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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Analysis of a generalization of the Signorini problems. Contact boundary conditions and frictions laws

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ABSTRACT. The contact conditions between two deformable bodies are approximated by a generalization of the Signorini problem due to the presence of a second deformable body. In the formulation of the contact problems, we must introduce a new notational framework in which the contact areas, the contact forces and the motions of associated boundaries are unknown beforehand, and must be determined as part of the solution. We obtain inequations which describe a restriction of the points from the contact boundary, supposing that these points move in a normal direction at one of the boundaries in contact. In this paper the strong and the variational of the boundaries contact conditions is presented, and we will formulate of the contact conditions and of the friction contact laws between two deformable bodies.

1. INTRODUCTION

Two main lines can be followed to impose contact conditions in normal direction: these are the non-penetration condition as geometrical constraints and constitutive laws for the micromechanical approach within the contact area. The interfacial behavior in the tangential direction (frictional response) is even more complicated. The most frequently used constitutive equation is the classical law of Coulomb. The purpose of the modelling of the contact conditions and of the friction contact laws between two deformable bodies, will be parametered by means of two applications one to one, we will also mention, under a functional framework, the transmission of forces in contact area, the contact stress and the friction law. The objective of this paper to emphasize dependency of the frictional coefficient with respect to the velocity of sliding, the difference between the adherence friction coefficient (fix contact) and the slide coefficient (sliding contact), and to give the variational form of the contact conditions with friction. However, other frictional laws are available which take into account local, micromechanical phenomena within the contact interface, see e.g. [3]. An extensive overview may be found in [6], and for the physical background see e.g. [7]. During the few last years frictional phenomena have also been considered within the framework of the theory of plasticity, this leads to non-associative slip rules.

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2. FORMULATION OF THE PROBLEM

Let us consider two linear elastic bodies that at a given time $t = 0$ occupy domains Ω^1 and $\Omega^2 \subset \mathbb{R}^d$ respectively, where $d = 2$ or $d = 3$. The boundary of each body, is divided into three subregions such that

$$\partial\Omega^1 = \Gamma^1 = \overline{\Gamma}^1_U \cup \overline{\Gamma}^1_N \cup \overline{\Gamma}^1_C \text{ and } \partial\Omega^2 = \Gamma^2 = \overline{\Gamma}^2_U \cup \overline{\Gamma}^2_N \cup \overline{\Gamma}^2_C,$$

which are topologically open, and disjoint, only Γ^1_C and Γ^2_C being accepted to have common points:

$$\Gamma^i_U \cap \Gamma^j_N = \emptyset \text{ for } (U,i) \neq (N,j),\ (U,N) \neq (C,C) \text{ and mess } (\Gamma^i_N) > 0,\ i,j = 1,2.$$

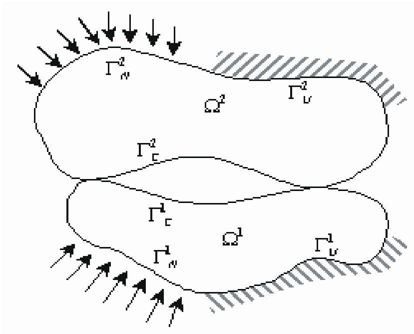

Figure 1. The contact of two elastic bodies

The displacement $\overline{u}(t,x)$ will be prescribed on $\Gamma_U = \overline{\Gamma}^1_U \cap \overline{\Gamma}^2_U$ and traction $\overline{h}(t,x)$ is to given on $\Gamma_N = \Gamma^1_N \cup \Gamma^2_N$. For the beginning, the boundary $\Gamma_N = \Gamma^1_N \cup \Gamma^2_N$ is considered without tensions. At the same time the stress vector $\sigma^{(n)}(u)$ is defined, oriented towards the exterior of the boundary $\partial\Omega = \partial\Omega^1 \cup \partial\Omega^2$. The initial displacement $u(0,x) = u_0(x)$, the initial velocity $\dot{\mathbf{u}}(0,x) = u_1(x)$ and the density of body force f are also given.

