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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} anniversary

Reciprocally associability of a n-group operation with a binary operation

VASILE POP

ABSTRACT. By juxtaposition of a binary operation with a n-ary operation are obtain a $(n+1)$-ary operation. In this paper we study the conditions for the obtained operation to determine a $(n+1)$-group structure.

1. INTRODUCTION

In [1], J. Dombres determines the binary operations which are associative, associable and commutable with a quasi-group and monoid operation by using functional equations.

In [6] we have determined the conditions in which a group operation is reciprocally associable with a binary operation.

In this paper we study the associability of a n-groups operation with a binary operation and the conditions for extending a n-group to a $(n+1)$-group using a binary operation.

Let us notice that the word of extension from the title of the paper has nothing to do with Hosszu type extensions given by Dudek and Michalski, extensions which are made only from a $(n+1)$-group to a $(k \cdot n+1)$-group.

2. MAIN RESULTS

We define the reciprocally associability of a binary operation by a n-ary operation, the operation obtained by juxtaposing a binary and an n - ary operation and we find the condition for the obtained operation, to determine an $(n+1)$ - group (Theorem 2.3).

Definition 2.1. If $''*''$: $G^2 \longrightarrow G$ is a binary operation and $f : G^{n+1} \longrightarrow G$ is a $(n+1)$-ary operation $(n \geq 2)$, we say that the operations $''*''$ and f are reciprocal associable if the following conditions hold:

a) $x_1 * f(x_2, x_3, \ldots, x_{n+2}) = f(x_1 * x_2, x_3, \ldots, x_{n+2})$;

b) $f(x_1, \ldots, x_n, x_{n+1}) * x_{n+2} = f(x_1, \ldots, x_n, x_{n+1} * x_{n+2})$,

for all $x_i \in G,\ i = \overline{1, n+2}$.

Theorem 2.1. *If (G, f) is a $(n+1)$-group and $''*''$ is a binary operation on G, having a right (or left) unit element, and operations $''*''$ and f are reciprocally associable, then $(G, *)$ is a group.*

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Proof. Let e be a right unit element in $(G,*)$. If we consider the reduced Hosszu group $(G,\circ)=Red_e(G,f)$, we have:

$$x\circ y=f(x,\underset{n-2}{e},\overline{e},y)=f(x*e,\underset{n-2}{e},\overline{e},y)=x*f(e,\underset{n-2}{e},\overline{e},y)=x*y.$$

Since $(G,\circ)$ is a group and the operation $''*''$ coincides with the operation $''\circ''$, it follows that $(G,*)$ is a group. □

Remark 2.1. If (G,f) is a $(n+1)$- group and $(G,*)$ is a group with the unit element e and the operations $''f''$ and $''*''$ are reciprocal associable, then

$$(G,*)=Red_e(G,f).$$

Theorem 2.2. *If (G,f) is a $(n+1)$-group and $(G,*)$ is a group, then the operations $''f''$ and $''*''$ are reciprocal associable if and only if (G,f) is an Hosszu type extension of the bigroup $(G,*)$.*

Proof. Let $(G,f)\in Ext^{n+1}(G,*)$. By [2] there exists $\alpha\in Aut(G,*)$, an automorphism of $(G,*)$, $a\in G$ such that $\alpha^n(x)=a*x*a^{-1}$ and $(G,f)=Ext_{\alpha,a}(G,*)$. Then

$$f(x_1,\dots,x_{n+1})=x_1*\alpha(x_2)\cdots\alpha^{n-1}(x_n)*a*x_{n+1}$$

We have:

$$x_1*f(x_2,x_3,\dots,x_{n+2})=x_1*x_2*\alpha(x_3)\cdots\alpha^{n-1}(x_{n+1})*a*x_{n+2}$$
$$=(x_1*x_2)*\alpha(x_3)*\cdots*\alpha^{n-1}(x_{n+1})*a*x_{n+2}=f(x_1*x_2,x_3,\cdots,x_{n+2})$$

and similarly

$$f(x_1,\dots,x_n,x_{n+1})*x_{n+2}=f(x_1,\cdots,x_n,x_{n+1}*x_{n+2}).$$

Thus, the operations are reciprocal associable.

