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Volterra-Fredholm nonlinear systems with modified argument via weakly Picard operators theory

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ABSTRACT.
In the present paper we consider the following system of nonlinear Volterra-Fredholm integral equations with modified argument:

$$u(t,x) = g(t,x,u(t,x),u(0,a)) + \int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds, \quad u \in C(\overline{D},\mathbb{R}^m).$$

For this system, we will prove: the existence of the solution, the data dependence of the solution, comparison theorems and a lower and upper subsolutions theorem.

1. INTRODUCTION

In this paper we consider the following system of nonlinear integral equation of Volterra-Fredholm (VF on short) type:

$$u(t,x) = g(t,x,u(t,x),u(0,a)) + \int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \tag{1.1}$$

for all $(t,x) \in [0,T] \times [a,b] := \overline{D}$; $u \in C(\overline{D},\mathbb{R}^m)$, where $b > a > 0$ and $T > 0$. Volterra-Fredholm integral equations often arise from the mathematical modelling of the spreading, in space and time, of some contagious diseases, in the theory of nonlinear parabolic boundary value problem and in many physical and biological models.

Most results for VF equation establish numerical approximation of the solutions, e.g. [8], [9], [24], [4], [11], [5], [7].

In [23], H. R. Thieme considered a model for the spatial spread of an epidemic consisting of a nonlinear integral equation of Volterra-Fredholm type having an unique solution. The author showed that this solution has a temporally asymptotic limit which describes the final state of the epidemic and is the minimal solution of another nonlinear integral equation.

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In [6], O. Diekmann described, derived and analysed a model of spatio-temporal development of an epidemic. The model considered leads (see [14]) to the following nonlinear integral equation of Volterra-Fredholm type:

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g(u(t-\tau,\xi))S_0(\xi)A(\tau,x,\xi)d\xi d\tau \tag{1.2}$$

for all $(t,x) \in [0,\infty] \times \Omega$, where Ω is a bounded domain in $\mathbb{R}^n$.

In [14] B. G. Pachpatte considered the integral equation

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega g(t,x,s,y,u(s,y))dyds \tag{1.3}$$

for all $(t,x) \in [0,T] \times \Omega = D$, where Ω is a bounded domain in $\mathbb{R}^n$. Using the contraction mapping principle, the author proved that, under appropriate assumptions, (1.3) has a unique solution in a subset S of $C(D,\mathbb{R}^n)$. The result was then applied to show the existence and uniqueness of solutions to certain nonlinear parabolic differential equations and mixed Volterra-Fredholm integral equations occurring in specific physical and biological problems (e.g. a reliable treatment of the Diekmann's model mentioned above is given).

In [10], D. Mangeron and L. E. Krivošein obtained existence, uniqueness and stability conditions for the solutions of a class of boundary problems for linear and nonlinear heat equation with delay. Under certain conditions, this problem is equivalent with the following nonlinear VF equation:

$$\begin{aligned} u(t,x) = n(t,x) + \int_0^t \int_0^a \Big[& G(x,\xi,t-\alpha) g\big(\xi,\alpha,u(\xi,\alpha),u\big(\xi,\alpha-r_1(\alpha)\big)\big) \\ & + \int_0^a \int_0^\alpha K(\xi,\alpha,s,y) g\big(s,y,u(s,y),u\big(s,y-r_2(s)\big)\big) dyds \Big] d\xi d\alpha, \end{aligned}$$

where

$$n(t,x) = \int_0^a \left[\frac{2}{a} \sum_{i=1}^{\infty} e^{-(\frac{\pi i}{a})^2 t} \cdot sin\frac{\pi i x}{a} \cdot sin\frac{\pi i \xi}{a} \cdot \varphi_0(\xi) \right] d\xi.$$

Applying the contraction mapping principle, an existence and uniqueness theorem is obtained.

In [15], the following problem is considered:

$$\begin{cases} u_t(t,x) = a^2 u_{xx}(t,x) + g\big(u(t,x),u(x,[t])\big) \\ u(x,0) = \varphi(x) \quad t \in \mathbb{R} \end{cases}$$

where $[t]$ means the integer part of t. Using integration by parts twice for the equation above, under appropriate conditions, the problem is equivalent with a VF equation and the successive approximation method is applied.

