^2) \frac{n+1}{n+2}\, p_n,\ n = 1, 2, \cdots \tag{2.5}$$

Using (2.3) for $k \in \{1, 2\}$ we get the following statements:

$$\mathbb{P}\{S_n = 1\} = \frac{1}{2\gamma}(p_{n+1} - \beta p_n); \tag{2.6}$$

$$\mathbb{P}\{S_n = 2\} = \frac{1}{2\gamma^2} \left(\frac{2\alpha\gamma n + \beta^2}{n+2}\, p_n - \frac{\beta}{n+2}\, p_{n+1} \right).$$

The following lemma states the asymptotic behaviour of the probability of the event $S_n = 0$.

Lemma 2.3. *For $\beta = 0$ and n an even number we have:*

$$p_n = \frac{d}{2\sqrt{n\pi}}(\beta + 2\sqrt{\alpha\gamma})^{n+\frac{1}{2}}(\alpha\gamma)^{-\frac{1}{4}}\left(1 + O(n^{-1})\right),\ as\ n \to \infty,$$

where

$$d = \begin{cases} 1 & if \quad \beta > 0 \\ 2 & if \quad \beta = 0. \end{cases}$$

Proof. From (2.4) we obtain:

$$p_n = \frac{(\beta + 2\sqrt{\alpha\gamma})^n}{\pi}\int_0^\pi (b + 2a\cos t)^n\, dt,$$

where $a = \frac{\sqrt{\alpha\gamma}}{\beta+2\sqrt{\alpha\gamma}} > 0$ and $b = \frac{\beta}{\beta+2\sqrt{\alpha\gamma}} \geq 0$ such that $2a + b = 1$. Using Laplace's method for integrals (see [10]), we obtain:

$$\frac{1}{\pi}\int_0^\pi (b + 2a\cos t)^n\, dt \;=\; \frac{d}{2\sqrt{an\pi}}\,(\,1 +\ O(n^{-1})\,),\ n \to \infty$$

where d is given above. Thus, the lemma has been proved. □

We get the following consequence.

Corollary 2.1. *For a fixed k we have:*

$$\frac{p_{n+k}}{p_n} = (\beta + 2\sqrt{\alpha\gamma})^k + O(n^{-1})\ ,\ as\ n \to \infty. \tag{2.7}$$

Indeed,

$$\frac{p_{n+k}}{p_n} = (\beta+2\sqrt{\alpha\gamma})^k\sqrt{\frac{n}{n+k}}\left(1 + O(n^{-1})\right) = (\,\beta+2\sqrt{\alpha\gamma})^k + O(n^{-1}),\ \text{as}\ n \to \infty.$$

We express now the asymptotic behaviour of S_n.

Lemma 2.4. *The following assertions hold:*

$$\begin{array}{lll} \mathbb{P}\{S_n \geq 0\} \;\leq\; (\beta + 2\sqrt{\alpha\gamma})^n \stackrel{n\to\infty}{\longrightarrow} 0, & & if\ \alpha < \gamma \\ \mathbb{P}\{S_n \leq 0\} \leq (\beta + 2\sqrt{\alpha\gamma}\,)^n \stackrel{n\to\infty}{\longrightarrow} 0, & & if\ \alpha > \gamma \\ \mathbb{P}\{S_n \geq 0\} = \mathbb{P}\{S_n \leq 0\} \;=\; 1/2 + O(n^{-1/2}), & & if\ \alpha = \gamma. \end{array}$$

Proof. We observe that $\beta+2\sqrt{\alpha\gamma} = 1-(\sqrt{\alpha}-\sqrt{\gamma})^2 \leq 1$, with equality if and only if $\alpha = \gamma$. Assume $\alpha < \gamma$. Let h be the increasing function $h(t) = \left(\frac{\gamma}{\alpha}\right)^{t/2},\ t \in \mathbb{R}$. Then, according to Tchebyshev's inequality, we find $\mathbb{P}\{S_n \geq 0\} \leq \frac{\mathbb{E}(h(S_n))}{h(0)}$. But $h(0) = 1$ and $\mathbb{E}(h(S_n)) = \mathbb{E}\left(\prod_{i=1}^n h(X_i)\right) = \mathbb{E}^n\left(h(X_1)\right) = (\beta + 2\sqrt{\alpha\gamma}\,)^n$. Since $\beta + 2\sqrt{\alpha\gamma} < 1$, $(\beta + 2\sqrt{\alpha\gamma}\,)^n \stackrel{n\to\infty}{\longrightarrow} 0$. Similarly, we get the second assertion. Remark that the first two statements also follow from the central limit theorem.