So long as the two bodies do not touch each other, the field of the displacements will be the solution of a boundary value problem of the differential equations of elastodynamics. If the two bodies touch one another, then in the contact boundary there are forces strong enough to prevent the interaction (penetration) of the two bodies. The condition that needs to be expressed in order to describe this process is called "the contact condition". Beside these forces there may appear in the contact area friction forces as well, which a law of friction can describe. The contact problem in a time interval with, has the following form.

The elastodynamic equation on $\Omega = \Omega_1 \cup \Omega_2$

$$\rho\ddot{\mathbf{u}}(t,x) - \sigma_{ij,j}(\mathbf{u}(t,x)) = \mathbf{f}(t,x) \text{ on } [0,t_E] \times \Omega. \tag{2.1}$$

The boundary conditions

$$\mathbf{u}(t,x) = \overline{\mathbf{u}}(t,x) \text{ on } [0,t_E] \times \Gamma_U, \tag{2.2}$$

$$\sigma^{(n)}(\mathbf{u})(t,x) = \overline{\mathbf{h}}(t,x) \text{ on } [0,t_E] \times \Gamma_N. \tag{2.3}$$

The contact condition and the friction law on $[0, t_E] \times \Gamma_C$ will be further presented, and the initial conditions are

$$u(0, x) = u_0(x) \text{ and } \dot{u}(0, x) = u_1(x).$$

3. THE CONTACT CONDITION

The contact condition will have to contain the condition of non-penetration of one body in the other (or their intersection (penetration), according to another law), the relative slip in the contact area, and the correct description of the transmission of forces between bodies. These processes must be expressed mathematically correct so that they could be approached by means of variational methods. Because of the difficulty that appears, the contact condition is approximated by the Signorini condition [1]. To approximate the contact conditions, we will situate in the linear elasticity theory.

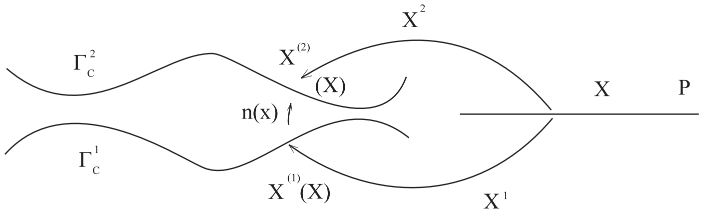

Fig. 2 The parametrization of the contact area

We will start by parametrization of the two contact boundaries Γ_C^1 and Γ_C^2 assumed to be disjoint. To this end, let us consider two one to one applications $x^{(1)} : P \to \Gamma_C^1$ and $x^{(2)} : P \to \Gamma_C^2$ of a domain P of C^1 class with the dimension $d-1$ for each contact zone.

So, in any $x \in P$ we can define the next notions:

– normal to Γ_C^1

$$n(x) := \frac{x^2(x) - x^1(x)}{|x^2(x) - x^1(x)|}, \tag{3.4}$$

– initial gap

$$g(x) := |x^2(x) - x^1(x)|, \tag{3.5}$$

– relative displacement, defined for a displacement field u given by

$$u^R(x) = u^{(1)}\left(x^{(1)}(x)\right) - u^{(2)}\left(x^{(2)}(x)\right), \tag{3.6}$$

where $u^{(j)} = u\big|_{\partial\Omega}$ represents the trace of u on $\partial\Omega$.

Further, we define the components $u_N := u \cdot n$ in direction n for a vectorial field $u : P \to \mathbb{R}^d$ and $u_T = u - u_N$ orthogonal on n.

The non-penetration condition of the two bodies will be determined as a geometrical contact condition. It is approximated by equation

$$u_N(x, t) \leq g(x).$$

This inequality describes the contact condition if the points on the contact zone move in the direction given by $n(x)$. We will analyze the error that results from this approximation. For this purpose the parameterizations $x^{(1)}$, $x^{(2)}$ and the boundaries Γ_C^1, Γ_C^2, which may touch one another must fulfill the conditions grouped under:

Hypothesis 3.1. Assume the next hypothesis:

H_1) Displacements u_i and deformations e_{ij} must be small, i.e.,

$$u_i(x) \le \varepsilon \text{ and } |e_{ij}(x)| \le \varepsilon, \tag{3.7}$$

where ε is a small positive parameter.

