CARPATHIAN J. MATH.
23 (2007), No. 1 - 2, 172 - 176

Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Existence and data dependence of fixed points and strict fixed points for multivalued Y-contractions

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ABSTRACT. The purpose of this paper is to study the existence and data dependence of the fixed points and strict fixed points of some multivalued Y-contractions in complete ordered metric spaces.

1. INTRODUCTION

Throughout this paper, the standard notations and terminologies in nonlinear analysis (see [11], [13]) are used. For the convenience of the reader we recall some of them.

Let (X, d) be a metric space. By $\tilde{B}(x_0, r)$ we denote the closed ball centered at $x_0 \in X$ with radius $r > 0$.

Also, we will use the following symbols:

$P(X) := \{Y \subset X | \ Y \text{ is nonempty}\}$, $P_{cl}(X) := \{Y \in P(X)| \ Y \text{ is closed}\}$,

$P_b(X) := \{Y \in P(X)| \ Y \text{ is bounded }\}$, $P_{b,cl}(X) := P_{cl}(X) \bigcap P_b(X)$.

Let A and B be nonempty subsets of the metric space (X, d). The gap between these sets is

$$D(A, B) = \inf\{d(a, b)| \ a \in A, \ b \in B\}.$$

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

The Pompeiu-Hausdorff generalized distance between the nonempty closed subsets A and B of the metric space (X, d) is defined by the following formula:

$$H(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

If $A, B \in P_{b,cl}(X)$, then one denotes

$$\delta(A, B) := \sup\{d(a, b)| \ a \in A, \ b \in B\}.$$

The symbol $T : X \multimap Y$ means $T : X \to P(Y)$, i. e. T is a set-valued operator from X to Y. We will denote by $Graf(T) := \{(x, y) \in X \times Y | y \in T(x)\}$ the graph of T. Recall that the set-valued operator is called closed if $Graf(T)$ is a closed subset of $X \times Y$.

For $T : X \to P(X)$ the symbol $Fix(T) := \{x \in X| \ x \in T(x)\}$ denotes the fixed point set of the set-valued operator T, while $SFix(T) := \{x \in X| \ \{x\} = T(x)\}$ is the strict fixed point set of T.

Received: 05.11.2006; In revised form: 19.01.2007; Accepted: 19.02.2007
2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. *Strict fixed point, multivalued operator, data dependence, generalized contraction.*

If (X, d) is a metric space, $T : X \to P_{cl}(X)$ is called a multivalued a-contraction if $a \in]0, 1[$ and $H(T(x_1), T(x_2)) \leq a \cdot d(x_1, x_2)$, for each $x_1, x_2 \in X$.

Let $T : X \to P_{cl}(X)$ be a multivalued operator. A sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

is called the sequence of successive approximations of T starting from x_0.

The following open question was considered by I. A. Rus in [10]:

Open problem. *Let (X, d) be a metric space and $Y \subseteq X \times X$. The operator $T : X \to P_{cl}(X)$ is called a set-valued (Y, a)-contraction if:*

(1.1) $$a \in]0, 1[\text{ and } H(T(x_1), T(x_2)) \leq a \cdot d(x_1, x_2), \text{ for each } (x_1, x_2) \in Y.$$

Construct a fixed point theory for set-valued (Y, a)-contractions.

In the previous definition, one can also consider generalized type conditions for T, as well as, similar assumptions with respect to δ instead of H.

The purpose of this paper is to study the existence and data dependence of the fixed points and strict fixed points of some multivalued Y-contractions in complete ordered metric spaces.

Our results generalize and extend some results given in [1], [6], [10] and [12].

2. Data dependence of the fixed point set

Let $(X, \leq)$ be an partially ordered set. Denote $X_{\leq} := \{(x, y) \in X \times X | x \leq y \text{ or } y \leq x\}$. Also, if $x, y \in X$, with $x \leq y$ then by $[x, y]_{\leq}$ we will denote the ordered segment joining x and y, i. e. $[x, y]_{\leq} := \{z \in X | x \leq z \leq y\}$.

Definition 2.1. Let X be a nonempty set. Then, by definition, $(X, d, \leq)$ is an ordered metric space if and only if:

(i) (X, d) is a metric space

(ii) $(X, \leq)$ is a partially ordered set

(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leq y_n$, for each $n \in \mathbb{N} \Rightarrow x \leq y$.

The following result was proved in [12]:

Theorem 2.1. *Let $(X, d, \leq)$ be an ordered complete metric space and $T : X \to P_{cl}(X)$ a set-valued operator. Suppose that the following assertions hold:*

(i) there exists $x_0 \in X$ such that if $y \in T(x_0)$ then $(x_0, y) \in X_{\leq}$;

(ii) for each $x, y \in X$, with $(x, y) \notin X_{\leq}$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_{\leq}$ and $(y, c(x, y)) \in X_{\leq}$;

(iii) $(x, y) \in X_{\leq}$ implies ($u \in T(x)$ and $v \in T(y)$ then $(u, v) \in X_{\leq}$);

(iv) T is a closed set-valued operator;

(v) T is a set-valued $(X_{\leq}, a)$-contraction.

