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Dedicated to Professor Ioan A. RUS on the occasion of his 70^{th} *anniversary*

Data dependence of the solutions for set differential equations

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ABSTRACT. Let $\mathbb{R}^n$ be the real n-dimensional Euclidian space and $P_{cp,cv}(\mathbb{R}^n)$ the family of all nonempty compact, convex subsets of $\mathbb{R}^n$ endowed with the Pompeiu-Hausdorff metric H.

Let $I = [a, b]$ and $X : I \to P_{cp,cv}(\mathbb{R}^n)$ be an operator. Hukuhara derivative of X might be introduced in the following way:

$$DX(t) = \lim_{h\to 0^+} \frac{X(t+h) - X(t)}{h} = \lim_{h\to 0^+} \frac{X(t) - X(t-h)}{h}.$$

We consider the Cauchy problem of the type

$$\begin{cases} DX = F(t, X) \\ X(0) = X_0 \end{cases}$$

where $F : I \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$.

The main purpose of the note is to study the data dependence of the solutions of the above problem.

1. INTRODUCTION

Let $\mathbb{R}^n$ be the real n-dimensional Euclidian space and $P_{cp,cv}(\mathbb{R}^n)$ the family of all nonempty compact, convex subset of $\mathbb{R}^n$ endowed with the Pompeiu-Hausdorff metric H.

Let $X : I \to P_{cp,cv}(\mathbb{R}^n)$, $I = [a, b]$ be an operator. Hukuhara derivative of X is defined in the following way [4]:

$$DX(t) = \lim_{h\to 0^+} \frac{X(t+h) - X(t)}{h}.$$

We consider the following Cauchy problem:

$$(PC) \begin{cases} DX &= F(t, X) \\ X(a) &= X_0 \end{cases}$$

where $F : I \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$.

The purpose of the note is to study the data dependence of the solutions of the above problem.

The paper is organized as follows. Next section, Preliminaries, contains some basic notations and notions used through the paper. Then, the third section presents a data dependence result for the solution of the previous problem.

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2. Preliminaries

The aim of this section is to present some notions and symbols used in the paper.

Let us define the following generalized functionals:

(1) $D : P(\mathbb{R}^n) \times P(\mathbb{R}^n) \to \mathbb{R}_+,\ D(A,B) = \inf\{d(a,b) \mid a \in A,\ b \in B\}$.
D is called the gap functional between A and B. In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.

(2) $\rho : P(\mathbb{R}^n) \times P(\mathbb{R}^n) \to \mathbb{R}_+ \cup \{+\infty\}$, $\rho(A,B) = \sup\{D(a,B) \mid a \in A\}$.
ρ is called the (generalized) excess functional.

(3) $H : P(\mathbb{R}^n) \times P(\mathbb{R}^n) \to \mathbb{R}_+ \cup \{+\infty\}$, $H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$.
H is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{cp,cv}(\mathbb{R}^n), H)$ is a complete metric space ([1]).

Lemma 2.1. ([3]) *Let X be a Banach space. Then $H(A+C, B+D) \leq H(A,B) + H(C,D)$, for $A, B, C, D \in P(X)$.*

Proof. Let $\varepsilon > 0$. From the definition of H it follows that there exists $(a+c) \in A+C$ such that $D(a+c, B+D) \geq H(A+C, B+D) - \varepsilon$ or exists $(b+d) \in B+D$ such that $D(b+d, A+C) \geq H(A+C, B+D) - \varepsilon$.

Let us consider the first case. For a, c we get $b \in B, d \in D$ such that $\|a-b\| \leq H(A,B) + \frac{\varepsilon}{2}$, $\|c-d\| \leq H(C,D) + \frac{\varepsilon}{2}$. Then $H(A+C, B+D) - \varepsilon \leq D(a+c, B+D) \leq \|(a+c)-(b+d)\|$ we obtain that $H(A+C, B+D) - \varepsilon \leq H(A,B) + H(C,D) + \varepsilon$, proving the desired inequality. ☐

Through the paper the operator $F : I \times P(\mathbb{R}^n) \to P(\mathbb{R}^n)$,(where $I = [a,b]$, for each $a, b \in \mathbb{R}$ with $a < b$), will be assumed to be continuous.

Definition 2.1. The operator $X : I \to P(\mathbb{R}^n)$ is said to be a *solution* of (PC) if it is continuously differentiable and satisfies (PC) everywhere on I.

We associate with the problem (PC) the following:

$$(I)\ \ X(t) = X_0 + \int_a^t F(s, X(s))ds,\ \ t \in [a,b],$$

where the integral which appears is the Hukuhara integral (see [4]).

Lemma 2.2. ([2]) *Let $X : I \to P(\mathbb{R}^n)$ be a continuously differentiable operator. We have:*

$$X(t) = X_0 + \int_a^t DX(s)ds,\ \ t \in [a,b].$$

Proof. The Hukuhara integral may be identified with the Bochner integral in cone of a Banach space B. Denote by $\xi : I \to B$

$$\xi(t) = \xi_0 + \int_a^t \dot{\xi}(s)ds,\ \ \text{for each}\ \ t \in [a,b].$$

where $\xi(a)$ corresponds to $X(a)$, $\dot{\xi}$ is the strong derivative of ξ and the integral is a Bochner integral. ☐

Lemma 2.3. ([2]) *The problem (PC) and the equation (I) are equivalent.*

3. Main results

The following result appeared in A. J. Brandao, F. S. De Blasi (see[2]). For the convenience of the reader we recall it here.