H_2) The contact boundaries Γ_C^1 and Γ_C^2 are in neighborhood, i.e.

$$|g(x)| \le 2\varepsilon,\ x \in P.$$

H_3) The curvatures of the two contact boundaries Γ_C^1 and Γ_C^2 are bounded.

H_4) The direction vector $n(x)$ must not deviate much from the normal $n^{(1)}(x^{(1)})$ at Γ_C^1 (normal oriented towards the exterior) than normal $n^{(2)}(x^{(2)})$ at Γ_C^2 and vice-versa:

$$\left|n(x) - n^{(1)}(x^{(1)}(x))\right| \le \left|n^{(1)}(x^{(1)}(x)) + n^{(2)}(x^{(2)}(x))\right| \text{ for } x \in P, \tag{3.8}$$

$$\left|n(x) + n^{(2)}(x^{(2)}(x))\right| \le \left|n^{(1)}(x^{(1)}(x)) + n^{(2)}(x^{(2)}(x))\right| \text{ for } x \in P. \tag{3.9}$$

The different signs before $n^{(1)}$ and $n^{(2)}$ are caused by the different orientation of these values.

Lemma 3.1. *Under the conditions of Hypothesis* 3.1 *the condition "Γ_C^1 and Γ_C^2 do not intersect" is equivalent to the inequality*

$$u_N^R(x) \le g(x) + r(x),$$

where $r(x)$ is the error that satisfies the condition $|r(x)| \le K\varepsilon^{3/2}$.

The proof of this result can be found in [2].

The contact condition has to correctly describe the transmission of forces between bodies and fulfill the next hypothesis, too:

1°. Newton's law regarding the equilibrium of forces must be valid, that is to say that force F^{12}, which is exercised by body Ω^1 upon body Ω^2, must be contrary to force F^{21} exercised by Ω^2 upon Ω^1.

2°. On the contact area only compressive forces can be transmitted.

3°. The forces can be transmitted only in such places where the bodies touch each other.

Condition 1° means

$$\sigma^{(n)}(x^{(1)}(x))J_1(x) = -\sigma^{(n)}(x^{(2)}(x))J_2(x) =: \sigma(x),\ x \in P, \tag{3.10}$$

with Gram determinants of the parameterizations $x^{(1)}$ and $x^{(2)}$:

$$J_k(x) \left| \det\left(\left(\left\langle \frac{\partial x^{(k)}}{\partial x_i}, \frac{\partial x^{(k)}}{\partial x_j} \right\rangle \right)_{i,j=1}^{d} \right) \right|, \quad k = \overline{1,2}.$$

Equation (3.10) can also be expressed as: let $x \in P$ and a surface $\Delta s \subset P$, $x \in \Delta s$. The surface element Δs is built by parameterizations $x^{(1)}$, $x^{(2)}$ and corresponds to the surface elements $\Delta s^{(1)} \subset \Gamma_C^1$ and $\Delta s^{(2)} \subset \Gamma_C^2$. Force F^{21} transmitted by Ω^2 through the surface element $\Delta s^{(1)}$ upon Ω^1 is

$$F^{21} = \int\limits_{\Delta s^{(1)}} \sigma^{(n)}(y)ds.$$

Accordingly, Ω^1 transmits through $\Delta s^{(2)}$ upon Ω^2 the force

$$F^{12} = \int\limits_{\Delta s^{(2)}} \sigma^{(n)}(y)ds.$$

After the transformation of the integral on the parameterized domain P, the equilibrium of forces is

$$\int\limits_{\Delta s} \sigma^{(n)}(x^{(1)}(x))J_1(x)ds = -\int\limits_{\Delta s} \sigma^{(n)}(x^{(2)}(x))J_2(x)ds.$$

Dividing by Δs and letting $|\Delta s| \to 0$, it results (3.10). The vectorial field $\sigma(x)$ defined in this way is given by the contact stress.

Condition 2° can be expressed in the following way:

$$\sigma_N(x) \leq 0 \text{ for } x \in P,$$

where $\sigma_N = \sigma_j n_j$ represents the contact stress component in the direction of vector n.