The purpose of the present paper is to give results concerning the following problems related to system (1.1): the existence of the solution, the data dependence of the solution, lower and upper subsolutions, comparison theorems.

Notations and basic notions

For any $x,y \in \mathbb{R}^m$, $x = (x_1,x_2,...,x_m)$, $y = (y_1,y_2,...,y_m)$: $x \le y \Longleftrightarrow x_i \le y_i$ for all $i \in \{1,2,...,m\}$;

$|x| := (|x_1|,|x_2|,...,|x_m|)$;

$\max\{x,y\} := \big(\max\{x_1,y_1\}, \max\{x_2,y_2\}, ..., \max\{x_m,y_m\}\big).$

Definition 1.1. Let X be a nonempty set and $d : X \times X \to \mathbb{R}^m$ such that:

(i) $d(x,y) \geq 0 \in \mathbb{R}^m$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$ for any $x, y \in X$;

(iii) for any $x, y, z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$.

Then (X,d) is said to be a *generalized a metric space* (g.m.s. on short).

Let (X,d) be a metric space (generalized or not) and $A : X \to X$ an operator.

$$F_A := \{x \in X : A(x) = x\};$$

$$A^0 := 1_X\ ,\ A^{n+1} := A \circ A^n \text{ for all } n \in \mathbb{N}.$$

and if $(X,d,\leq)$ is an ordered metric space (generalized or not):

$$(LF)_A := \big\{x \in X\ :\ x \leq A(x)\big\}$$

$$(UF)_A := \big\{x \in X\ :\ x \geq A(x)\big\}$$

Because the tool used in the present paper is the weakly Picard operators theory, for the convenient of the reader, we present some results concerning this important class of operators.

2. Weakly Picard operators

Definition 2.2. (Rus [16]) Let (X,d) be a metric space (generalized or not). An operator $A : X \to X$ is said to be:

(i) **weakly Picard** if for any $x_0 \in X$ we have: $A^n(x_0) \to x_0^*$, where $x_0^* \in F_A$ may depend on x_0.

(ii) **Picard** if $F_A = \{x^*\}$ and for any $x_0 \in X$ we have: $A^n(x_0) \to x^*$.

For a Po A, consider the mapping A^∞ defined by:

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Notice that $A^\infty(X) = F_A$. If A is a Po, then $A^\infty(x) = x^*$ for all $x \in X$, where x^* is the unique fixed point of A.

Example 2.1. Any operator satisfying the conditions of Perov fixed point theorem (see [17]) is a Po.

The following characterization theorem of wPo represents a basic tool in the study of this class of operators.

Theorem 2.1. *(Rus* [16]*) Let (X,d) be a metric space (generalized or not) and $A : X \to X$ an operator. Then A is wPo if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ such that:*

(i) for any $\lambda \in \Lambda$, we have $X_\lambda \in I(A)$;

(ii) for any $\lambda \in \Lambda$, the restriction $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is Po.

Data dependence of the fixed points of wPo

In order to study the data dependence of the solutions, the following abstract result (which generalizes a Rus theorem - see [17]) is required:

Definition 2.3. Let (X, d) be a g.m.s. The generalized Pompeiu - Hausdorff functional is

$$H : P(X) \times P(X) \to \mathbb{R}^m_+, \quad H = (H_1, H_2, ..., H_m)$$

defined by:

$$H_i(Y, Z) := max\left\{ \sup_{y \in Y} \inf_{z \in Z} d_i(y, z), \sup_{z \in Z} \inf_{y \in Y} d_i(y, z) \right\} \quad \text{for all } i = \overline{1, m}.$$

for any $Y, Z \in P(X)$.

Remark 2.1. With H defined above, $(P_{cl,b}(X), H)$ is a g.m.s., where $P_{cl,b}(X) = \{Y \subset X : Y$ nonempty, closed and bounded$\}$.