Then for each $x \in X$ there is a sequence $(x_n)n \in \mathbb{N}$ of successive approximations of T starting from x, that converges to a fixed point of T.

A data dependence theorem is the following.

Theorem 2.2. *Let $(X, d, \leq)$ be an ordered complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued operators. Let us suppose that the following assumptions hold:*

(i) there is $x_0 \in X$ such that, if $y \in T_1(x_0)$ then $(x_0, y) \in X_{\leq}$;

(ii) there is $y_0 \in X$ such that, if $y \in T_2(y_0)$ then $(y_0, y) \in X_\leq$;

(iii) for every $x, y \in X$, with $(x, y) \notin X_\leq$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_\leq$ and $(y, c(x, y)) \in X_\leq$;

(iv) if $(x, y) \in X_\leq$ then the following implication is true ($u \in T_i(x)$ şi $v \in T_i(y)$, then $(u, v) \in X_\leq$), for $i \in \{1, 2\}$;

(v) T_i is closed, for $i \in \{1, 2\}$;

(vi) T_i is a multivalued $(X_\leq, a_i)$-contraction, i. e.

$$a_i \in]0,1[\text{ and } H(T_i(x), T_i(y)) \leq a_i \cdot d(x, y), \text{ for each } x, y \in X, x \leq y,$$

for $i \in \{1, 2\}$;

(vii) if $x_0^* \in FixT_1$ then for each $y \in T_2(x_0^*)$ we have $(x_0^*, y) \in X_\leq$;

$(viii)$ if $y_0^* \in FixT_2$ then for each $y \in T_1(y_0^*)$ we have $(y_0^*, y) \in X_\leq$;

(ix) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for each $x \in X$.

Then:

(a) $Fix(T_i) \neq \emptyset$, for $i \in \{1, 2\}$;

(b) $H(Fix(T_1), Fix(T_2)) \leq \frac{\eta}{1-\max\{a_1,a_2\}}$.

Proof. From Theorem 2.1 we have that $Fix(T_i) \neq \emptyset$, for $i \in \{1, 2\}$. Moreover, since T_i is closed, for each $i \in \{1, 2\}$, we immediately get that $Fix(T_i)$ is closed, for $i \in \{1, 2\}$.

We will show now the assertion (b). Let $q > 1$ and $x_0^* \in Fix(T_1)$. Then, there exists $x_1 \in T_2(x_0^*)$ such that $d(x_0^*, x_1) \leq qH(T_1(x_0^*), T_2(x_0^*)) \leq q\eta$. For $x_1 \in T_2(x_0^*)$ there exists $x_2 \in T_2(x_1)$ such that $d(x_1, x_2) \leq qH(T_2(x_0^*), T_2(x_1)) \leq qa_2 d(x_0^*, x_1)$, since $(x_0^*, x_1) \in X_\leq$. We get that the sequence $(x_n)_{n\in\mathbb{N}}$ has the properties:

(1) $x_{n+1} \in T(x_n)$, $n \in \mathbb{N}$;

(2) $d(x_n, x_{n+1}) \leq (qa_2)^n d(x_0^*, x_1)$, for each $n \in \mathbb{N}$.

Then $d(x_n, x_{n+p}) \leq (qa_2)^n \frac{1}{1-qa_2} d(x_0^*, x_1)$, for each $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$. So, by choosing $q \in]1, a_2^{-1}[$ we have that the sequence (x_n) is convergent and its limit is a fixed point for T_2. Let us denote this fixed point by x_2^*.

When $p \to +\infty$ we have that $d(x_n, x_2^*) \leq (qa_2)^n \frac{1}{1-qa_2} d(x_0^*, x_1)$, for each $n \in \mathbb{N}$. For $n = 0$, we have that $d(x_0^*, x_2^*) \leq \frac{q\eta}{1-qa_2}$.

By a similar argument with respect to T_1 we are able to construct a sequence of successive approximations for T_1 starting from $y_0^* \in Fix(T_2)$. It follows that there exists $x_1^* \in Fix(T_1)$ such that $d(y_0^*, x_1^*) \leq \frac{q\eta}{1-qa_1}$.

From the above two relations we deduce

$$H(Fix(T_1), Fix(T_2)) \leq \frac{q\eta}{1 - q\max\{a_1, a_2\}}.$$

If we take $q \searrow 1$ the conclusion follows. □

3. Strict fixed points

We are now interested for the existence of the strict fixed points for a multivalued operator satisfying a contractive type condition with respect to δ.

A first result in this direction is the following.