Condition 3° represents:

$$\sigma_N(x)\left(u_N^R(x) - g(x)\right) = 0, \ \forall\, x \in P.$$

In conclusion, the contact conditions can be modelled in the following way:

$$(\sigma^{(n)} \circ x^{(1)})J_1 = -(\sigma^{(n)} \circ x^{(2)})J_2 =: \sigma, \tag{3.11}$$

$$u_N^R \leq g;\ \sigma_N \leq 0;\ \sigma_N(u_N^R - g) = 0. \tag{3.12}$$

The variational expression of condition (3.12) has the form

$$u_N^R \leq g;\ \sigma_N\left(v_N^R - u_N^R\right) \geq 0,\ \forall\, v_N^R \in T. \tag{3.13}$$

So far, the stress on the boundary has been considered continuous functions in the local system. In the case of weak solutions of the elasticity equations, the boundaries stresses are generally defined as functions in the Sobolev space

$$H^{-1/2}(\Gamma_C^1, \mathbb{R}^d) \times H^{-1/2}(\Gamma_C^2, \mathbb{R}^d).$$

We will show how one can extend the operations with stresses on the contact boundaries Γ_C^1 and Γ_C^2 upon the parametrical surface P. We will define the contact stress and will express the balance of forces for a functional which models the contact conditions as well.

For this the following hypothesis is necessary upon $x^{(1)}$ and $x^{(2)}$.

Hypothesis 3.2. Let be the Lipschitz continuous parameterizations $x^{(1)}$ and $x^{(2)}$, with continuous inverses and the Gram determinants of the transformations J_1

and J_2 be bounded, which have bounded inverses, too. This means that the there exist the constants $0 < c_0 \le C_0$ and $0 < c_1 \le C_1$ such that

$$c_0|x-y| \le \left|x^{(k)}(x) - x^{(k)}(y)\right| \le C_0|x-y|, \quad \forall\, x, y \in P,\ k = 1, 2, \tag{3.14}$$

$$c_1 \le |J_1(x)|,\ |J_2(x)| \le C_1. \tag{3.15}$$

For $0 < \alpha < 1$ the function spaces with indexes α have the form

$$\overline{H}^{\alpha}_{\Gamma_U}(\Gamma^k_C) := \left\{ u\big|_{\Gamma^k_C} : u \in H^{\alpha}(\Gamma^k),\ u = 0 \quad \text{on} \quad \Gamma^k_U \right\},\ k = 1, 2$$

and

$$\overline{H}^{\alpha}_k(P) := \left\{ u \circ x^{(k)} : u \in H^{\alpha}_{\Gamma_U}(\Gamma^k_C) \right\},$$

and their dual

$$H^{-\alpha}_{\Gamma_U}(\Gamma^k_C) := \left(\overline{H}^{\alpha}_{\Gamma_U}(\Gamma^k_C)\right)^* \text{ and } H^{-\alpha}_k(P) := \left(\overline{H}^{\alpha}_k(P)\right)^*.$$

Now, we can define the operators

$$x^*_k : \overline{H}^{\alpha}(\Gamma^k_C) \to \overline{H}^{\alpha}_k(P), \quad u \to u \circ x^k. \tag{3.16}$$

We notice that x^*_1 and x^*_2 are linear operators. The continuity results from the property of the Sobolev-Slobodeckij norm.

$$\|x^*_1(u)\|^2_{H^{\alpha}(p)} = \int\limits_P |u(x^{(1)}(x))|^2 ds_x + \int\limits_P \int\limits_P \frac{|u(x^{(1)}(x)) - u(x^{(1)}(y))|^2}{|x-y|^{d-1+2\alpha}} ds_x ds_y$$

$$= \int\limits_{\Gamma^1_C} |u(x')|^2 J_1^{-1}(x') ds_{x'}$$

$$+ \int\limits_{\Gamma^1_C}\int\limits_{\Gamma^1_C} \frac{|u(x') - u(y')|^2}{|(x^{(1)})^{-1}(x') - (x^{(1)})^{-1}(y')|^{d-1+2\alpha}} J_1^{-1}(x') J_1^{-1}(y') ds_{x'} ds_{y'}$$

$$\le c_1^{-1} \int\limits_{\Gamma^1_C} |u(x')|^2 ds_{x'} + \frac{C_0^{d-1+2\alpha}}{c_1^2} \int\limits_{\Gamma^1_C}\int\limits_{\Gamma^1_C} \frac{|u(x') - u(y')|^2}{|x'-y'|^{d-1+2\alpha}} ds_{x'} ds_{y'}$$