If $\alpha = \gamma$, from Lemma 2.2 and Lemma 2.3 we obtain:

$$\mathbb{P}\{S_n \geq 0\} = \mathbb{P}\{S_n \leq 0\} = \frac{1 + p_n}{2} = \frac{1}{2} + O(n^{-\frac{1}{2}}),\ \ n \to \infty.$$

The lemma has been proved. □

3. Admissible paths with given number of maxima

Let us denote $U_n = \{0\} \times \{-1,0,1\}^n$. To each vector $x = (x_j)_0^n \in U_n$ we associate the n-steps path $s = \sigma(x)$, $s = (s_i)_0^n \in \mathbb{Z}^{n+1}$, defined by $s_i = \sum_{j=0}^{i} x_j$.

For a path s we define:

(3.8) $m_s = max\{s_i\};\ V_s = \{i\ :\ s_i = m_s\};\ V_s^* = \{i \in V_s\ :\ i = 0 \text{ or } s_i - s_{i-1} = 1\}.$

The elements of the set V_s are the points where the path s reaches its maximum m_s. Each element of the set V_s^* is an isolated maximum point or the first one point of a group of consecutive maximum points.

We intend to classify the vectors x of U_n after their composition and number of maximum points of the associated paths $\sigma(x)$. Thus, we say that $x \in U_n$ has $(a,b,c)-composition$ if its structure is the following: a components 1, $b+1$ components 0 and c components -1. The associated path $s = \sigma(x)$ reaches after $n = a+b+c$ steps the end value $s_n = a - c$.

Let us denote $U_{a,b,c}$ the set of all vectors x with $(a,b,c)-composition$. Hence:

$$U_n = \bigcup_{a,b,c \in \mathbb{N};\ a+b+c=n} U_{a,b,c}.$$

Denoting $N_{a,b,c} = \text{card}\,(U_{a,b,c})$ and supposing $a+b+c=n$, we have:

$$N_{a,b,c} = \begin{pmatrix} n \\ a,b,c \end{pmatrix}. \tag{3.9}$$

For positive integers a,b,c,r,v, such that $1 \le v \le r \le a+b+c+1$, we denote

$$U_{a,b,c}^{r,v} = \{x \in U_{a,b,c}\ :\ \text{card}\left(V_{\sigma(x)}\right) = r;\ \text{card}\left(V_{\sigma(x)}^*\right) = v\} \tag{3.10}$$

and

$$N_{a,b,c}^{r,v} = \text{card}\left(U_{a,b,c}^{r,v}\right).$$

Firstly, we refer to the case $b=0$. Here we have $r=v$. There holds the following well-known result (see [3], [9] or [8]).

Lemma 3.5. *We have*

$$U_{a,0,c}^{v,v} \neq \emptyset \ \Leftrightarrow\ 1 \le v \le 1 + min\{a,c\} \tag{3.11}$$

and in this case

$$N_{a,0,c}^{v,v} = \begin{cases} 1 & if\ \ a = c = 0 \\ N_{a-1,0,c-v+1} & if\ \ \max\{a,c\} = a \ge 1 \\ N_{a-v+1,0,c-1} & if\ \ \max\{a,c\} = c \ge 1. \end{cases}$$

Now, using a standard method, namely so-called *Goodman-Narayana*'s technique, we translate the counting result for $\{-1,1\}$ paths into the corresponding result for $\{-1,0,1\}$ paths (see [4]).