Definition 2.4. [1] Let (X, d) be a g.m.s. An operator $A : X \to X$ is said to be C**-weakly Picard** if it is wPo and there exists a matrix $C \in \mathcal{M}_m(\mathbb{R}_+)$ such that

$$d\left(x, A^\infty(x)\right) \le Cd\left(x, A(x)\right) \text{ for all } x \in X.$$

Furthermore, if A is Po, then it is said to be C**-Picard**

Example 2.2. [1] Let (X, d) be a complete g.m.s. and $A : X \to X$ an orbitally continuous operator. If there exists $M \in \mathcal{M}_m(\mathbb{R}_+)$ which converges to zero such that:

$$d\left(A^2(x), A(x)\right) \le Md\left(A(x), x\right) \text{ for all } x \in X,$$

then A is C-wPo with $C = (I - M)^{-1}$

Theorem 2.2. [1] *Let (X, d) be a complete g.m.s. and $A, B : X \to X$ two operators. Assume that:*
(i) there exist $C, D \in \mathcal{M}_m(\mathbb{R}_+)$ such that A is C-weakly Picard and B is D-weakly Picard;
(ii) there exists $\eta \in \mathbb{R}^m_+$ such that

$$d\left(A(x), B(x)\right) \le \eta \quad \text{for all } x \in X.$$

Then

$$H(F_A, F_B) \le \max\{C\eta, D\eta\}.$$

WPo in ordered metric spaces

Lemma 2.1. (Rus [19]) *Let $(X, d, \le)$ be an ordered metric space (generalized or not) and $A : X \to X$ such that:*
(i) *A is increasing;* (ii) *A is wPo.*
Then A^∞ is increasing.

Lemma 2.2. *(Abstract comparison lemma; Rus* [19]*) Let $(X, d, \le)$ be an ordered metric space (generalized or not) and $A, B, C : X \to X$ such that:*
(i) $A \le B \le C$; (ii) A, B, C are wPo; (iii) B is increasing.
If $x, y, z \in X$, with $x \le y \le z$, then

$$A^\infty(x) \le B^\infty(y) \le C^\infty(z).$$

Lemma 2.3. (Gronwall abstract lemma; Rus [19])
*Let $(X, d, \leq)$ be an ordered metric space (generalized or not) and $A : X \to X$ such that: (i) A is increasing; (ii) A is Po; Let $F_A = \{x_A^*\}$.
Then:*

$$(LF)_A \leq x_A^* \leq (UF)_A.$$

Applications of wPo theory in the study of various differential or integral equations may be found in: [21], [20], [19], [22], [13], [12], [2].

3. Existence theorem

Consider the system (1.1).

Theorem 3.3. *Assume that the following conditions are satisfied:*
(c1) *$g \in C(\overline{D} \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$, $K \in C(\overline{D} \times \overline{D} \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$, $\varphi_1 \in C(\overline{D}, [0, T])$ and $\varphi_2 \in C(\overline{D}, [a, b])$;*
(c2) *there exists a matrix $L_g \in \mathcal{M}_m(\mathbb{R}_+)$ such that:*

$$|g(t, x, u, \overline{u}) - g(t, x, v, \overline{v})| \leq L_g\left(|u - v| + |\overline{u} - \overline{v}|\right) \tag{3.4}$$

for all $(t, x) \in \overline{D}$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}^m$;
(c3) *there exists a matrix $L_K \in \mathcal{M}_m(\mathbb{R}_+)$ such that:*

$$|K(t, x, s, y, u, \overline{u}) - K(t, x, s, y, v, \overline{v})| \leq L_K\left(|u - v| + |\overline{u} - \overline{v}|\right) \tag{3.5}$$

for all $(t, x, s, y) \in \overline{D} \times \overline{D}$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}^m$;
(c4) *the matrix L_g converges to zero;*
(c5) *$g(0, a, \Lambda, \Lambda) = \Lambda$ for all $\Lambda \in \mathbb{R}^m$;*
(c6) *there exists $\tau > 0$ such that the matrix L defined by:*

$$L := L_g + \frac{b - a}{\tau} L_K + \max\left\{\int_0^t \int_a^b e^{\tau[\varphi_2(s,y) - t]} dy ds : t \in [0, T]\right\} L_K \tag{3.6}$$

converges to zero. Then (1.1) has a non finite number of solutions in $C(\overline{D}, \mathbb{R}^m)$.