$$\le k(C_0, c_1, d, \alpha)\|u\|^2_{H^{\alpha}(\Gamma^k_C)}.$$

From (3.14), x^*_k is bijection application and the inverse $(x^*_k)^{-1} = u \circ (x^{(k)})^{-1}$ applies the space $\overline{H}^{\alpha}_k(P)$ on the space $\overline{H}^{\alpha}_{\Gamma_U}(\Gamma^k_C)$. With the help of x^*_1 and x^*_2 we can define the transformations

$$\overline{x}_k : \overline{H}^{-\alpha}_{\Gamma_U}(\Gamma^k_C) \to \overline{H}^{-\alpha}_k(P),\ k = 1, 2$$

by

$$\langle \overline{x}_k f, x^*_k u\rangle_P := \langle f, u\rangle_{\Gamma^k_C},\ \forall\ f \in H^{-\alpha}_{\Gamma_U}(\Gamma^k_C),\ \forall\ u \in H^{\alpha}_{\Gamma_U}(\Gamma^k_C). \tag{3.17}$$

Operators x^*_k, $k = 1, 2$ are well defined and because they are invertible and continuous, from (3.17) results that $\overline{x}_k$ are continuous as well.

For a function $f \in L_2(\Gamma_C^k)$ we have:

$$\langle \overline{x}_k f, u\rangle_P = \left\langle f, (x_k^*)^{-1} u\right\rangle_{\Gamma_C^k} = \int\limits_{\Gamma_C^k} f(x) u\left(x_k^{-1}(x)\right) ds_x = \int\limits_P f\left(x^k(x)\right) u(x) J_k(x) ds_x$$

so $(\overline{x}_k f)(x) = f\left(x^{(k)}(x)\right) J_k(x)$ which is the definition $\overline{x}_k$ and is in accordance with the definition of the boundaries Γ_C^k on a parametrization field P. Thus, the balance of forces for the stress on the contact area

$$\sigma^n \in H_{\Gamma_U}^{-1/2}\left(\Gamma_C^1, \mathbb{R}^d\right) \times H_{\Gamma_U}^{-1/2}\left(\Gamma_C^2, \mathbb{R}^d\right)$$

is given by

$$\overline{x}_1\left(\sigma^n|_{\Gamma_C^1}\right) = -\overline{x}_2\left(\sigma^n|_{\Gamma_C^2}\right)$$

and the contact stress σ is given by $\sigma = \overline{x}_1\left(\sigma^{(n)}|_{\Gamma_C^2}\right)$.

4. The friction law

The oldest friction law (historically speaking) is the Coulomb friction law, which states that force F_R necessary for the movement of a body on a solid foundation is proportional with force F_N with which the body presses normal on the base.

In order to correctly describe the friction law and to mark out the difference between the adherence friction coefficient (in the case of the fixed contract) and the sliding friction coefficient (in the case of the sliding contact), the friction coefficient F must depend upon the velocity $\dot{\mathbf{u}}_T^R(x)$ with which the bodies slide one by the other in point x.

The friction law will be:

$$\left\{ \begin{array}{l} \dot{\mathbf{u}}_T^R = 0 \Rightarrow \sigma_T = F(0)|\sigma_N| \dfrac{\dot{\mathbf{u}}_T^R}{|\dot{\mathbf{u}}_T^R|} \\ \dot{\mathbf{u}}_T^R \neq 0 \Rightarrow \sigma_T = F(\dot{\mathbf{u}}_T^R)|\sigma_n| \dfrac{\dot{\mathbf{u}}_T^R}{|\dot{\mathbf{u}}_T^R|}. \end{array} \right. \tag{4.18}$$

The law describes the dependence of the tangential stress upon the normal stress and upon the velocity of sliding:

– if the bodies are adherent to one another (fix contact) in point x, we have $\dot{\mathbf{u}}_T^R = 0$ and so the value of the tangential forces density is smaller than the result of the multiplication of the adherence friction coefficient and the value of the normal forces density. This defines the adherent friction (fix contact);

– if the bodies slide in x one by the other, then $\dot{\mathbf{u}}_T^R \neq 0$ and the friction also causes a density of the tangential force which opposes the sliding force and whose value is equal to the result of the multiplication of the friction coefficient and the value of the normal force density.