Lemma 3.6. *We have*

$$U_{a,b,c}^{r,v} \neq \emptyset \ \Leftrightarrow\ \begin{cases} \max\{a,c\} \ge 1 \\ 1 \le v \le 1 + \min\{a,c\} \\ v \le r \le b+v \end{cases} \ \ or\ \ \begin{cases} a = c = 0 \\ v = 1 \\ r = b+1. \end{cases} \tag{3.12}$$

In this case, supposing $\max\{a,c\} \geq 1$,

$$N^{r,v}_{a,b,c} = \binom{r-1}{v-1}\binom{a+b+c-r}{a+c-v} N^{v,v}_{a,0,c}. \tag{3.13}$$

From the two previous lemmas and (3.9) we obtain the following statement.

Lemma 3.7. *Under the conditions (3.12) we have:*

$$N^{r,v}_{a,b,c} = \begin{cases} 1 & if \;\; a=c=0 \\ \binom{r-1}{v-1}\binom{a+b+c-r}{a-1,\, b+v-r,\, c-v+1} & if \;\; \max\{a,c\} = a \geq 1 \\ \binom{r-1}{v-1}\binom{a+b+c-r}{a-v+1,\, b+v-r,\, c-1} & if \;\; a < c. \end{cases}$$

This above result clear up the counting of the admissible paths with given number of maxima.

4. Main results

Now, we translate the counting results of Lemma 3.7 into the calculus of a probability distribution.

Theorem 4.1. *Let n and r be two positive integers, with $r \leq n+1$. Also let $k \in \mathbb{Z}$.*
(i) *If* $\mathbb{P}\{\, Q_n = r;\; S_n = k\} \; > 0$ *then*

$$|k| \leq n+1-r. \tag{4.14}$$

(ii) *Independently on above condition, there holds:*

$$\mathbb{P}\{Q_n = r;\; S_n = k\} \; = \begin{cases} [z^k]\, \{\alpha z(\beta+\gamma z^{-1})^{r-1}\varphi^{n-r}(z)\}, & if\; k \geq 0 \\ [z^{-k}]\, \{\gamma z(\beta+\alpha z^{-1})^{r-1}\varphi^{n-r}(z^{-1})\}, & if\; k < 0, \end{cases}$$

where φ is the generating function of the distribution of X_1.

Proof. (i) Using the definitions (3.10), we have:

$$\mathbb{P}\{Q_n = r;\; S_n = k\} = \sum_{1\leq v\leq r} \;\; \sum_{\substack{a+b+c=n;\\ a-c=k}} N^{r,v}_{a,b,c}\, \alpha^a\beta^b\gamma^c. \tag{4.15}$$

Let us suppose that $\mathbb{P}\{Q_n = r;\; S_n = k\} > 0$. From (4.15) and Lemma 3.7 there exists a vector (a,b,c,r,v) with natural components such that $a+b+c = n,\; a-c = k$ and (3.12) holds. Hence, if $\max\{a,c\} \geq 1$ then $|k| = |a-c| \leq a+c-\min\{a,c\} = n-b-\min\{a,c\} \leq n+1-b-v \leq n+1-r$. If $a=c=0$ then $|k| = 0 = n+1-r$.

(ii) Let us consider that (4.14) holds. Suppose $k \geq 0$.

If $r = n+1$ then $k=0$ and we have:

$$[z^0]\, \{\alpha z(\beta+\gamma z^{-1})^n\varphi^{-1}(z)\} = [z^0]\, \{\alpha z^{-1}(\beta+\gamma z)^n\varphi^{-1}(z^{-1})\} =$$

$$= [z^0]\, \{(\beta+\gamma z)^n\} - [z^0]\, \left\{\frac{z(\beta+\gamma z)^{n+1}}{\alpha+\beta z+\gamma z^2}\right\} = \beta^n = \mathbb{P}\{Q_n = n+1;\; S_n = 0\}.$$