Proof. Let the space $C(\overline{D}, \mathbb{R}^m)$ be endowed with a Bielecki-Chebysev suitable norm

$$\|u\|_{BC} = (\|u_1\|_{BC}, \|u_2\|_{BC}, ..., \|u_m\|_{BC}),$$

where

$$\|u_i\|_{BC} := \sup\{|u_i(t, x)| e^{-\tau t} : t \in [0, T], x \in [a, b]\}, \quad \tau > 0 \quad i = \overline{1, m}. \tag{3.7}$$

Consider the operator $A : C(\overline{D}, \mathbb{R}^m) \to C(\overline{D}, \mathbb{R}^m)$ defined by:

$$\begin{aligned} A(u)(t, x) :&= g(t, x, u(t, x), u(0, a)) \\ &+ \int_0^t \int_a^b K\Big(t, x, s, y, u(s, y), u\big(\varphi_1(s, y), \varphi_2(s, y)\big)\Big) dy ds \end{aligned} \tag{3.8}$$

for all $u \in C(\overline{D}, \mathbb{R}^m)$ for all $(t, x) \in \overline{D}$. For any $\Lambda \in \mathbb{R}^m$, consider the sets:

$$X_\Lambda := \left\{u \in C(\overline{D}, \mathbb{R}^m) : u(0, a) = \Lambda\right\} \tag{3.9}$$

It is easy to show that $C(\overline{D},\mathbb{R}^m) = \bigcup\limits_{\Lambda\in\mathbb{R}^m} X_\Lambda$ is a partition of the space $C(\overline{D},\mathbb{R}^m)$ and X_Λ are closed subsets for all $\Lambda \in \mathbb{R}^m$. Moreover, for all $u \in X_\Lambda$, we have:

$$A(u)(0,a) = g(0,a,\Lambda,\Lambda) = \Lambda, \quad \text{i.e. } A(u) \in X_\Lambda$$

So, X_Λ is invariant under A, for all $\Lambda \in \mathbb{R}^m$.

We will show that, for all $\Lambda \in \mathbb{R}^m$, $A|_{X_\Lambda} : X_\Lambda \to X_\Lambda$ is Po. Let $u, v \in X_\Lambda$. We have (see [3]):

$$\|A(u) - A(v)\|_{BC}$$

$$\leq \left[L_g + \frac{b-a}{\tau}L_K + \max\left\{\int_0^t\int_a^b e^{\tau[\varphi_2(s,y)-t]}dyds : t \in [0,T]\right\}L_K\right]\|u-v\|_{BC}$$

From **(c6)**, by Example 2.1 it follows that the operator $A : X_\Lambda \to X_\Lambda$ is a Po.
We are in the conditions of Theorem 2.1, so $A : C(\overline{D},\mathbb{R}^m) \to C(\overline{D},\mathbb{R}^m)$ is a wPo.
For all $u_0 \in C(\overline{D},\mathbb{R}^m)$, $A|_{X_{u_0(0,a)}} : X_{u_0(0,a)} \to X_{u_0(0,a)}$ is Po, so there exists a unique solution in $X_{u_0(0,a)}$. Let $u^*(u_0)$ be this solution; obviously it depends on u_0. □

Remark 3.2. Condition **(c6)** from Theorem 3.3 can be replaced by:

$$\textbf{(c7)} \quad \varphi_2(t,x) \leq t \quad \text{for all } (t,x) \in \overline{D}$$

In this case the operator A given by (3.8) is a $\overline{L}$-contraction, with

$$\overline{L} = L_g + \frac{2(b-a)}{\tau}L_K \tag{3.10}$$

and $\overline{L}$ converges to zero for a suitable chosen τ.

4. Data dependence of the solutions

In order to prove the dependence of the solutions of (1.1) on g and K, let us consider one more VF system:

$$\begin{aligned} u(t,x) &= h(t,x,u(t,x),u(0,a)) \\ &+ \int_0^t\int_a^b N\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds, \end{aligned} \tag{4.11}$$

for all $(t,x) \in \overline{D}$; $u \in C(\overline{D},\mathbb{R}^m)$, with $h \in C(\overline{D}\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$ and $N \in C(\overline{D}\times\overline{D}\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$.