This can be summarized in the following way: for higher sliding velocities, F is equal with the friction coefficient, sliding coefficient or is asymptotically approaching it, for lower velocities, because F is a continuous function, in point $\dot{\mathbf{u}}_T^R = 0$ will take the value of the adherence friction (fix contact). The results of the experimental research, which confirm a continuous dependence of the friction coefficient upon the sliding velocity, can be found in [7].

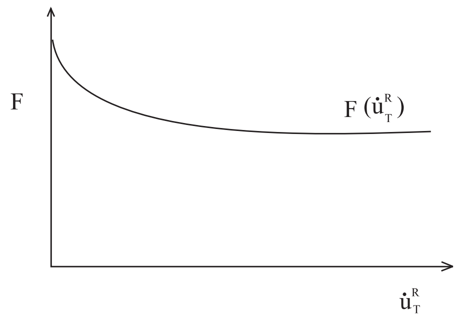

Fig. 3. Dependency of the friction coefficient with respect to velocity of sliding

If the friction coefficient is considered to be independent from the sliding velocity, then the conditions found in the specialty literature have the form:

$$|\boldsymbol{\sigma}_T| \leq F \quad |\boldsymbol{\sigma}_N| \Rightarrow \dot{\mathbf{u}}_T^R = 0, \quad |\boldsymbol{\sigma}_T| = F \quad |\sigma_N| \Rightarrow \exists\, \lambda > 0,\ \dot{\boldsymbol{u}}_T^R = \lambda \cdot \boldsymbol{\sigma}_T. \tag{4.19}$$

The variational form of the friction law has the form:

$$\boldsymbol{\sigma}_T \left(\boldsymbol{v}_T^R - \dot{\mathbf{u}}_T^R\right) + F\left(\dot{\mathbf{u}}_T^R\right)|\boldsymbol{\sigma}_N|\left(|\boldsymbol{v}_T^R| - |\dot{\mathbf{u}}_T^R|\right) \geq 0,\ \forall\, \boldsymbol{v}_T^R \in T, \tag{4.20}$$

where $T = T(x)$ is for $x \in P$ orthogonal subspace at $n(x)$, $T \subset \mathbb{R}^d$.

Proposition 4.1. *The formulation of friction law given by* (4.18) *and* (4.20) *are equivalent.*

Proof. (i) (4.18) $\Rightarrow$ (4.20). In the case $\dot{\mathbf{u}}_T^R = 0$ we have $\dot{\mathbf{u}}_T^R = 0$, so for each ν_T^R results

$$\begin{aligned} &\boldsymbol{\sigma}_T \left(v_T^R - \dot{\mathbf{u}}_T^R\right) + F\left(\dot{u}_T^R\right)|\boldsymbol{\sigma}_N|\left(|v_T^R| - |\dot{\mathbf{u}}_T^R|\right) \\ &= \boldsymbol{\sigma}_T \cdot \boldsymbol{v}_T^R + F(0)|\boldsymbol{\sigma}_N| v_T^R \geq \left(-|\boldsymbol{\sigma}_N| + F(0)|\boldsymbol{\sigma}_N|\right)|\boldsymbol{v}_T^R| \geq 0. \end{aligned}$$

For $\dot{\mathbf{u}}_T^R \neq 0$ we have $|\sigma_T| = -F\left(\dot{\mathbf{u}}_T^R\right)|\boldsymbol{\sigma}_{\boldsymbol{N}}|\frac{\dot{\mathbf{u}}_T^R}{|\dot{\mathbf{u}}_T^R|}$, hence for $\forall\, \boldsymbol{v}_T^R \in T$ results

$$\begin{aligned} &\boldsymbol{\sigma}_T \left(\boldsymbol{v}_T^R - \dot{\mathbf{u}}_T^R\right) + F\left(\dot{u}_T^R\right)|\boldsymbol{\sigma}_N|\left(|\boldsymbol{v}_T^R| - |\dot{\mathbf{u}}_T^R|\right) \\ &= F\left(\dot{\mathbf{u}}_T^R\right)|\boldsymbol{\sigma}_N|\left(\frac{-\dot{\mathbf{u}}_T^R}{|\dot{u}_T^R|}\boldsymbol{v}_T^R + |v_T^R|\right) \geq F\left(\dot{u}_T^R\right)|\boldsymbol{\sigma}_N|\left(-|\boldsymbol{v}_T^R| + |\boldsymbol{v}_T^R|\right) = 0 \end{aligned}$$

so (4.20) is proved.