On $C(I, P_{cp,cv}(X))$ we consider the metric $H_*(X,Y) := \max_{t\in[a,b]} H(X(t),Y(t))$. The pair $(C(I, P_{cp,cv}(X)), H_*)$ is a Banach space.

Theorem 3.1. *Let $F : I \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ be an operator. Suppose that:*

$i)$ F is continuous on $I \times P_{cp,cv}(\mathbb{R}^n)$;

$ii)$ $F(t,\cdot)$ is Lipschitzian, for every $t \in I$, i.e., there exists $k \geq 0$ such that $H(F(t,X), F(t,Y)) \leq kH(X,Y)$, for all $X, Y \in P_{cp,cv}(\mathbb{R}^n)$ and all $t \in I$.

Then the problem (PC) has exactly one solution.

Proof. Consider the map Γ which associates to each continuous function $X : I \to P_{cp,cv}(\mathbb{R}^n)$ the continuous function $\Gamma : I \to P_{cp,cv}(\mathbb{R}^n)$ defined for each $t \in I$ by

$$\Gamma X(t) = X_o + \int_a^t F(s, X(s))ds.$$

Condition ii) and the Bielecki norm technique allow us to apply the Banach contraction principle. □

A data dependence result is:

Theorem 3.2. *Let $F, G : I \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$, be continuous. Consider the following problems*

$$(PC_1)\begin{cases} DX = F(t,X) \\ X(a) = X_0 \end{cases}$$

$$(PC_2)\begin{cases} DX = G(t,X) \\ X(a) = Y_0. \end{cases}$$

Suppose:

$i)$ $H(F(t,X), F(t,Y)) \leq k_1 H(X,Y)$, for all $X, Y \in P_{cp,cv}(\mathbb{R}^n)$, for all $t \in I$, where $k_1 > 0$. Denote by X_F^* the unique solution of the problem (PC_1);

$ii)$ There exists $\eta_i > 0$, $i = 1,2$ such that

$$H(F(t,X), G(t,X)) \leq \eta_1, \text{ for all } (t,X) \in I \times P_{cp,cv}(\mathbb{R}^n)$$

and $H(X_0, Y_0) \leq \eta_2$;

$iii)$ There exists X_G^* a solution of the problem (PC_2).

Then

$$H_*(X_F^*, X_G^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - k_1(b-a)}.$$

Proof. Consider the following integral equations

$$\Gamma_1 X(t) = X_o + \int_a^t F(s, X(s))ds. \tag{3.1}$$

$$\Gamma_2 X(t) = Y_o + \int_a^t G(s, X(s))ds. \tag{3.2}$$

Then from Lemma 2.3 , (3.1) is equivalent with (PC_1), while (3.2) is equivalent with (PC_2). We have:

$$\begin{aligned} H(X_F^*(t), X_G^*(t)) &= H(\Gamma_1 X_F^*(t), \Gamma_2 X_G^*(t)) \\ &\le H(\Gamma_1 X_F^*(t), \Gamma_1 X_G^*(t)) + H(\Gamma_1 X_G^*(t), \Gamma_2 X_G^*(t)) \\ &= H(X_o + \int_a^t F(s, X_F^*(s))ds, X_o + \int_a^t F(s, X_G^*(s))ds) \\ &+ H(X_o + \int_a^t F(s, X_G^*(s))ds, Y_o + \int_a^t G(s, X_G^*(s))ds) \\ &\le H(X_0, X_0) + H(\int_a^t F(s, X_F^*(s))ds, \int_a^t F(s, X_G^*(s))ds) \\ &+ H(X_0, Y_0) + H(\int_a^t F(s, X_G^*(s))ds, \int_a^t G(s, X_G^*(s))ds) \\ &\le H(X_0, Y_0) + \int_a^t H(F(s, X_F^*(s)), F(s, X_G^*(s)))ds + \int_a^t \eta_1 ds \\ &\le \eta_2 + \eta_1(b-a) + k_1(b-a)H(X_F^*(t), X_G^*(t)). \end{aligned}$$

Then we have:

$$\begin{aligned} \max_{t\in[a,b]} H(X_F^*(t), X_G^*(t)) &\le \max_{t\in[a,b]} (\eta_2 + \eta_1(b-a) + k_1(b-a)H(X_F^*(t), X_G^*(t))) \\ &\le \eta_2 + \eta_1(b-a) + k_1(b-a) \max_{t\in[a,b]} H(X_F^*(t), X_G^*(t)), \end{aligned}$$

$$\max_{t\in[a,b]} H(X_F^*(t), X_G^*(t)) \le \frac{\eta_2 + \eta_1(b-a)}{1 - k_1(b-a)}$$

$$H_*(X_F^*, X_G^*) \le \frac{\eta_2 + \eta_1(b-a)}{1 - k_1(b-a)}.$$

□

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