Theorem 4.4. *Consider both systems* (1.1) *and* (4.11) *under the conditions* **(c1)**-**(c5)** *and* **(c7)**, *with* $\overline{L}_1$ *and* $\overline{L}_2$ *the matrices from* **(c7)**. *Assume there exist* $\eta_1, \eta_2 \in \mathbb{R}^m_+$ *such that:*
$|g(t,x,u,\overline{u}) - h(t,x,u,\overline{u})| \leq \eta_1$ *for all* $(t,x,u,\overline{u}) \in \overline{D}\times\mathbb{R}^m\times\mathbb{R}^m$;
$|K(t,x,s,y,u,\overline{u}) - N(t,x,s,y,u,\overline{u})| \leq \eta_2$ *for all* $(t,x,s,y,u,\overline{u}) \in \overline{D}\times\overline{D}\times\mathbb{R}^m\times\mathbb{R}^m$.
If S_1 *and* S_2 *are the sets of solutions of the systems in* $C(\overline{D},\mathbb{R}^m)$, *then:*
a) $S_1 \neq \emptyset$, $S_2 \neq \emptyset$ *and*

b) $H(S_1,S_2) \leq \max\left\{(I-\overline{L}_1)^{-1}\Big[\eta_1 + T(b-a)\eta_2\Big], (I-\overline{L}_2)^{-1}\Big[\eta_1 + T(b-a)\eta_2\Big]\right\}$, *where $H = (H_1, H_2, ..., H_m)$ is the Pompeiu-Hausdorff functional*

$$H : P\big(C(\overline{D}, \mathbb{R}^m)\big) \times P\big(C(\overline{D}, \mathbb{R}^m)\big) \to \mathbb{R}^m_+,$$

defined by:

$$H(U,V) := \begin{pmatrix} \max\big\{ \sup\limits_{u\in U} \inf\limits_{v\in V} \|u_1 - v_1\|_{BC}, \sup\limits_{v\in V} \inf\limits_{u\in U} \|u_1 - v_1\|_{BC} \big\} \\ \vdots \\ \max\big\{ \sup\limits_{u\in U} \inf\limits_{v\in V} \|u_m - v_m\|_{BC}, \sup\limits_{v\in V} \inf\limits_{u\in U} \|u_m - v_m\|_{BC} \big\} \end{pmatrix} \tag{4.12}$$

for all $U, V \in P\big(C(\overline{D}, \mathbb{R}^m)\big)$.

Proof. Consider the operators $A_1, A_2(\overline{D}, \mathbb{R}^m) \to C(\overline{D}, \mathbb{R}^m)$ defined by:

$$\begin{aligned} A_1(u)(t,x) :&= g(t,x,u(t,x),u(0,a)) \\ &+ \int_0^t \int_a^b K\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \end{aligned}$$

and

$$\begin{aligned} A_2(u)(t,x) :&= h(t,x,u(t,x),u(0,a)) \\ &+ \int_0^t \int_a^b N\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds. \end{aligned}$$

The systems (1.1) and (4.11) are fulfil conditions of Theorem 3.3, so A_1 and A_2 are wPo. Moreover, by Example 2.2, A_1 is C_1-wPo, with $C_1 = \big(I - \overline{L}_1\big)^{-1}$ and A_2 is C_2-wPo, with $C_2 = \big(I - \overline{L}_2\big)^{-1}$.

From hypotheses, it follows that for all $u \in C(\overline{D}, \mathbb{R}^m)$ we have:

$$\|A_1(u) - A_2(u)\|_{BC} \le \eta_1 + T(b-a)\eta_2$$

We can apply now Theorem 2.2 and the conclusion follows. □

5. Data dependence: monotonicity

In this section we give two comparison theorems and a Gronwall type theorem. Consider the system (1.1).

Theorem 5.5. *Assume conditions (**c1**)-(**c5**) and (**c7**) be satisfied.*

Moreover: $g(t,x,\cdot,\cdot)$ is increasing for all $(t,x) \in \overline{D}$ and $K(t,x,s,y,\cdot,\cdot)$ is increasing for all $(t,x,s,y) \in \overline{D} \times \overline{D}$.

If u and v are solutions of (1.1), with $u(0,a) \le v(0,a)$, then:

$$u \le v$$

Proof. Consider the operator $A : C(\overline{D}, \mathbb{R}^m) \to C(\overline{D}, \mathbb{R}^m)$ given by (3.8), for all $(t,x) \in \overline{D}$. For any $\Lambda \in \mathbb{R}^m$, consider the sets X_Λ given by (3.9). We are in the conditions of Theorem 3.3, so $A|_{X_\Lambda}$ is Po for all $\Lambda \in \mathbb{R}^m$, and A is wPo. From hypotheses, A is increasing. Therefore, we are in the conditions of Lemma 2.1.