(ii) (4.20) $\Rightarrow$ (4.18). If we take $v_T^R = -\lambda \cdot \sigma_T$, in (4.20) by division with λ, for $\lambda \to \infty$ and by division with $|\sigma_T|$, results

$$|\sigma_T| \leq F\left(\dot{\mathbf{u}}_T^R\right)|\sigma_N|. \tag{4.21}$$

For $\dot{\mathbf{u}}_T^R = 0$ results the statement (4.18). If $\dot{\mathbf{u}}_T^R \neq 0$, we put $\boldsymbol{v}_T^R = 2\dot{\mathbf{u}}_T^R$ and $\boldsymbol{v}_T^R = \frac{1}{2}\,\dot{\mathbf{u}}_T^R$ in (4.20) and we obtain

$$\boldsymbol{\sigma}_T \dot{\mathbf{u}}_T^R + F\left(\dot{\mathbf{u}}_T^R\right)|\boldsymbol{\sigma}_N| \cdot |\dot{\mathbf{u}}_T^R| = 0. \tag{4.22}$$

Using (4.22) and (4.21) we obtain

$$|\boldsymbol{\sigma}_T| = -|\boldsymbol{\sigma}_T| \frac{\dot{\mathbf{u}}_T^R}{|\dot{\mathbf{u}}_T^R|} ,$$

this means that $\boldsymbol{\sigma}_T$ is has to be oriented to $\dot{\mathbf{u}}_T^R$, so we will have:

$$\boldsymbol{\sigma}_T \cdot \dot{\mathbf{u}}_T^R = -|\boldsymbol{\sigma}_T| \, |\dot{\mathbf{u}}_T^R|.$$

Using (4.22) and $|\boldsymbol{\sigma}_T| = F\left(\dot{\mathbf{u}}_T^R\right) |\boldsymbol{\sigma}_N|$ we have

$$\boldsymbol{\sigma}_T = -F\left(\dot{\mathbf{u}}_T^R\right) |\boldsymbol{\sigma}_N| \frac{\dot{\mathbf{u}}_T^R}{|\dot{\mathbf{u}}_T^R|} .$$

With this we are able to completely express the dynamic contact problem. One other concept over of the friction law, is associated friction law equivalent with (4.19):

$$\text{(4.23)} \qquad \begin{cases} |\boldsymbol{\sigma}_T| \leq F\left(\dot{\boldsymbol{u}}_T^R\right) p(|R\boldsymbol{\sigma}_N|), \\ |\boldsymbol{\sigma}_N| < F(0) p(|R\boldsymbol{\sigma}_N|) \Rightarrow \dot{\mathbf{u}}_T^R = 0, \\ |\boldsymbol{\sigma}_T| < F\left(\dot{\mathbf{u}}_T^R\right) p(R\boldsymbol{\sigma}_N|) \Rightarrow \exists\, \lambda \ s.t.\ \boldsymbol{\sigma}_T = -\lambda \dot{\mathbf{u}}_T^R \end{cases}$$

Here R is a *normal regularization operator* that is, a linear and continuous operator $R : H^{-\frac{1}{2}}(P) \to L^2(P)$ need it to regularize the normal trace of the stress tensor on P. The function p is a non-negative function, the so-called *friction bound*. This friction law (4.23) states that tangential shear cannot exceed the maximum frictional resistance $F\left(\dot{\mathbf{u}}_T^R\right) p(|R\sigma_N|$, and non-local smoothing operator R is introduced for technical reasons, since the trace of the stress tensor on the boundary is too rough.