If $1 \le r \le n$ we apply (4.15), (4.14) and Lemma 3.7. But $v = 2v - v \le 2(1+c) + (b-r) = n-k-r+2$ and (from the assumption) $n-k-r+2 \ge 1$. Therefore:

$$\mathbb{P}\{Q_n = r; S_n = k\} = \sum_{1\le v\le min\{r,n-k-r+2\}} \sum_{\substack{a,b,c\in\mathbb{N};\\ a+b+c=n;\ a-c=k}} N^{r,v}_{a,b,c}\alpha^a\beta^b\gamma^c$$

$$= \sum_{1\le v\le min\{r,n-k-r+2\}} \begin{pmatrix} r-1 \\ v-1 \end{pmatrix} \alpha\beta^{r-v}\gamma^{v-1} \sum_{\substack{s,t,u\in\mathbb{N};\\ s+t+u=n-r;\\ s-u=k+v-2}} \begin{pmatrix} n-r \\ s,t,u \end{pmatrix} \alpha^s\beta^t\gamma^u$$

$$= \alpha \sum_{v=1}^{min\{r,n-k-r+2\}} [z^{1-v}]\ \{(\beta+\gamma z^{-1})^{r-1}\}\ .\ [z^{k+v-2}]\ \{(\alpha z+\beta+\gamma z^{-1})^{n-r}\}$$

$$= [z^k]\ \{\alpha z(\beta+\gamma z^{-1})^{r-1}(\alpha z+\beta+\gamma z^{-1})^{n-r}\}.$$

Now, replacing α with γ and also k with $-k$ we derive the result for $k < 0$. When $|k| > n+1-r$, the conclusion also holds because all terms are null. Thus, the statements of the theorem are proved. □

From Theorem 4.1 and Lemma 2.2 we find the distribution function of the random variable $[\,Q_n \mid S_n = 0]$.

Corollary 4.2. *Assuming* $\mathbb{P}\{S_n = 0\} > 0$ *and* $1 \le r \le n+1$*, we have:*

$$\mathbb{P}\{\ Q_n = r \mid S_n = 0\ \} = \frac{[z^0]\ \{(\alpha z+\beta+\gamma z^{-1})^{n-r}(\beta+\gamma z^{-1})^{r-1}(\alpha z)\}}{[z^0]\ \{\ (\alpha z+\beta+\gamma z^{-1})^n\ \}}.$$

Example 4.1. For the classical random walk with $\alpha = \gamma = \frac{1}{2}$ we obtain:

$$\mathbb{P}\{Q_{2m} = r \mid S_{2m} = 0\}\ = \frac{[z^0]\ \{z^{2-r}(z+\frac{1}{z})^{2m-r}\}}{[z^0]\ \{(z+\frac{1}{z})^{2m}\}} =$$

$$= \frac{\begin{pmatrix} 2m-r \\ m-1 \end{pmatrix}}{\begin{pmatrix} 2m \\ m \end{pmatrix}} = \frac{1}{2^r}\frac{\int_0^\pi \cos(r-2)t\cos^{2m-r} t\, dt}{\int_0^\pi \cos^{2m} t\, dt},\quad r = 1, 2, \cdots m+1.$$

Remark 4.1. The above combinatorial expression can be found in [2], pp. 1047.

To estimate the moments of the random variable $[\,Q_n \,|\, S_n = 0\,]$ we need some technical results. Thus, let us start from the well-known identity:

$$r^k = \sum_{j=1}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} r^{\underline{j}}\ ,\quad k \in \mathbb{N}^*, \tag{4.16}$$

where $r^{\underline{j}} := r(r-1)\cdots(r-j+1)$, and $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ designates the Stirling's numbers of the second kind. Referring to the sum $\sum_{r=1}^{\infty}\ r^k z^r$, we shall prove a "finite version" of a well-known identity (see [5]).

Lemma 4.8. *For the natural numbers k and n, such that $k \le n+1$, the following identity holds:*

$$\sum_{r=1}^{n+1} r^k z^r = \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \left(\frac{j! z^j}{(1-z)^{j+1}} + \frac{z^{n+2}}{(1-z)^{j+1}} \Theta_j(z) \right), \; z \in \mathbb{C}, \; z \neq 1,$$

where $\Theta_j(z)$ is a polynomial function of degree j.