For any $\alpha \in \mathbb{R}^m$, consider the function $\tilde{\alpha}$ given by:

$$\tilde{\alpha} : \overline{D} \to \mathbb{R}; \quad \tilde{\alpha}(t,x) := \alpha \text{ for all } (t,x) \in \overline{D}.$$

Obviously, $u \in F_A$. But $u \in X_{u(0,a)}$, so u is the unique fixed point of A in $X_{u(0,a)}$. It is clear that $\widetilde{u(0,a)} \in X_{u(0,a)}$; from here it follows that $A^\infty(\widetilde{u(0,a)}) = u$. In the same way, $A^\infty(\widetilde{v(0,a)}) = v$. We have $\widetilde{u(0,a)} \le \widetilde{v(0,a)}$ and, applying Lemma 2.1, we obtain

$$A^\infty(\widetilde{u(0,a)}) \le A^\infty(\widetilde{v(0,a)}) \quad \text{so:} \quad u \le v. \qquad \square$$

Consider now three VF systems:

$$u(t,x) = g_i(t,x,u(t,x),u(0,a)) + \int_0^t \int_a^b K_i\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds \quad i=1,2,3 \tag{5.13}$$

for all $(t,x) \in \overline{D}$; $u \in C(\overline{D},\mathbb{R}^m)$.

Theorem 5.6. *Assume that the systems* (5.13) *are fulfil conditions* **(c1)-(c5)** *and* **(c7)**. *Moreover, assume that*

(c8) $g_2(t,x,\cdot,\cdot)$ *is increasing for all* $(t,x) \in \overline{D}$ *and* $K_2(t,x,s,y,\cdot,\cdot)$ *is increasing for all* $(t,x,s,y) \in \overline{D} \times \overline{D}$;

(c9) $g_1 \le g_2 \le g_3$ *and* $K_1 \le K_2 \le K_3$.

If u, v and w are solutions of (5.13) *with* $u(0,a) \le v(0,a) \le w(0,a)$, *then:*

$$u \le v \le w.$$

Proof. By Theorem 3.3 $A_i : C(\overline{D},\mathbb{R}^m) \to C(\overline{D},\mathbb{R}^m)$ given by:

$$A_i(u)(t,x) := g_i(t,x,u(t,x),u(0,a)) + \int_0^t \int_a^b K_i\Big(t,x,s,y,u(s,y),u\big(\varphi_1(s,y),\varphi_2(s,y)\big)\Big)dyds,\ i=1,2,3$$

are wPo.

From (c8), A_2 is increasing, and from (c9) it follows that

$$A_1 \le A_2 \le A_3$$

Therefore, the conditions of Lemma 2.2 are fulfilled.

As in the proof of Theorem 5.5, we have (by Lemma 2.2),

$$A_1^\infty(\widetilde{u(0,a)}) \le A_2^\infty(\widetilde{v(0,a)}) \le A_3^\infty(\widetilde{w(0,a)})$$

and

$$A_1^\infty(\widetilde{u(0,a)}) = u,\ A_2^\infty(\widetilde{v(0,a)}) = v,\ A_3^\infty(\widetilde{w(0,a)}) = w$$

It follows that $u \le v \le w$. $\square$

For the system (1.1) we have the following Gronwall type theorem.

Theorem 5.7. *Assume that the conditions* **(c1)-(c5)** *and* **(c7)** *are fulfilled. Let u be a solution, v a lower subsolution and w an upper subsolution.*

If $v(0,a) \le u(0,a) \le w(0,a)$, *then:*

$$v \le u \le w$$

Proof. By Theorem 3.3, $A : C(\overline{D}, \mathbb{R}^m) \to C(\overline{D}, \mathbb{R}^m)$ given by (3.8) is Po. Because u is solution of the system, it follows that $A^\infty(\widetilde{u(0,a)}) = u$.

$v(0,a) \le u(0,a)$ implies $A^\infty(\widetilde{v(0,a)}) \le A^\infty(\widetilde{u(0,a)})$. But v is a lower subsolution so, applying Lemma 2.3, it follows that: $v \le A^\infty(\widetilde{v(0,a)})$.