The friction law (4.23), was used with

$$p(r) = r \qquad \text{or}$$
$$p(r) = r(1 - \alpha r)_+,$$

where α is a small positive coefficient related to the wear and hardness of surface and $r_+ = \max\{0, r\}$. This friction law was derived from thermodynamic considerations and means that the normal stress is too large, that is, it exceeds $1/\alpha$, the surface disintegrates and offers no resistance to the motion, see e.g. [8], [9] and [10].

We are looking for the solution u of the differential equations system

$$(\rho(x)\ddot{\mathbf{u}}(t,x) - \boldsymbol{\sigma}_{ij,j}(\boldsymbol{u})(t,x) = f(t,x) \text{ on } [0,t_E] \times \Omega$$

with boundary conditions

$$\boldsymbol{u}(t,x) = \overline{\boldsymbol{u}}(t,x) \text{ on } [0,t_E] \times \Gamma_U,$$
$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u})(t,x) = \overline{\boldsymbol{h}}(t,x) \text{ on } [0,t_E] \times \Gamma_N,$$

$$\left.\begin{array}{r} \left(\boldsymbol{\sigma}^{(n)} \circ \boldsymbol{x}^{(1)}\right) J_1 = -\left(\boldsymbol{\sigma}^{(n)} \circ \boldsymbol{x}^{(2)}\right) J_2 =: \boldsymbol{\sigma} \\ \boldsymbol{u}_T^R \leq \boldsymbol{g},\ \boldsymbol{\sigma}_N = 0,\ \boldsymbol{\sigma}_N \left(\mathbf{u}_N^R - \boldsymbol{g}\right) = 0 \\ \dot{\mathbf{u}}_T^R = 0 \Rightarrow |\boldsymbol{\sigma}_T| \leq F(0)|\boldsymbol{\sigma}_N| \\ \dot{\mathbf{u}}_T^R \neq 0 \Rightarrow \boldsymbol{\sigma}_T = -F\left(\dot{\mathbf{u}}_T^R\right) |\boldsymbol{\sigma}_N| \frac{\dot{\mathbf{u}}_T^R}{|\dot{\mathbf{u}}_T^R|} \end{array}\right\} \quad \text{on } [0,t_E] \times P,$$

and initial conditions

$$u(x,0) = u_0(x), \quad \dot{\mathbf{u}}(0,x) = u_1(x).$$

□

The objective of this paper to emphasize dependency of the frictional coefficient with respect to the velocity of sliding, the difference between the adherence friction coefficient (fix contact) and the slide coefficient (sliding contact), and to give the variational form of the contact conditions with friction.

A correct physical interpretation of the static problem is possible only when it is considered as an incremental step in a temporal discretisation of the dynamic problem.

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REFERENCES

[1] Cocu, M., *Unilateral contact problems with friction for an elastoviscoplastic material with internal state variable*, Proc. Contact Mechanics Int. Symp., Ed. A. Curnier; PPUR, 1992, 207-216

[2] Drabla, S. and Sofonea, M., *Analysis of a Signorini problem with friction*, IMA J. of App. Math., **63** (1999) 113-130

[3] Duvaut, G., *Loi de frottement non locale*, J. Mec. The. Appl. Special Issue, 1982, 73-78

[4] Eck, C., *Existenz und Regularitat der Losungen fur Kontaktproblememit Reiburg*, Dissertation, Univ. Stuttgart, 1966

[5] Laursen, T. A., *Computational contact impact mechanics. Fundamentals of modeling interfacial phenomena in nonlinear finite element analysis*, Springer-Verlag Berlin Heidelberg New York, 2003

[6] Oden, J. T. and Martins, J. A. C., *Models and computational methods for dynamic friction phenomena*, Comp. Meth. Appl. Mech. Eng., **52** (1986), 527-634

[7] Pop, N., *Aplicatii ale inecuatiilor variationale în probleme de contact cu frecare*, Cub Press 22, 1998

[8] Signorini, A., *Sopra alcune questioni di elastostatica*, Atti. Soc. Ital. Progr. Sci., 1933

[9] Tabor, D., *Friction - The present state of our understanding*, J. Lubr. Technol., **103** (1981), 169-170

[10] Woo, K. L. and Thomas, T. R., *Contact of rough surface: A review of experimental work*, Wear, **58** (1980), 331-340

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