If f and $''*''$ are associable, from Remark 2.1 we have:

$$(G,*)=Red_e(G,f)$$

and so

$$(G,f)=Ext_{\alpha,a}(G,*),$$

where

$$\alpha(x)=f(e,x,\underset{n-2}{e},\bar{e})\quad\text{and}\quad a=f(\underset{n+1}{e}).$$

□

Definition 2.2. If f is a $(n+1)$-operation on G, reciprocal associable with the binary operation $''*''$, then we can define the $(n+2)$-ary operations in G as follows

(2.1) $$[*,f](x_1,x_2,\dots,x_{n+2})=x_1*f(x_2,\dots,x_{n+2})$$

(2.2) $$[f,*](x_1,\dots,x_{n+1},x_{n+2})=f(x_1,\dots,x_{n+1})*x_{n+2}.$$

Each of the operations defined in (2.1) and (2.2) are called juxtaposition of the two operations.

The problem which occurs in what follows is: under which conditions, the two operations on polygroups give a $(n+2)$-group operation. In the following we give a solution to this problem.

Definition 2.3. If (G, f) is a $(n+1)$- group, $(G, *)$ a bigroup the operations f and $''*''$ are reciprocal associable and $(G, [f, *])$ is $(n+2)$-group, then it is called the extension of the $(n+1)$-group (G, f) by the group $(G, *)$.

Definition 2.4. If $''*''$ is a binary operation and f a $(n+1)$-ary operation on G, we say that the operations are commutable if

$$f(x_1, x_2 \dots, x_{n+1}) * x_{n+2} = x_1 * f(x_2, \dots, x_{n+1}, x_{n+2})$$

for all $x_i \in G, \quad i = \overline{1, n+2}$.

Remark 2.2. Two binary operations $''*''$ and $''\circ''$ on G are commutable if:

$$x * (y \circ z) = (x \circ y) * z; \quad x, y, z \in G.$$

Remark 2.3. The reciprocal associable operations $''f''$ and $''*''$ are commutable if their juxtapositions coincide, that is:

$$[*, f] = [f, *].$$

Theorem 2.3. *If (G, f) is a $(n+1)$-group, $(G, *)$ is a group and the operations f and $''*''$ are reciprocal associable, then the following statements are equivalent:*

$i)$ *Operations $''*''$ and f are commutable,*

$ii)$ *$(G, [f, *])$ is a $(n+2)$-group.*

Proof. $i) \implies ii)$ Since the operations f and $''*''$ are reciprocal associable, from Theorem 2.2 follows that $(G, f) \in Ext^{n+1}(G, *)$, so

$$f(x_1, x_2, \dots, x_n, x_{n+1}) = x_1 * \alpha(x_2) * \dots * \alpha^{n-1}(x_n) * a * x_{n+1},$$

where $\alpha \in Aut(G, *), \quad \alpha(a) = a, \quad \alpha^n(x) = a * x * a^{-1}$.

The condition of commutability gives:

$$x_1 * x_2 * \alpha(x_2) * \dots * \alpha^{n-1}(x_{n+1}) * a * x_{n+2}$$
$$= x_1 * \alpha(x_2) * \dots * \alpha^{n-1}(x_n) * a * x_{n+1} * x_{n+2},$$

so

(2.3) $$x_2 * \alpha(x_3) * \dots * \alpha^{n-1}(x_{n+1}) * a = \alpha(x_2) * \dots * \alpha^{n-1}(x_n) * a * x_{n+1}$$

Taking in (2.3) $x_3 = \dots = x_{n+1} = e$, we obtain $x_2 = \alpha(x_2)$, whence

$$\alpha(x) = x, \quad x \in G.$$

Taking in (2.3) $x_2 = \dots = x_n = e$, we obtain $x_{n+1} * a = a * x_{n+1}$, so $a \in Z(G, *)$, the center of the group $(G, *)$. Hence

$$f(x_1, x_2, \dots, x_{n+1}) = x_1 * x_2 * \dots * x_{n+1} * a.$$

Denoting by g the $(n+2)$-ary operation, $G = (f, *)$ we obtain:

$$g(x_1, x_2, \dots, x_{n+1}, x_{n+2}) = x_1 * x_2 * \dots x_{n+1} * x_{n+2} * a,$$

which is obvious a $(n+2)$-particular group operation (derived group).