We obtain: $v \le A^\infty(\widetilde{v(0,a)}) \le A^\infty(\widetilde{u(0,a)}) = u$. In the same way it can be showed that $u \le w$. $\square$

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References

[1] Bacoţiu, Claudia, *Data Dependence of the Fixed Points Set of Weakly Picard Operators in Generalized Metric Spaces*, Stud. Univ. Babeş-Bolyai Math. **49** (1) (2004), 15-17

[2] Bacoţiu, Claudia, *Existence, Uniqueness and Data Dependence for the Solutions of Volterra-Fredholm Integral Equations in L^2-spaces*, Mathematica **48** (1) (2006), 9-12

[3] Bacoţiu, Claudia, *Volterra-Fredholm Nonlinear Integral Equations Via Picard Operators Theory*, Mathematica (to appear)

[4] Brunner, H. and Messina, E., *Time-stepping methods for Volterra-Fredholm integral equations*, Rend. Mat. Appl. **23** (2003), 329-342

[5] Cardone, A., Messina, E. and Russo, E., *A fast iterative method for discretized Volterra-Fredholm integral equations*, J. Comput. Appl. Math. **189** (2006), 568-579

[6] Diekmann, O., *Thresholds and travelling waves for the geographical spread of infection*, J. Math. Biol. **6** (1978), 109-130

[7] Hącia, L., *On integral equations in space-time*, Demonstr. Math. **32** (4) (1999), 795-805

[8] Hadizadeh, M., *Posteriori Error Estimates for the Nonlinear Volterra-Fredholm Integral Equations*, J. Comput. Appl. Math. **45** (2003), 677-687

[9] Maleknejad, K. and Hadizadeh, M., *A New Computational Method for Volterra-Fredholm Integral Equations*, J. Comput. Appl. Math. **37** (1999), 1-8

[10] Mangeron, D. and Krivošein, L. E., *Sistemi policalorici a rimanenza ed a argomento ritardato; problemi al contorno per le equazioni integro-differenziali con operatore calorico ed argomento ritardato*, Rend. Sem. Mat., Univ. Padova (1965), 1-24

[11] Maleknejad, K. and Fadaei Yami, M. R., *A computational method for system of Volterra-Fredholm integral equations*, Appl. Math. Comput. **183** (2006), 589-595

[12] Mureşan, V., *Existence, Uniqueness and Data Dependence for the Solutions of some Integro-Differential Equations of Mixed Type in Banach Space*, J. Anal. Appl. **23** (2004), 205-216

[13] Olaru, M., *Data dependence for some integral equation via weakly Picard operators*, Gen. Math. **12** (3) (2004), 31-36

[14] Pachpatte, B. G., *On mixed Volterra-Fredholm type integral equations*, Indian J. Pure Appl. Math. **17** (1986), 448-496

[15] Poorkarimi, H. and Wiener, J., *Bounded solutions of nonlinear parabolic equations with time delay*, Electron. J. Diff. Eq. **2** (1999), 87-91

[16] Rus, I. A., *Weakly Picard mappings*, Comment. Math. Univ. Carolinae **34** (1993), no. 4, 769-773

[17] Rus, I. A., *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001

[18] Rus, I. A., *A delay integral equation from biomathematics*, Preprint no. 3, 1989, 87-90

[19] Rus, I. A., *Picard operators and applications*, Sci. Math. Jpn. **58** (1), (2003), 191–219

[20] Rus, I. A., *Weakly Picard operators and applications*, Seminar on Fixed Point Theory Cluj Napoca **2** (2001), 41-58

[21] Rus, I. A., *Fiber Picard operators and applications*, Stud. Univ. Babeş-Bolyai Math. **44** (1999), 89-98

[22] Rus, I. A., *Iterative functional-differential equations via weakly Picard operators*, ICAM5 Baia Mare, Sept. 21-24 2006

[23] Thieme, H. R., *A Model for the Spatial Spread of an Epidemic*, J. Math. Biol. **4** (1977), 337-351

[24] Wazwaz, A. M., *A reliable treatment for mixed Volterra- Fredholm integral equations*, Appl. Math. Comput. **127** (2002), 405-414

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