Proof. We prove by induction that there exist some polynomial functions $\Theta_j(z)$ of degree $j \le n+1$ such that:

$$\frac{d^j}{dz^j} \left(\sum_{r=0}^{n+1} z^r \right) = \frac{j! + z^{n+2-j}\Theta_j(z)}{(1-z)^{j+1}}, \; z \in \mathbb{C}, \; z \neq 1.$$

From (4.16) we get:

$$\sum_{r=0}^{n+1} r^k z^r = \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \sum_{r=j}^{n+1} r^{\underline{j}} z^r = \sum_{j=1}^{k} z^j \left\{ \begin{array}{c} k \\ j \end{array} \right\} \frac{d^j}{dz^j} \left(\sum_{r=0}^{n+1} z^r \right).$$

Therefore we obtain the conclusion. □

Theorem 4.2. *The random variable $[Q_n \,|\, S_n = 0]$ has the following k-moments, for $1 \le k \le n+1$:*

$$\mathbb{E}\,(\, Q_n^k \mid S_n = 0 \,) = \frac{1}{p_n} [z^0] \left\{ \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \frac{j!}{\alpha^j z^j} (\beta + \gamma z^{-1})^{j-1} \varphi^{n+1}(z) \right\}.$$

Proof. From Corollary 4.2 and the property $[z^0]\,\{f(z)\} = [z^0]\,\{f(z^{-1})\}$ of rational complex functions f, we find:

$$\mathbb{P}\,\{\, Q_n = r \mid S_n = 0 \,\} = \frac{1}{p_n} [z^0] \left\{ \frac{\alpha(\alpha + \beta z + \gamma z^2)^n}{z^{n+1}(\beta + \gamma z)} R^r(z) \right\},$$

where $R(z) := (\beta z + \gamma z^2)(\alpha + \beta z + \gamma z^2)^{-1}$. From Lemma 4.8 we obtain:

$$\mathbb{E}\,(\, Q_n^k \mid S_n = 0 \,) = \sum_{r=1}^{n+1} r^k \mathbb{P}(\, Q_n = r \mid S_n = 0 \,) =$$

$$= \frac{1}{p_n} \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} \left([z^0] \left\{ \frac{j!}{\alpha^j z^{n+1-j}} (\alpha + \beta z + \gamma z^2)^{n+1} (\beta + \gamma z)^{j-1} \right\} + \right.$$

$$\left. + [z^0] \left\{ \frac{z}{\alpha^j} (\beta + \gamma z)^{n+1} (\alpha + \beta z + \gamma z^2)^{j-1} \Theta_j(R(z)) \right\} \right).$$

But all second terms of the parenthesis of above sum are null.
Hence, the theorem is proved. □

Now, we formulate one of the main results of the paper.

Theorem 4.3. *The random variable $[Q_n \,|\, S_n = 0]$ has the following mean and variance:*

$$\mathbb{E}(Q_n \mid S_n = 0) = \frac{\mathbb{P}\{S_{n+1} = 1\}}{\mathbb{P}\{S_1 = 1\}\mathbb{P}\{S_n = 0\}} = 2 + \frac{\beta}{\sqrt{\alpha\gamma}} + O(n^{-1}), \; \textit{as } n \to \infty$$

$$\mathbb{V}(\,Q_n \mid S_n = 0) = \frac{2\,\mathbb{P}\{S_{n+2} = 2\}}{\mathbb{P}^2\{S_1 = 1\}\mathbb{P}\{S_n = 0\}} - m(m+1)$$

$$= 2 + \frac{\beta^2 + 3\beta\sqrt{\alpha\gamma}}{\alpha\gamma} + 0(n^{-1}),\ \textit{as}\ n \to \infty,$$

where $m = \mathbb{E}(Q_n \mid S_n = 0)$ *(taking only even naturals* n*, if* $\beta = 0$*).*

Proof. We choose $k = 1$ in Theorem 4.2 and we obtain the exact formula of the mean:

$$m = \frac{1}{\alpha p_n}\,[z^0]\ \left\{\ z^{-1}(\alpha z + \beta + \gamma z^{-1})^{n+1}\ \right\} = \frac{\mathbb{P}\{S_{n+1} = 1\}}{\mathbb{P}\{S_1 = 1\}\mathbb{P}\{S_n = 0\}}.$$

Also, from (2.6) and (2.7) we find the asymptotic behavior of the mean. In a similar way, using (2.6), (2.7) and Theorem 4.2 ($k = 2$), we estimate the variance of the r. v. $[Q_n \mid S_n = 0]$. □

Remark 4.2. For $\alpha = \gamma = 1/2$ we find the well-known results due to Katzenbeisser and Panny [7], pp. 308.