ii)$\Longrightarrow$i) First we prove that if the operations $''*''$ and f are reciprocal associable, then the operations $''*''$ and $g = [*, f]$ are reciprocal associable. We have:

$$x * g(y_1, \dots, y_{n+2}) = x * y_1 * f(y_2, \dots, y_{n+2})$$
$$= (x * y_1) * f(y_2, \dots, y_{n+2}) = g(x * y_1, y_2, \dots, y_{n+2})$$

and similarly

$$g(x_1, \ldots, x_{n+2}) * y = g(x_1, \ldots, x_{n+1}, x_{n+2} * y).$$

Using the Remark 2.1 for the operations "$*$" and $[*, f] = g$, we obtain

$$x * y = g\Big(x, \underset{n-1}{e}, \overline{\overline{e}}, y\Big),$$

where $\overline{\overline{e}}$ is the skew element of e in (G, g).
We have

$$x * y = x * f\Big(\underset{n-1}{e}, \overline{\overline{e}}, y\Big) = x * f\Big(e, \underset{n-2}{e}, \overline{\overline{e}}, y\Big)$$
$$= f\Big(x * e, \underset{n-2}{e}, \overline{\overline{e}}, y\Big) = f\Big(x, \underset{n-2}{e}, \overline{\overline{e}}, y\Big).$$

From $x * y = f\Big(x, \underset{n-2}{e}, \overline{e}, y\Big)$ we obtain $\overline{e} = \overline{\overline{e}}$ (since the skew elements of the unit element e, in (G, f) and (G, g) coincide).

The extension morphism from $(G, *)$ to (G, g) is

$$\beta(x) = g\Big(e, x, \underset{n-1}{e}, \overline{e}\Big) = e * f\Big(x, \underset{n-2}{e}, \overline{e}\Big) = e * x = x,$$

so

$$g(x_1, \ldots, x_{n+2}) = x_1 * \cdots * x_{n+2} * a, \quad a \in Z(G, *).$$

From $g(x_1, x_2, \ldots, x_{n+2}) = x_1 * f(x_2, \ldots, x_{n+2})$ we obtain

$$f(x_2, \ldots, x_{n+2}) = x_2 * \cdots * x_{n+2} * a.$$

The operation denoted by $h = [f, *]$ will be

$$h(x_1, \ldots, x_{n+2}) = f(x_1, \ldots, x_{n+1}) * x_{n+2}$$
$$= x_1 * \cdots * x_{n+1} * a * x_{n+2} = x_1 * \cdots * x_{n+1} * x_{n+2} * a,$$

so the operations obtained by juxtaposition g and f coincide. □

Corollary 2.1. *If (G, f) is a $(n+1)$- group and $(G, *)$ is a group, then the operations f and "$*$" are reciprocal associable and commutable if and only if the $(n+1)$ group (G, f) is derived by the group $(G, *)$.*

Proof. From the definition of the derived $(n+1)$ group, it follows that if $(G, f) \in Der^{n+1}(G, *)$, there exists $a \in Z(G, *)$ such that $(G, f) = Ext_{1_G, a}(G, *)$. Thus $f(x_1, \ldots, x_{n+1}) = x_1 * \cdots * x_{n+1} * a$, relation which has been obtained in the proof of the previous theorem. □

Remark 2.4. Every $(n+2)$-group obtained by the extension of a $(n+1)$-group by a group is a $(n+2)$ derived group.

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TECHNICAL UNIVERSITY OF CLUJ-NAPOCA
DEPARTMENT OF MATHEMATICS
C. DAICOVICIU 15, 400020 CLUJ-NAPOCA,
ROMANIA
E-mail address: vasile.pop@math.utcluj.ro