Theorem 3.3. *Let $(X, d, \leq)$ be an ordered complete metric space and $T : X \to P_{b,cl}(X)$ a set-valued operator. Suppose that the following assertions hold:*

(i) there exists $x_0 \in X$ such that if $y \in T(x_0)$ then $(x_0, y) \in X_\leq$;

(ii) $(x, y) \in X_{\leq}$ *implies (*$u \in T(x)$ *and* $v \in T(y)$ *then* $(u, v) \in X_{\leq}$*);*

(iii) T *is a closed set-valued operator;*

(iv) *there exist* $a, b, c \in \mathbb{R}_+$*, with* $a + b + c \in]0, 1[$ *such that*

$$\delta(T(x), T(y)) \leq ad(x, y) + b\delta(x, T(x)) + c\delta(y, T(y)) \text{ for each } x, y \in X, \text{ with } x \leq y.$$

Then $Fix(T) = SFix(T) \neq \emptyset$*.*

Proof. Let $q > 1$. Then there exists $x_1 \in T(x_0)$ such that $\delta(x_0, T(x_0)) \leq qd(x_0, x_1)$. Then we have successively:

$\delta(x_1, T(x_1)) \leq \delta(T(x_0), T(x_1)) \leq ad(x_0, x_1) + b\delta(x_0, T(x_0)) + c\delta(x_1, T(x_1)) \leq ad(x_0, x_1) + bqd(x_0, x_1) + c\delta(x_1, T(x_1))$, since $x_0 \leq x_1$. Hence

$$\delta(x_1, T(x_1)) \leq \frac{a + bq}{1 - c} \cdot d(x_0, x_1).$$

We can construct a sequence $(x_n)n \in \mathbb{N}$ of successive approximations for T having the property:

$$d(x_n, x_{n+1}) \leq \delta(x_n, T(x_n)) \leq \left[\frac{a + bq}{1 - c}\right]^n d(x_0, x_1), \text{ for } n \in \mathbb{N}.$$

By choosing $q > \frac{b}{1-a-c}$, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence convergent to a certain $x^* \in X$. Since T is closed, we have that $x^* \in Fix(T)$.

Let us establish now the relation $Fix(T) \subset SFix(T)$. For this purpose, let $x^* \in Fix(T)$. Using (iv) for $x = y = x^*$ we get:

$$\delta(T(x^*)) \leq (b + c)\delta(x^*, T(x^*)) \leq (b + c)\delta(T(x^*)).$$

If we suppose $card(\delta(T(x^*))) > 1$ then we obtain the contradiction $b + c \geq 1$. Hence $\delta(T(x^*)) = 0$ and so, since $x^* \in Fix(T)$ we have got the conclusion $\{x^*\} = T(x^*)$. □

In a similar way to Theorem 2.2 we can prove the following data dependence theorem for the strict fixed point set of a multivalued operator satisfying a contractive type condition with respect to δ.

Theorem 3.4. *Let* $(X, d, \leq)$ *be an ordered complete metric space and* $T_1, T_2 : X \to P_{b,cl}(X)$ *two multivalued operators. Let us suppose that the following assumptions hold:*

(i) *there is* $x_0 \in X$ *such that, if* $y \in T_1(x_0)$ *then* $(x_0, y) \in X_{\leq}$*;*

(ii) *there is* $y_0 \in X$ *such that, if* $y \in T_2(y_0)$ *then* $(y_0, y) \in X_{\leq}$*;*

(iii) *if* $(x, y) \in X_{\leq}$ *then the following implication is true (*$u \in T_i(x)$ *şi* $v \in T_i(y)$*, then* $(u, v) \in X_{\leq}$*), for* $i \in \{1, 2\}$*;*

(iv) T_i *is closed, for* $i \in \{1, 2\}$

(v) *there exist* $a_i, b_i, c_i \in \mathbb{R}_+$*, with* $a_i + b_i + c_i \in]0, 1[$ *such that*

$$\delta(T_i(x), T_i(y)) \leq a_i d(x, y) + b_i\delta(x, T_i(x)) + c_i\delta(y, T_i(y)),$$

for each $x, y \in X$*, with* $x \leq y$ *and for* $i \in \{1, 2\}$*;*

(vi) *if* $x_0^* \in FixT_1$ *then for each* $y \in T_2(x_0^*)$ *we have* $(x_0^*, y) \in X_{\leq}$*;*

(vii) *if* $y_0^* \in FixT_2$ *then for each* $y \in T_1(y_0^*)$ *we have* $(y_0^*, y) \in X_{\leq}$*;*

$(viii)$ *there exists* $\eta > 0$ *such that* $\delta(T_1(x), T_2(x)) \leq \eta$*, for each* $x \in X$*.*

Then:

(a) $Fix(T_i) = SFix(T_i) \in P_{cl}(X)$*, for* $i \in \{1, 2\}$*;*

$(b)\ H(SFix(T_1), SFix(T_2)) \le \frac{(1-\min\{c_1,c_2\})\eta}{1-\max\{a_1+b_1+c_1, a_2+b_2+c_2\}}$.

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