Finally, we give exact and asymptotic formulas for $\mathbb{E}(Q_n)$.

Theorem 4.4. *The following estimations hold:*

$$\mathbb{E}(Q_n) = \frac{\mathbb{P}\{S_{n+1} > 0\}}{\mathbb{P}\{S_1 = 1\}} + \frac{\mathbb{P}\{S_{n+1} < 0\}}{\mathbb{P}\{S_1 = -1\}} - \frac{\mathbb{P}\{S_{n+1} = 1\}}{\mathbb{P}\{S_1 = 1\}}$$

$$= \frac{1}{\max\{\alpha, \gamma\}} + O(n^{-\frac{1}{2}}),\ \ \textit{as}\ n \to \infty.$$

Proof. We have :

$$\mathbb{E}(Q_n) = \sum_{|k| \le n} \sum_{r=1}^{n+1-|k|} r\mathbb{P}\{Q_n = r;\ S_n = k\}.$$

From Theorem 4.1 we get:

$$\sum_{r=1}^{n+1-|k|} r\mathbb{P}(Q_n = r;\ S_n = k) = \begin{cases} \frac{1}{\alpha}\,[z^{k+1}]\,\{(\alpha z + \beta + \gamma z^{-1})^{n+1}\}, & k \ge 0 \\ \frac{1}{\gamma}\,[z^{-k+1}]\,\{(\gamma z + \beta + \alpha z^{-1})^{n+1}\}, & k \le 0. \end{cases}$$

But (cf. (2.3)) we have:

$$\sum_{r=1}^{n+1-|k|} r\mathbb{P}\{Q_n = r;\ S_n = k\} = \begin{cases} \frac{\mathbb{P}\{S_{n+1}=k+1\}}{\mathbb{P}\{S_1=1\}}, & k \ge 0 \\ \frac{\mathbb{P}\{S_{n+1}=k-1\}}{\mathbb{P}\{S_1=-1\}}, & k \le 0. \end{cases}$$

By summing these relations we obtain the exact formula of the mean of the r.v. Q_n. The asymptotic behaviour of the mean follows from (2.6), Corollary 2.1 and Lemma 2.4. □

REFERENCES

[1] Cox, D. R. and Miller H. D., *The Theory of Stochastic Processes*, Methuen, London, 1965
[2] Dwass, M., *Simple random walk and rank order statistics*, Ann. Math. Statist., **38** (1967), 1042-1053
[3] Feller, W., *An Introduction to Probability Theory and its Applications*, **I**, 3rd.ed., Wiley, New York, 1968, pp. 90
[4] Goodman, E. and Narayama T. V., *Lattice paths with diagonal steps*, Canadian. Math. Bull., **12** (1969), 847-855
[5] Graham, R. L., Knuth, D. E. and Patashnik, O., *Concrete mathematics*, 2-edition, Addison-Wesley, Massachusetts, 1994, pp. 337
[6] Katzenbeisser, W. and Panny, W. *Asymptotic results on the maximal deviation of simple random walks*, Stoch. Proc. Appl., **18** (1984), 263-275
[7] Katzenbeisser, W. and Panny, W., *On the number of times where a simple random walk reaches its maximum*, J. Appl. Prob., **29** (1992), 305-312
[8] Katzenbeisser, W. and Panny, W., *Simple random walk statistics. Part I: Discrete time results*, J. Appl. Prob., **33** (1996), 311-330
[9] Mohanty, S. G., *Lattice Path Counting and Applications*, Academic Press, New York, 1979, pp. 90
[10] Panny, W., *The maximal deviation of lattice paths*, Athenäum Hain, 1984, pp. 15
[11] Révész, P., *Random walk in random and non-random environments*, Wold Scient. Publish. Co., Singapore, 1990, pp. 157-160

TRANSILVANIA UNIVERSITY OF BRAŞOV
DEPARTMENT OF ANALYSIS AND PROBABILITIES
IULIU MANIU ST. 50, BRAŞOV, ROMANIA
E-mail address: epaltanea@unitbv.ro